BULLETIN OF THE
HELLENIC MATHEMATICAL SOCIETY
Volume 64, 2020 (32–55)

NON-ASYMPTOTIC ℓ_1 SPACES WITH UNIQUE ℓ_1 ASYMPTOTIC MODEL

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Abstract. A recent result of Freeman, Odell, Sari, and Zheng [FOSZ] states that whenever a separable Banach space not containing ℓ_1 has the property that all asymptotic models generated by weakly null sequences are equivalent to the unit vector basis of c_0 then the space is Asymptotic c_0. We show that if we replace c_0 with ℓ_1 then this result is no longer true. Moreover, a stronger result of B. Maurey - H. P. Rosenthal [MR] type is presented, namely, there exists a reflexive Banach space with an unconditional basis admitting ℓ_1 as a unique asymptotic model whereas any subsequence of the basis generates a non-Asymptotic ℓ_1 subspace.

1. Introduction

In this paper we study the question whether the uniqueness of asymptotic models, or equivalently, the uniform uniqueness of joint spreading models in a given Banach space implies that the space must be Asymptotic ℓ_p. This is a coordinate free version from [MMT] of the notion of asymptotic ℓ_p spaces with a Schauder basis by Milman and Tomczak-Jaegermann from [MT]. The question draws its motivation from the following Problem of Halbeisen and Odell from [HO] and a subsequent remarkable result from [FOSZ]. Given a Banach space X, let $\mathscr{F}_0(X)$ denote the class of normalized weakly null sequences and $\mathscr{F}_b(X)$ denote the class of all normalized block sequences of a fixed basis, if X has one.

Problem 1 ([HO]). *Let X be a Banach space that admits a unique asymptotic model with respect to $\mathscr{F}_0(X)$, or with respect to $\mathscr{F}_b(X)$ if X has a basis. Does X contain an Asymptotic ℓ_p, $1 \leq p < \infty$, or an Asymptotic c_0 subspace?*

An asymptotic model is a notion which describes the asymptotic behavior of an array of sequences $(x^i_j)_j$, $i \in \mathbb{N}$. On the contrary a space is Asymptotic ℓ_p, for $1 \leq p < \infty$, (resp. Asymptotic c_0) if the asymptotic behavior of the whole space resembles that of ℓ_p (resp. c_0). Remarkably, in some cases unique asymptotic array structure implies that a space is Asymptotic c_0.

Theorem 1.1 ([FOSZ]). *Let X be a separable Banach space that does not contain ℓ_1 and admits a unique c_0 asymptotic model with respect to $\mathscr{F}_0(X)$. Then the space X is Asymptotic c_0.*

It was observed by Baudier, Lancien, Kalton, the third author, and Schlumprecht in [BLMS, Section 9.2] that Theorem 1.1 no longer holds if we replace c_0 with ℓ_p for any $1 < p < \infty$. The counterexamples are spaces very similar to the space defined by Szlenk in [S]. The main purpose of this paper is to provide an answer for the remaining case $p = 1$. Note that the main obstruction in this case is the fact that the ℓ_1-norm is the largest one and hence, assuming

Date: Submitted Oct. 5, 2019; accepted Feb. 6, 2020.
2010 *Mathematics Subject Classification.* 46B03, 46B06, 46B25, 46B45.
Key words and phrases. Asymptotic-ℓ_p spaces, joint spreading models, separation of asymptotic structures.
The third author's research was supported by the National Science Foundation under Grant Numbers NSF DMS-1600600 and NSF DMS-1912897.

that the space admits a unique ℓ_1 asymptotic model which means a very strong presence of asymptotic ℓ_1 structure, it is not obvious how to preserve a tree structure in the space which has norm smaller than ℓ_1.

Theorem 1.2. *There exists a reflexive Banach space X with an unconditional basis that admits a unique ℓ_1 asymptotic model with respect to $\mathscr{F}_0(X)$, whereas it is not an Asymptotic ℓ_1 space.*

In fact, for every countable ordinal ξ, there is a space T^ξ_{inc}, as in Theorem 1.2, that contains a weakly null ℓ_2-tree of height ω^ξ. An easy modification of T^ξ_{inc} can yield a space containing a weakly null ℓ_p-tree, for any $1 < p < \infty$ with $p \neq 2$, or a weakly null c_0-tree of height ω^ξ. Furthermore, the following analogue of the classical B. Maurey - H. P. Rosenthal [MR] result is proved, which yields a stronger separation of the two properties than Theorem 1.2.

Theorem 1.3. *There exists a reflexive Banach space X with an unconditional basis that admits a unique ℓ_1 asymptotic model with respect to $\mathscr{F}_0(X)$, whereas any subsequence of the basis generates a non-Asymptotic ℓ_1 subspace.*

More specifically, for every countable ordinal ξ, there is a space $T^\xi_{ess\text{-}inc}$ as in the theorem above such that the space generated by any infinite subsequence of its basis contains a block c_0-tree of height ω^ξ. It is possible to modify $T^\xi_{ess\text{-}inc}$ to contain ℓ_p-trees for any $1 < p < \infty$, instead of c_0-trees.

In the final part of this paper we show that, for $1 < r < p < \infty$, certain spaces $JT^\xi_{r,p}$, similar to those defined by Odell and Schlumprecht in [OS, Example 4.2] (see also [O2, page 66]), admit a unique ℓ_p asymptotic model but are not Asymptotic ℓ_p. These are spaces with an unconditional Schauder basis $(e_t)_{t\in\mathcal{T}_\xi}$ indexed over a well-founded and infinite branching countable tree $\mathcal{T}_\xi$ of height ω^ξ. The norm of $JT^\xi_{r,p}$ is defined as follows: if $x = \sum_{t\in\mathcal{T}_\xi} a_t e_t$ and S is a segment of $\mathcal{T}_\xi$ define $\|S(x)\|_r^r = \sum_{t\in S} |a_t|^r$ and

$$\|x\|_{JT^\xi_{r,p}} = \sup\left\{\Big(\sum_{i=1}^n \|S_i(x)\|_r^p\Big)^{1/p} : (S_i)_{i=1}^n \text{ disjoint segments of } \mathcal{T}_\xi\right\}. \tag{1}$$

The space T^ξ_{inc} from Theorem 1.2 is defined on the same tree. We say that two segments S_1, S_2 of $\mathcal{T}_\xi$ are incomparable if any node of S_1 is incomparable to any node of S_2. We relabel the basis of the Tsirelson space T as $(e_t)_{t\in\mathcal{T}_\xi}$ so that the order is compatible with the initial one and define the norm of T^ξ_{inc} as follows : for $x = \sum_{t\in\mathcal{T}_\xi} a_t e_t$ define $\|S(x)\|_2^2 = \sum_{t\in S} |a_t|^2$ and

$$\|x\|_{T^\xi_{inc}} = \sup\left\{\Big\|\sum_{i=1}^n \|S_i(x)\|_2 e_{\min S_i}\Big\|_T : (S_i)_{i=1}^n \text{ incomparable segments of } \mathcal{T}_\xi\right\}.$$

However, we will not use the above description of the norms. Instead we revert to the notion of norming sets and norming functionals. This makes some parts of the proof easier and it can also be potentially useful to show similar results on more complicated spaces based on these norms.

Finally, we should mention that Problem 1 is only one of several concerning the separation of different asymptotic structures in Banach space theory. For example, in [AM3] the first and third author showed that there exist spaces with a uniformly unique spreading model, which can be chosen to be any ℓ_p or c_0, that have no Asymptotic ℓ_p or c_0 subspace. This answers a question by Odell in [O1] and Junge, Kutzarova, and Odell in [JKO]. Moreover, in [KM]

Kutzarova and the third author showed that certain spaces by Beanland, the first author, and the third author from [ABM] are asymptotically symmetric and have no Asymptotic ℓ_p or c_0 subspaces, answering a question from [JKO].

Notation. *By* $\mathbb{N} = \{1, 2, \ldots\}$ *we denote the set of all positive integers. We will use capital letters as* $L, M, N, \ldots$ *(resp. lower case letters as* $s, t, u, \ldots$*) to denote infinite subsets (resp. finite subsets) of* $\mathbb{N}$*. For every infinite subset* L *of* $\mathbb{N}$*, the notation* $[L]^\infty$ *(resp.* $[L]^{<\infty}$*) stands for the set of all infinite (resp. finite) subsets of* L*. For every* $s \in [\mathbb{N}]^{<\infty}$*, by* $|s|$ *we denote the cardinality of* s*. For* $L \in [\mathbb{N}]^\infty$ *and* $k \in \mathbb{N}$*,* $[L]^k$ *(resp.* $[L]^{\leq k}$*) is the set of all* $s \in [L]^{<\infty}$ *with* $|s| = k$ *(resp.* $|s| \leq k$*). For every* $s, t \in [\mathbb{N}]^{<\infty}$*, we write* $s < t$ *if either at least one of them is the empty set, or* $\max s < \min t$*. Also for* $\emptyset \neq M \in [\mathbb{N}]^\infty$ *and* $n \in \mathbb{N}$ *we write* $n < M$ *if* $n < \min M$*. For* $s = \{n_1 < \ldots < n_k\} \in [\mathbb{N}]^{<\infty}$ *and for each* $1 \leq i \leq k$*, we set* $s(i) = n_i$*.*

Moreover, we follow [LT] *for standard notation and terminology concerning Banach space theory.*

2. Asymptotic structures

Let us recall the definitions of the asymptotic notions that appear in the results of this paper and were mentioned in the introduction. Namely, asymptotic models, joint spreading models and the notions of Asymptotic ℓ_p and Asymptotic c_0 spaces. For a more thorough discussion, including several open problems and known results, we refer the reader to [AM3, Section 3].

Definition 2.1 ([HO])**.** An infinite array of sequences $(x_j^i)_j$, $i \in \mathbb{N}$, in a Banach space X, is said to generate a sequence $(e_i)_i$, in a seminormed space E, as an asymptotic model if for every $\varepsilon > 0$ and $n \in \mathbb{N}$, there is a $k_0 \in \mathbb{N}$ such that for any natural numbers $k_0 \leq k_1 < \cdots < k_n$ and any choice of scalars $a_1, \ldots, a_n$ in $[-1, 1]$ we have that

$$\left| \Big\| \sum_{i=1}^n a_i x_{k_i}^i \Big\| - \Big\| \sum_{i=1}^n a_i e_i \Big\| \right| < \varepsilon.$$

A Banach space X is said to admit a unique asymptotic model with respect to a family $\mathscr{F}$ of normalized sequences in X if whenever two infinite arrays, consisting of sequences from $\mathscr{F}$, generate asymptotic models then those must be equivalent. Typical families under consideration are those of normalized weakly null sequences, denoted $\mathscr{F}_0(X)$, normalized Schauder basic sequences, denoted $\mathscr{F}(X)$, or the family all normalized block sequences of a fixed basis of X, if it has one, denoted $\mathscr{F}_b(X)$.

The notion of plegma families was first introduced by Kanellopoulos, Tyros, and the first author in [AKT]. We will use the slightly modified definition of from [AGLM].

Definition 2.2 ([AGLM])**.** Let $M \in [\mathbb{N}]^\infty$ and $k \in \mathbb{N}$. A plegma (resp. strict plegma) family in $[M]^k$ is a finite sequence $(s_i)_{i=1}^l$ in $[M]^k$ satisfying the following.

(i) $s_{i_1}(j_1) < s_{i_2}(j_2)$ for every $1 \leq j_1 < j_2 \leq k$ and $1 \leq i_1, i_2 \leq l$.
(ii) $s_{i_1}(j) \leq s_{i_2}(j)$ $\big($resp. $s_{i_1}(j) < s_{i_2}(j)\big)$ for all $1 \leq i_1 < i_2 \leq l$ and $1 \leq j \leq k$.

For each $l \in \mathbb{N}$, the set of all sequences $(s_i)_{i=1}^l$ which are plegma families in $[M]^k$ will be denoted by $Plm_l([M]^k)$ and that of the strict plegma ones by $S\text{-}Plm_l([M]^k)$.

Definition 2.3 ([AGLM])**.** A finite array of sequences $(x_j^i)_j$, $1 \leq i \leq l$, in a Banach space X, is said to generate another array of sequences $(e_j^i)_j$, $1 \leq i \leq l$, in a seminormed space E, as a joint spreading model if for every $\varepsilon > 0$ and $n \in \mathbb{N}$, there is a $k_0 \in \mathbb{N}$ such that for any

$(s_i)_{i=1}^l \in S\text{-}Plm([\mathbb{N}]^n)$ with $k_0 \leq s_1(1)$ and any $l \times n$ matrix $A = (a_{ij})$ with entries in $[-1, 1]$ we have that

$$\left| \Big\| \sum_{i=1}^{l} \sum_{j=1}^{n} a_{ij} x^i_{s_i(j)} \Big\| - \Big\| \sum_{i=1}^{l} \sum_{j=1}^{n} a_{ij} e^i_j \Big\| \right| < \varepsilon.$$

A Banach space X is said to admit a uniformly unique joint spreading model with respect to a family of normalized sequences $\mathscr{F}$ in X, if there exists a constant C such that whenever two arrays $(x^i_j)_j$ and $(y^i_j)_j$, $1 \leq i \leq l$, of sequences from $\mathscr{F}$ generate joint spreading models then those must be C-equivalent. Moreover, a Banach space admits a uniformly unique joint spreading model with respect to a family $\mathscr{F}$ if and only if it admits a unique asymptotic model with respect to $\mathscr{F}$ (see, e.g., [AGLM, Remark 4.21] or [AM3, Proposition 3.12]).

It was proved in [AGLM] that whenever a Banach space admits a uniformly unique joint spreading model with respect to some family satisfying certain stability conditions, then it satisfies a property concerning its bounded linear operators, called the Uniform Approximation on Large Subspaces property (see [AGLM, Theorem 5.17] and [AGLM, Theorem 5.23]).

Definition 2.4 ([MT] and [MMT]). A Banach space X is called Asymptotic ℓ_p, $1 \leq p < \infty$, (resp. Asymptotic c_0) if there exists a constant C such that in a two-player n-turn game $G(n, p, C)$, where in each turn $k = 1, \ldots, n$ player (S) picks a finite codimensional subspace Y_k of X and then player (V) picks a normalized vector $x_k \in Y_k$, player (S) has a winning strategy to force player (V) to pick a sequence $(x_k)_{k=1}^n$ that is C-equivalent to the unit vector basis of ℓ_p^n (resp. ℓ_n^∞).

The typical example of a non-classical Asymptotic ℓ_p space is the Tsirelson space as defined by Figiel and Johnson in [FJ]. This is a reflexive Asymptotic ℓ_1 space and it is the dual of Tsirelson's original space from [T] that is Asymptotic c_0. Finally, whenever a Banach space is Asymptotic ℓ_p or Asymptotic c_0, then it admits a uniformly unique joint spreading model with respect to $\mathscr{F}_0(X)$ (see, e.g., [AGLM, Corollary 4.12]).

3. A family of non-asymptotic ℓ_1 spaces admitting uniformly unique ℓ_1 joint spreading models

In this section we define the spaces T^ξ_{inc}, for each countable ordinal ξ, and we prove that they admit a uniformly unique ℓ_1 joint spreading model with respect to $\mathscr{F}_b(T^\xi_{inc})$ and are not Asymptotic ℓ_1. The spaces are defined in terms of norming sets and norming functionals as this is more convenient to prove the desired result.

3.1. **Measures on Well-Founded Countable Compact Trees.** We start with a key result that will be used later to prove that T^ξ_{inc} admits a uniformly unique joint spreading model equivalent to the unit vector basis of ℓ_1.

Let $\preceq$ be a partial order on some infinite subset M of the naturals, which is compatible with the standard order, i.e. $n \preceq m$ implies $n \leq m$, for all $n, m \in M$. Assume that, for each $n \in M$, the set $S_n = \{m \in M : m \preceq n\}$ is finite and totally ordered with respect to $\preceq$, that is, $\mathcal{T} = (M, \preceq)$ is a tree. Let us also assume that the tree $\mathcal{T}$ is well-founded, i.e., it contains no infinite totally ordered sets, and infinite branching, i.e., every non-maximal node has infinitely many immediate successors.

Observe that $\widetilde{\mathcal{T}} = (\{S_t : t \in \mathcal{T}\}, \subset)$ is also a tree and is in fact isomorphic to $\mathcal{T}$ via the mapping $t \mapsto S_t$. Given $t \in \mathcal{T}$, we will denote S_t by $\tilde{t}$. Moreover, two nodes $\tilde{t}_1, \tilde{t}_2$ are incomparable in $\widetilde{\mathcal{T}}$ if and only if the nodes t_1, t_2 are incomparable in $\mathcal{T}$, i.e. not comparable in the respective order. For $\tilde{t} \in \mathcal{T}$, we denote by $V_{\tilde{t}}$ the set consisting of $\tilde{t}$ and all of its successors.

Note that $\widetilde{\mathcal{T}}$ is a countable compact space when equipped with the pointwise convergence topology and hence $\mathcal{M}(\widetilde{\mathcal{T}})$, the set of all regular measures on $\widetilde{\mathcal{T}}$, is isometric to $\ell_1(\widetilde{\mathcal{T}})$. In particular, each $\mu \in \mathcal{M}(\widetilde{\mathcal{T}})$ is of the form $\mu = \sum_{\tilde{t}\in\widetilde{\mathcal{T}}} a_{\tilde{t}}\delta_{\tilde{t}}$, where $\delta_{\tilde{t}}$ is the Dirac measure centered on $\tilde{t}$, and $\|\mu\| = \sum_{\tilde{t}\in\widetilde{\mathcal{T}}} |a_{\tilde{t}}|$. Finally, the support of μ is defined as $\text{supp}\mu = \{\tilde{t} \in \widetilde{\mathcal{T}} : a_{\tilde{t}} \neq 0\}$. We will prove the following proposition, starting with Lemma 3.2

Proposition 3.1. *Let $(\mu_j)_j$ be a sequence of positive regular measures on $\widetilde{\mathcal{T}}$ with disjoint finite supports and let $c > 0$ be such that $\mu_j(\widetilde{\mathcal{T}}) < c$ for all $j \in \mathbb{N}$. Then, for every $\varepsilon > 0$, there is an $L \in [\mathbb{N}]^\infty$ and a subset G_j of $\text{supp}\mu_j$ for each $j \in L$, satisfying the following.*

(i) *$\mu_j(\widetilde{\mathcal{T}} \setminus G_j) \leq \varepsilon$ for every $j \in L$.*
(ii) *The sets G_j, $j \in \mathbb{N}$, are pairwise incomparable.*

Lemma 3.2. *Let $(\mu_j)_j$ be a sequence of positive regular measures on $\widetilde{\mathcal{T}}$ with disjoint finite supports and let $c > 0$ be such that $\mu_j(\widetilde{\mathcal{T}}) < c$ for all $j \in \mathbb{N}$. Assume that $w^*\text{-}\lim_j \mu_j = \mu = \sum_{\tilde{t}\in\widetilde{\mathcal{T}}} a_{\tilde{t}}\delta_{\tilde{t}}$. Then, for every $\tilde{t} \in \text{supp}\mu$ and $\varepsilon > 0$, there is an $L \in [\mathbb{N}]^\infty$ and a subset $G_j^{\tilde{t}}$ of $\text{supp}\mu_j$ for each $j \in L$, satisfying the following.*

(i) *$G_j^{\tilde{t}} \subset V_{\tilde{t}}$ for every $j \in L$.*
(ii) *$|\mu_j(G_j^{\tilde{t}}) - a_{\tilde{t}}| < \varepsilon$ for every $j \in L$.*
(iii) *The sets $G_j^{\tilde{t}}$, $j \in L$, are pairwise incomparable.*

Proof. Recall that the nodes of $\mathcal{T}$ are in fact naturals numbers. Hence identifying $\{t : \tilde{t} \in \text{supp}\mu_j\}$, $j \in \mathbb{N}$, as subsets of the naturals and passing to a subsequence, we may assume that they are successive.

Let $(\tilde{t}_j)_j$ be an enumeration of the immediate successors of $\tilde{t}$ and for each $j \in \mathbb{N}$ define $W_j^{\tilde{t}} = V_{\tilde{t}} \setminus \cup_{i=1}^j V_{\tilde{t}_i}$. Observe that $(W_j^{\tilde{t}})_j$ is a decreasing sequence of clopen subsets of $\widetilde{\mathcal{T}}$ with $\cap_j W_j^{\tilde{t}} = \{\tilde{t}\}$ and hence $\lim_j \mu(W_j^{\tilde{t}}) = a_{\tilde{t}}$ and $\lim_j \mu_j(W_i^{\tilde{t}}) = \mu(W_i^{\tilde{t}})$ for all $i \in \mathbb{N}$. We can thus find $N \in [\mathbb{N}]^\infty$ and pass to a subsequence of $(\mu_j)_j$, which we relabel for convenience, so that $\lim_{j\in N} |\mu_j(W_j^{\tilde{t}}) - \mu(W_j^{\tilde{t}})| = 0$ and define $G_j^{\tilde{t}} = \text{supp}\mu_j \cap W_j^{\tilde{t}}$ for each $j \in N$. Note then that $\lim_{j\in N} \mu_j(G_j^{\tilde{t}}) = a_{\tilde{t}}$ and $\mu_j|_{G_j^{\tilde{t}}}(\cup_{i=1}^j V_{\tilde{t}_i}) = 0$ for all $j \in N$.

There is at most one $j \in N$ such that $\tilde{t} \in G_j^{\tilde{t}}$ and hence, passing to a subsequence, we may assume that $\tilde{t} \notin G_j^{\tilde{t}}$ for all $j \in N$. Moreover, since $\lim_{j\in N} \mu_j(G_j^{\tilde{t}}) = a_{\tilde{t}}$, we may even pass to a further subsequence such that $|\mu_j(G_j^{\tilde{t}}) - a_{\tilde{t}}| < \varepsilon$ for all $j \in N$. For the remaining part of the proof we will choose, by induction, an $L \in [N]^\infty$ such that $G_j^{\tilde{t}}$, $j \in L$, are pairwise incomparable. Set $l_1 = \min N$ and suppose that we have chosen $l_1 < \ldots < l_k$ in N, for some $k \in \mathbb{N}$, such that $G_{l_i}^{\tilde{t}}$ and $G_{l_j}^{\tilde{t}}$ are incomparable, $1 \leq i < j \leq k$. Pick $l_k < l_{k+1} \in N$ such that $\mu_{l_{k+1}}|_{G_{l_{k+1}}^{\tilde{t}}}(\cup\{V_{\tilde{s}} : \tilde{s} \in \cup_{i=1}^k G_{l_i}^{\tilde{t}}\}) = 0$. Then, if for some $1 \leq i \leq k$ the nodes $\tilde{s}_1 \in G_{l_i}^{\tilde{t}}$ and $\tilde{s}_2 \in G_{l_{k+1}}^{\tilde{t}}$ are comparable, we have that $\tilde{s}_2 \in V_{\tilde{s}_1}$ whereas $\mu_{l_{k+1}}(V_{\tilde{s}_1}) = 0$, which is a contradiction. Hence $G_{l_1}^{\tilde{t}}, \ldots, G_{l_{k+1}}^{\tilde{t}}$ are pairwise incomparable. $\square$

Proof of Proposition 3.1. Passing to a subsequence, since $\widetilde{\mathcal{T}}$ is compact with respect to the pointwise convergence topology and $(\mu_j)_j$ are uniformly bounded, we may assume that $(\mu_j)_j$ w^*-converges to some measure $\mu = \sum_{\tilde{t}\in\widetilde{\mathcal{T}}} a_{\tilde{t}}\delta_{\tilde{t}}$ in $\mathcal{M}(\widetilde{\mathcal{T}})$.

Let $\delta > 0$ be such $(1-\delta)(\mu(\widetilde{\mathcal{T}})-\delta) > \mu(\widetilde{\mathcal{T}}) - \varepsilon/2$ and pick $n_0 \in \mathbb{N}$ such that $\sum_{i=1}^{n_0} a_{\tilde{t}_i} \geq \mu(\widetilde{\mathcal{T}}) - \delta$. Applying the previous lemma successively for each $\tilde{t}_i$, $i = 1,\dots,n_0$, we obtain an $L \in [\mathbb{N}]^\infty$ and, for each $j \in L$ and $i = 1,\dots,n_0$, a subset G_j^i of suppμ_j satisfying items (i) - (iii) of Lemma 3.2 for $\tilde{t}_i$ and $\delta a_{\tilde{t}_i}$. Note that if $\tilde{t}_{i_1}, \tilde{t}_{i_2}$ are incomparable for some $1 \leq i_1, i_2 \leq n_0$, then by item (i), the sets $G_{j_1}^{i_1}$ and $G_{j_2}^{i_2}$ are also incomparable for any $j_1, j_2 \in L$. If the nodes $\tilde{t}_{i_1}, \tilde{t}_{i_2}$ are comparable, say $\tilde{t}_{i_1} \subset \tilde{t}_{i_2}$, then there exists at most one $j \in L$ such that $\tilde{t}_{i_2} \in G_j^{i_1}$. Hence by a finite induction argument, we may pass to a subsequence such that the sets G_j^i, for $i = 1,\dots,n_0$ and $j \in L$, are pairwise incomparable. Define $G_j = \cup_{i=1}^{n_0} G_j^i$, $j \in L$, and conclude that

$$\mu_j(G_j) = \sum_{i=1}^{n_0} \mu_j(G_j^i) \geq \sum_{i=1}^{n_0} a_{\tilde{t}_i} - \delta a_{\tilde{t}_i} \geq (\mu(\widetilde{\mathcal{T}}) - \delta)(1-\delta) > \mu(\widetilde{\mathcal{T}}) - \frac{\varepsilon}{2}.$$

Finally, passing to a further subsequence if necessary, we may also assume that $|\mu_j(\widetilde{\mathcal{T}}) - \mu(\widetilde{\mathcal{T}})| < \varepsilon/2$ and hence $\mu_j(\widetilde{\mathcal{T}} \setminus G_j) < \varepsilon$ for every $j \in L$. □

3.2. **Tsirelson Extension of a Ground Set.** In order to define T_{inc}^{ξ}, we first introduce some necessary concepts used in the construction of Tsirelson type spaces.

Definition 3.3. A subset W of $c_{00}(\mathbb{N})$ is called a norming set if it satisfies the following conditions.

(i) W is symmetric and $e_i^* \in W$ for every $i \in \mathbb{N}$.
(ii) $\|f\|_\infty \leq 1$ for every $f \in W$.
(iii) W is closed under the restriction of its elements to intervals of $\mathbb{N}$.

A norming set W induces a norm $\|\cdot\|_W$ on $c_{00}(\mathbb{N})$ defined as

$$\|x\|_W = \sup\{f(x) : f \in W\}.$$

Definition 3.4. Let G be a norming set on $c_{00}(\mathbb{N})$. The Tsirelson extension of G, denoted by W_G, is the minimal subset of $c_{00}(\mathbb{N})$ that contains G and is closed under the $(\mathcal{S}, 1/2)$-operation, i.e., if $f_1,\dots,f_n$ are in W_G and $n \leq \text{supp}f_1 < \dots < \text{supp}f_n$, then $1/2\sum_{i=1}^n f_i$ is also in W_G. We call G the ground set of W_G.

Note that W_G is a norming set on $c_{00}(\mathbb{N})$. Moreover, the induced norm $\|\cdot\|_{W_G}$ satisfies the following implicit equation

$$\|x\|_{W_G} = \max\left\{\|x\|_G, \frac{1}{2}\sup\sum_{i=1}^n \|E_i x\|_{W_G}\right\}$$

where the supremum is taken over all finite collections $E_1,\dots,E_n$ of successive intervals of $\mathbb{N}$ with $n \leq E_1$.

Definition 3.5. Let $f \in W_G$. For a finite tree $\mathcal{A}$, a family $(f_\alpha)_{\alpha\in\mathcal{A}}$ is said to be a tree analysis of f if the following are satisfied.

(i) $\mathcal{A}$ has a unique root denoted by 0 and $f_0 = f$.
(ii) For every maximal node $\alpha \in \mathcal{A}$ we have that $f_\alpha \in G$.

(iii) Let α be a non-maximal node of $\mathcal{A}$ and denote by $S(\alpha)$ set of immediate successors of α. Then $f_\alpha \in W_G$ and the ranges of f_s, $s \in S(\alpha)$, are disjoint and $f_\alpha = 1/2 \sum_{s\in S(\alpha)} f_s$.

It follows, by minimality, that every $f \in W_G$ admits a tree analysis.

Proposition 3.6. *Let $f \in W_G$ with a tree analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$ and denote by $\mathcal{M}$ the set of all maximal nodes of $\mathcal{A}$. Then the following hold.*

(i) *For every $\alpha \in \mathcal{M}$, there is a $k_\alpha \in \mathbb{N}\cup\{0\}$ such that $f = \sum_{\alpha\in\mathcal{M}} f_\alpha/2^{k_\alpha}$.*
(ii) *If $\mathcal{N} \subset \mathcal{M}$, then $g = \sum_{\alpha\in\mathcal{N}} f_\alpha/2^{k_\alpha}$ is in W_G and $g = f|_{\cup\{\text{supp} f_\alpha : \alpha\in\mathcal{N}\}}$.*

For an extensive review on Tsirelson's space we refer the reader to [CS].

3.3. **Definition of the space T^ξ_{inc}.** We define the space T^ξ_{inc} as the completion of $c_{00}(\mathbb{N})$ with respect to the norm induced by a norming set W_ξ. This norming set is a subset of the Tsirelson extension of a ground set G^ξ_2, the functionals of which satisfy a certain property. Both this property and G^ξ_2 are defined via an infinite branching well-founded tree $\mathcal{T}_\xi$ on the natural numbers.

We start by fixing a partition of the naturals $\mathbb{N} = \cup_{j=0}^\infty N_j$ into infinite sets and an injection $\phi : [\mathbb{N}]^{<\infty} \to \mathbb{N}$. Recall the definition of the Schreier families $(\mathcal{S}_\xi)_{\xi<\omega_1}$.

Definition 3.7. Let ξ be a countable ordinal. We define, by transfinite induction, the Schreier family $\mathcal{S}_\xi \subset [\mathbb{N}]^{<\infty}$ as follows.

(i) If $\xi = 0$, then $\mathcal{S}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$.
(ii) If $\xi = \alpha + 1$, then

$$\mathcal{S}_\xi = \{\cup_{j=1}^n E_j : n \in \mathbb{N},\ E_1 < \ldots < E_n \in \mathcal{S}_\alpha \text{ and } n \leq E_1\}.$$

(iii) If ξ is a limit ordinal we choose a fixed sequence $(\alpha(\xi, j))_j \subset [1, \xi)$ which increases to ξ and set

$$\mathcal{S}_\xi = \{E \subset \mathbb{N} : \text{ there exists } j \in \mathbb{N} \text{ such that } E \in S_{\alpha(\xi,j)} \text{ and } j \leq E\}.$$

We now define the tree $\mathcal{T}_\xi$, by defining a partial order $\preceq_\xi$ on $\mathbb{N}$.

Definition 3.8. Fix a countable ordinal ξ and define the partial order $\preceq_\xi$ on $\mathbb{N}$ as follows: $n \preceq_\xi m$ if there exists $\{n_0, \ldots, n_k\} \in \mathcal{S}_\xi$ such that

(i) $n_0 \in N_0$ and $n_i \in N_{\phi(n_0,\ldots,n_{i-1})}$ with $n_{i-1} < n_i$ for every $1 \leq i \leq k$,
(ii) $n = n_i$ and $m = n_j$ for some $0 \leq i \leq j \leq k$.

Remark 3.9. Note that $\mathcal{T}_\xi = (\mathbb{N}, \preceq_\xi)$ is an infinite branching tree and it is also well-founded since $\mathcal{S}_\xi$ is a compact family, i.e., $\{\chi_E : E \in \mathcal{S}_\xi\}$ is a compact subset of $\{0, 1\}^{\mathbb{N}}$. Moreover, the partial order $\preceq_\xi$ is compatible with the standard order on the naturals and finally, standard inductive arguments yield that $\mathcal{T}_\xi$ is of height ω^ξ.

Definition 3.10. Define the following norming set on $c_{00}(\mathbb{N})$

$$G^\xi_2 = \Big\{ \sum_{i\in S} a_i e^*_i : S \text{ is a segment of } \mathcal{T}_\xi \text{ and } \sum_{i\in S} a_i^2 \leq 1 \Big\}$$

and denote by W_ξ the subset of $W_{G^\xi_2}$ containing all f with tree analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$ such that there exist pairwise incomparable segments S_α of $\mathcal{T}_\xi$ with $\text{supp} f_\alpha \subset S_\alpha$ for every maximal node $\alpha \in \mathcal{A}$. Denote by T^ξ_{inc} the completion of $c_{00}(\mathbb{N})$ with respect to the norm $\|\cdot\|_{W_\xi}$ induced by the norming set W_ξ.

Remark 3.11. The unit vector basis $(e_j)_j$ of $c_{00}(\mathbb{N})$ forms a 1-unconditional Schauder basis for the space T^ξ_{inc}. Moreover it is boundedly complete, since T^ξ_{inc} admits ℓ_1 as a uniformly unique spreading model as shown in Proposition 3.12.

First, we show that T^ξ_{inc} admits a uniformly unique joint spreading model with respect to $\mathscr{F}_b(T^\xi_{inc})$, that is equivalent to the unit vector basis of ℓ_1.

Proposition 3.12. *The space T^ξ_{inc} admits a uniformly unique joint spreading model with respect to $\mathscr{F}_b(T^\xi_{inc})$, which is equivalent to the unit vector basis of ℓ_1.*

Proof. Let $(x^i_j)_j$, $1 \leq i \leq l$, be an array of normalized block sequences in T^ξ_{inc} and $\varepsilon > 0$. Passing to a subsequence, we assume that $\mathrm{supp} x^{i_1}_j < \mathrm{supp} x^{i_2}_{j+1}$ for every $i_1, i_2 = 1, \dots, l$ and $j \in \mathbb{N}$. For every $i = 1, \dots, l$ and $j \in \mathbb{N}$, pick a functional $f^i_j = \sum_{\alpha \in \mathcal{M}^i_j} f^i_{j,\alpha}/2^{k^i_{j,\alpha}}$ in W_ξ such that $f^i_j(x^i_j) \geq 1 - \varepsilon$ and $f^i_{j,\alpha}(x^i_j) > 0$ for every $\alpha \in \mathcal{M}^i_j$, where $\mathcal{M}^i_j$ denotes the set of all maximal nodes of a fixed tree analysis of f^i_j. For every $i = 1, \dots, l$, $j \in \mathbb{N}$ and $\alpha \in \mathcal{M}^i_j$, define $\lambda^i_{j,\alpha} = f^i_{j,\alpha}(x^i_j)/2^{k^i_{j,\alpha}}$ and $t^i_{j,\alpha} = \min \mathrm{supp} f^i_{j,\alpha}$. Moreover, for each $j \in \mathbb{N}$, define the probability measure

$$\mu_j = \frac{1}{l} \sum_{i=1}^{l} \frac{1}{f^i_j(x^i_j)} \sum_{\alpha \in \mathcal{M}^i_j} \lambda^i_{j,\alpha} \delta_{\tilde{t}^i_{j,\alpha}}.$$

Then, Proposition 3.1 yields an $L \in [\mathbb{N}]^\infty$ and a sequence $(G_j)_{j \in L}$ of pairwise incomparable subsets of $\widetilde{\mathcal{T}}_\xi$ such that $\mu_j(G_j) \geq 1 - \delta$ for every $j \in L$ and for some δ sufficiently small such that for any $i = 1, \dots, l$ and $j \in L$

$$\frac{1}{f^i_j(x^i_j)} \sum_{\alpha \in \mathcal{M}^i_j} \lambda^i_{j,\alpha} \delta_{\tilde{t}^i_{j,\alpha}}(G_j) \geq (1 - \varepsilon)^2. \tag{2}$$

Let $k \in \mathbb{N}$ and $(s_i)_{i=1}^l \in S\text{-}Plm_l([L]^k)$ with $kl \leq x_{s_1(1)}$. Then, for $i = 1, \dots, l$ and $j \in L$, if $\mathcal{N}^i_j = \{\alpha \in \mathcal{M}^i_{s_i(j)} : \tilde{t}^i_{j,\alpha} \in G_{s_i(j)}\}$, item (ii) of Proposition 3.6 yields that

$$g^i_j = \sum_{\alpha \in \mathcal{N}^i_j} \frac{1}{2^{k^i_{s_i(j),\alpha}}} f^i_{s_i(j),\alpha} \in W_\xi.$$

Moreover, (2) implies $g^i_j(x^i_{s_i(j)}) \geq (1 - \varepsilon)^2$ for all $i = 1, \dots, l$ and $j \in L$, and since G_j, $j \in L$, are pairwise incomparable, we have that $g = 1/2 \sum_{i=1}^l \sum_{j=1}^k g^i_j$ is in W_ξ. Then for any choice of scalars $(a_{ij})_{i=1,j=1}^{l,k}$, due to unconditionality, we conclude that

$$\Big\| \sum_{i=1}^{l} \sum_{j=1}^{k} a_{ij} x^i_{s_i(j)} \Big\| \geq \Big\| \sum_{i=1}^{l} \sum_{j=1}^{k} |a_{ij}| x^i_{s_i(j)} \Big\| \geq \frac{(1-\varepsilon)^2}{2} \sum_{i=1}^{l} \sum_{j=1}^{k} |a_{ij}|.$$

□

Proposition 3.13. *The space T^ξ_{inc} is reflexive.*

Proof. Since T^ξ_{inc} admits a boundedly complete unconditional Schauder basis, it does not contain c_0 (see [LT, Theorem 1.c.10]) and hence it suffices to show that it does not contain ℓ_1 as follows from [J1, Theorem 2].

Fix $n \in \mathbb{N}$. Let $(x_j)_j$ be a normalized block sequence in T^ξ_{inc} and $f = \sum_{i\in S} b_i e_i^*$ in G^ξ_2. For each $j = 1, \ldots, n$, define $I_j = \{i \in S : i \in \text{supp} x_j\}$ and note that

$$\Big(\sum_{i\in I_j} b_j x_j(i)\Big)^2 \leq \sum_{i\in I_j} b_i^2.$$

Then, for any choice of scalars $a_1, \ldots, a_n$, we have that

$$f\Big(\sum_{j=1}^n a_j x_j\Big) = \sum_{j=1}^n a_j \sum_{i\in S} b_i x_j(i) \leq \Big(\sum_{j=1}^n a_j^2\Big)^{\frac{1}{2}} \Big(\sum_{j=1}^n \Big(\sum_{i\in I_j} b_i x_j(i)\Big)^2\Big)^{\frac{1}{2}}$$
$$\leq \Big(\sum_{j=1}^n a_j^2\Big)^{\frac{1}{2}} \Big(\sum_{j=1}^n \sum_{i\in I_j} b_i^2\Big)^{\frac{1}{2}} \leq \Big(\sum_{j=1}^n a_j^2\Big)^{\frac{1}{2}}$$

and hence

$$\Big\|\sum_{j=1}^n a_j x_j\Big\|_{G^\xi_2} \leq \Big(\sum_{j=1}^n a_j^2\Big)^{\frac{1}{2}}.$$

That is, for any normalized block sequence $(x_j)_j$ in T^ξ_{inc}, there exists a block subsequence $(y_j)_j$ with $\|y_j\|_{G^\xi_2} \to 0$.

We show that T^ξ_{inc} does not contain ℓ_1 in a similar manner as in the proof of the reflexivity for the classical Tsirelson space [FJ]. Suppose that T^ξ_{inc} contains ℓ_1. Then James' ℓ_1 distortion theorem [J2] implies that, for $\varepsilon < 1/4$, there exists a normalized block sequence $(x_j)_j$ in T^ξ_{inc} such that

$$\Big\|\sum_{j=1}^n a_j x_j\Big\| \geq (1-\varepsilon)\sum_{j=1}^n |a_j|$$

for any $n \in \mathbb{N}$ and any choice of scalars $a_1, \ldots, a_n$. Applying the result of the previous paragraph, we may also assume that $\|x_j\|_{G^\xi_2} < 1/2$ for every $j \in \mathbb{N}$ and hence, for any $n \geq 2$, we have that

$$\Big\|x_1 + \frac{1}{n}\sum_{i=2}^{n+1} x_i\Big\| > \Big\|x_1 + \frac{1}{n}\sum_{i=2}^{n+1} x_i\Big\|_{G^\xi_2}. \tag{3}$$

Moreover, for any $n \in \mathbb{N}$, we have that

$$\Big\|x_1 + \frac{1}{n}\sum_{i=2}^{n+1} x_i\Big\| \geq 2(1-\varepsilon).$$

Observe that (3) implies that there exists $f = 1/2\sum_{j=1}^k f_j \in W_\xi \setminus G^\xi_2$ such that

$$f\Big(x_1 + \frac{1}{n}\sum_{i=2}^{n+1} x_i\Big) > \Big\|x_1 + \frac{1}{n}\sum_{i=2}^{n+1} x_i\Big\| - \varepsilon \geq \frac{5}{4}$$

and that $\min \text{supp} f_1 \leq \max \text{supp} x_1$, since otherwise

$$f\Big(x_1 + \frac{1}{n}\sum_{i=2}^{n+1} x_i\Big) = \frac{1}{n}\sum_{i=2}^{n+1} f(x_i) \leq 1.$$

Therefore, $k \leq \max \text{supp} x_1$. Note that there are at most k i's such that the support of x_i intersects the supports of at least two f_j's and hence

$$f\Big(x_1 + \frac{1}{n}\sum_{i=2}^{n+1} x_i\Big) \leq 1 + \frac{k}{n} + \frac{n-k}{2n} \leq 1 + \frac{n + \max \text{supp} x_1}{2n} \xrightarrow[n\to\infty]{} \frac{3}{2}.$$

This yields a contradiction for sufficiently large n since $3/2 < 2(1-\varepsilon)$. □

Proposition 3.14. *The space T^ξ_{inc} is not Asymptotic ℓ_1.*

Proof. Suppose that T^ξ_{inc} is C-Asymptotic ℓ_1 and let $n \in \mathbb{N}$ be such that $n > C^2$. Since T^ξ_{inc} is reflexive, we may assume that player (S) chooses tail subspaces (see [AGLM, Lemma 5.18]) throughout any winning strategy in the game $G(n, 1, C)$. Let us assume the role of player (V) and let Y_1 be the tail subspace with which player (S) initiates the game. Then, as player (V), we choose an element of the basis $e_{j_1} \in Y_1$, such that $|S| \geq n$ for every maximal segment S of $\mathcal{T}_\xi$ with $\min S = j_1$. Suppose that in the $k+1$ turn of the game, for $k < n$, player (S) chooses the subspace Y_{k+1}. Then, again as player (V), we choose a vector $e_{j_{k+1}} \in Y_{k+1}$ with j_{k+1} an immediate successor of j_k. Note that, in the final outcome of the game, we have chosen elements of the basis $e_{j_1}, \dots, e_{j_n}$ such that $\{j_1, \dots, j_n\}$ is a segment of $\mathcal{T}_\xi$ and hence $\{e_{j_1}, \dots, e_{j_n}\}$ is isometric to the standard basis of ℓ_2^n. We calculate

$$\Big\|\frac{1}{n}\sum_{i=1}^{n} e_{j_i}\Big\| = \Big\|\frac{1}{n}\sum_{i=1}^{n} e_{j_i}\Big\|_{G_2^\xi} = n^{-\frac{1}{2}}$$

whereas, since T^ξ_{inc} is C-Asymptotic ℓ_1, we have that

$$\frac{1}{C} \leq \Big\|\frac{1}{n}\sum_{i=1}^{n} e_{j_i}\Big\|$$

and this is a contradiction. □

Remark 3.15. For any $1 < p < \infty$, we may replace the norming set G_2^ξ with

$$G_p^\xi = \Big\{\sum_{i\in S} a_i e_i^* : \text{ S is a segment of } \mathcal{T}_\xi \text{ and } \sum_{i\in S}|a_i|^q \leq 1\Big\}$$

where $p^{-1} + q^{-1} = 1$, to obtain a reflexive Banach space admitting a uniformly unique ℓ_1 joint spreading model, that contains a weakly null ℓ_p-tree of height ω^ξ or a weakly null c_0-tree if we replace G_2^ξ with $G = \{\pm e_i^* : i \in \mathbb{N}\}$.

4. A stronger separation of the two properties

The spaces T^ξ_{inc} constructed in the previous section, yield a separation between the properties of being an Asymptotic ℓ_1 space and admitting a unique ℓ_1 asymptotic model. It is easy however to see that these spaces contain subsequences of their bases that generate Asymptotic ℓ_1 subspaces. For example, consider any subspace generated by a subsequence $(e_j)_{j\in M}$ of the basis of some T^ξ_{inc}, such that the elements of M are pairwise incomparable in $\mathcal{T}_\xi$. In this section we show that, for any countable ordinal ξ, there is a reflexive Banach space $T^\xi_{ess\text{-}inc}$ that admits a unique ℓ_1 asymptotic model with respect to $\mathscr{F}_b(T^\xi_{ess\text{-}inc})$ and any subsequence of its basis generates a non-Asymptotic ℓ_1 subspace. To some extent, this family of spaces is the Maurey - Rosenthal [MR] analogue of the two aforementioned properties.

Start by fixing a countable ordinal ξ and let $(m_j)_{j\geq 0}$, $(n_j)_{j\geq 0}$ be increasing sequences of natural numbers such that :

(i) $m_0 = 2$, $m_1 = 4$ and $m_j \geq m_{j-1}^2$ for every $j \geq 2$ and
(ii) $n_0 = 1$, $n_1 = 6$ and $n_j > \log_2 m_j^2 + n_{j-1}$ for every $j \geq 2$.

Let $\mathcal{Q}$ denote the collection of all finite sequences $((g_1, m_{j_1}), \ldots, (g_k, m_{j_k}))$, where $g_i : \mathbb{N} \to \{-1, 0, 1\}$ has finite support and $j_i \in \mathbb{N}$ for $1 \leq i \leq k$, and $m_{j_1} < \cdots < m_{j_k}$. Let $\sigma : \mathcal{Q} \to \{m_j : j \in \mathbb{N}\}$ be an injection so that each sequence $((g_1, m_{j_1}), \ldots, (g_k, m_{j_k}))$ is mapped to some m_j with $m_{j_k} < m_j$.

Definition 4.1. Let $\widetilde{\mathcal{T}}_\xi$ be the set of all finite sequences $((g_1, m_{j_1}), \ldots, (g_k, m_{j_k}))$ satisfying the following conditions.

(i) $g_i : \mathbb{N} \to \{-1, 0, 1\}$ for $i = 1, \ldots, k$ with $\text{supp} g_1 < \ldots < \text{supp} g_k$.
(ii) $\text{supp} g_i \in \mathcal{S}_{n_{j_i}}$ for $i = 1, \ldots, k$, where $n_{j_1} = n_1$ and $n_{j_1} < \ldots < n_{j_k}$.
(iii) $m_{j_1} = m_1$ and $m_{j_i} = \sigma((g_1, m_{j_1}), \ldots, (g_{i-1}, m_{j_{i-1}}))$ for every $i = 2, \ldots, k$.
(iv) $\{\min \text{supp} g_i : i = 1, \ldots, k\} \in \mathcal{S}_\xi$.

Note that item (iii) of the above definition implies that $\widetilde{\mathcal{T}}_\xi$, equipped with the partial order $\leq_{\widetilde{\mathcal{T}}_\xi}$ where $\tilde{t}_1 \leq_{\widetilde{\mathcal{T}}_\xi} \tilde{t}_2$ if $\tilde{t}_1$ is an initial segment of $\tilde{t}_2$, is a tree. Moreover, it is easy to see that it is infinite-branching, and as follows from item (iv) and standard inductive arguments, it is also well founded and of height ω^ξ. In particular, the above remain true if for an infinite subset of the naturals M we additionally require that $\text{supp} g_i \subset M$ for every $i = 1, \ldots, k$, in Definition 4.1.

We may also identify $\widetilde{\mathcal{T}}_\xi$ as a closed subset, with respect to the pointwise convergence topology, of $\big\{\{\pm m_j^{-1}\}_{j\in\mathbb{N}} \cup \{0\}\big\}^{\mathbb{N}}$ via the mapping

$$((g_1, m_{j_1}), \ldots, (g_k, m_{j_k})) \mapsto m_{j_1}^{-1}\, g_1 + \cdots + m_{j_k}^{-1}\, g_k.$$

The fact that $\lim_j m_j^{-1} = 0$ implies that $\big\{\{\pm m_j^{-1}\}_{j\in\mathbb{N}} \cup \{0\}\big\}^{\mathbb{N}}$ is compact with respect to the pointwise convergence topology of $[-1, 1]^{\mathbb{N}}$.

Observe that, as a consequence of item (iii), any $\tilde{t} = ((g_1, m_{j_1}), \ldots, (g_k, m_{j_k}))$ in $\widetilde{\mathcal{T}}_\xi$ is uniquely determined by the pair (g_k, m_{j_k}), which we will denote by (g_t, m_{j_t}) or just by t (i.e., $t = (g_t, m_{j_t})$). Taking advantage of this we may define $\mathcal{T}_\xi = \{(g_t, m_{j_t}) : \tilde{t} \in \widetilde{\mathcal{T}}_\xi\}$, which is in bijection with $\widetilde{\mathcal{T}}_\xi$ via the mapping $t = (g_t, m_{j_t}) \mapsto \tilde{t}$. Note that $\leq_{\widetilde{\mathcal{T}}_\xi}$ induces a natural order, denoted by $\leq_{\mathcal{T}_\xi}$, on $\mathcal{T}_\xi$, where $(g_{t_1}, m_{j_{t_1}}) \leq_{\mathcal{T}_\xi} (g_{t_2}, m_{j_{t_2}})$ if $\tilde{t}_1 \leq_{\widetilde{\mathcal{T}}} \tilde{t}_2$. Clearly, the tree $(\mathcal{T}_\xi, \leq_{\mathcal{T}_\xi})$ is isomorphic to $(\widetilde{\mathcal{T}}_\xi, \leq_{\widetilde{\mathcal{T}}_\xi})$ via the mapping $t = (g_t, m_{j_t}) \mapsto \tilde{t}$.

Definition 4.2. Let $\widetilde{\mathcal{W}_\xi}$ be the set of all finite sequences $(m_{j_1}, m_{j_2}, \ldots, m_{j_k})$ for which there exist $g_1, \ldots, g_k : \mathbb{N} \to \{-1, 0, 1\}$ with $((g_1, m_{j_1}), \ldots, (g_k, m_{j_k})) \in \widetilde{\mathcal{T}}_\xi$.

The initial segment order $\leq_{\widetilde{\mathcal{W}_\xi}}$ is a partial order on $\widetilde{\mathcal{W}_\xi}$ and is in fact naturally induced by the order $\leq_{\widetilde{\mathcal{T}}_\xi}$. Moreover, it is easy to verify that $(\widetilde{\mathcal{W}_\xi}, \leq_{\widetilde{\mathcal{W}_\xi}})$ is a well founded infinite-branching tree of height ω^ξ. It is also isomorphic to the tree $(\mathcal{W}_\xi, \leq_{\mathcal{W}_\xi})$, where $\mathcal{W}_\xi = \{m_{j_t} : \tilde{t} \in \widetilde{\mathcal{T}}_\xi\}$ and $m_{j_{t_1}} \leq_{\mathcal{W}_\xi} m_{j_{t_2}}$ if $\tilde{t}_1 \leq_{\widetilde{\mathcal{T}}_\xi} \tilde{t}_2$. This correspondence between $\widetilde{\mathcal{W}_\xi}$ and $\mathcal{W}_\xi$ is identical to that of $\widetilde{\mathcal{T}}_\xi$ and $\mathcal{T}_\xi$.

Remark 4.3. (i) If $m_{j_{t_1}}$, $m_{j_{t_2}}$ are incomparable nodes in $\mathcal{W}_\xi$, then for every $g_1, g_2 : \mathbb{N} \to \{-1, 0, 1\}$ such that $(g_1, m_{j_{t_1}})$ and $(g_2, m_{j_{t_2}})$ are in $\mathcal{T}_\xi$, these are also incomparable.

(ii) Note that there exist nodes $\tilde{t}_1$ and $\tilde{t}_2$ which are incomparable in $\widetilde{\mathcal{T}}_\xi$, whereas $m_{j_{t_1}}$ and $m_{j_{t_2}}$ are comparable in $\mathcal{W}_\xi$. To see this, consider any node $\tilde{t} = ((g_1, m_{j_1}), \dots, (g_k, m_{j_k}))$ in $\widetilde{\mathcal{T}}_\xi$ with $k > 1$ and, for each $i = 1, \dots, k-1$, let $h_i : \mathbb{N} \to \{-1, 0, 1\}$ be such that $h_i \neq g_i$ and $t_i = (h_i, m_{j_i})$ is in $\mathcal{T}_\xi$. Then, item (iii) of Definition 4.1 implies that the nodes $\tilde{t}_i$ and $\tilde{t}$ are incomparable whereas $m_{j_{t_i}}$ and m_{j_t} are comparable for every $i = 1, \dots, k-1$, since $\tilde{t} \in \widetilde{\mathcal{T}}_\xi$.

Definition 4.4. We say that a subset X of $\mathcal{T}_\xi$ is essentially incomparable if whenever $(g_{t_1}, m_{j_{t_1}})$, $(g_{t_2}, m_{j_{t_2}})$ are in X with $m_{j_{t_1}} <_{\mathcal{W}_\xi} m_{j_{t_2}}$ and $g : \mathbb{N} \to \{-1, 0, 1\}$ is the unique sequence such that $(g, m_{j_{t_1}}) \leq_{\mathcal{T}_\xi} (g_{t_2}, m_{j_{t_2}})$, then $\mathrm{supp} g < \mathrm{supp} g_{t_1}$.

Remark 4.5. Let X be an essentially incomparable subset of $\mathcal{T}_\xi$ and $h_t : \mathbb{N} \to \{-1, 0, 1\}$ with $\mathrm{supp} h_t \subset \mathrm{supp} g_t$ for every $t \in X$. Then $\{(h_t, m_{j_t}) : t \in X\}$ is also an essentially incomparable subset of $\mathcal{T}_\xi$.

The following lemma is an extension of Proposition 3.1 and is the main ingredient of the proof that the space $T^\xi_{\text{ess-inc}}$ admits a uniformly unique joint spreading model.

Lemma 4.6. *Let $(\mu_i)_i$ be a sequence of positive regular measures on $\widetilde{\mathcal{T}}_\xi$ with finite supports and let $C > 0$ be such that $\mu_i(\widetilde{\mathcal{T}}_\xi) < C$ for all $i \in \mathbb{N}$. Assume that the sets $\cup\{\mathrm{supp} g_t : \tilde{t} \in \mathrm{supp}\mu_i\}$, $i \in \mathbb{N}$, are disjoint. Then, for every $\varepsilon > 0$, there exists an $M \in [\mathbb{N}]^\infty$ and G_i^1, G_i^2 subsets of $\widetilde{\mathcal{T}}_\xi$ for each $i \in M$, such that*

(i) *G_i^1, G_i^2 are disjoint subsets of $\mathrm{supp}\mu_i$ for every $i \in M$,*
(ii) *$\mu_i(\widetilde{\mathcal{T}}_\xi \setminus G_i^1 \cup G_i^2) < \varepsilon$ for every $i \in M$,*
(iii) *$\{t \in \mathcal{T}_\xi : \tilde{t} \in \cup_{i \in M} G_i^1\}$ is essentially incomparable and*
(iv) *for every $i_1 \neq i_2$ in M, every $\tilde{t}_1 \in G_{i_1}^2$ and $\tilde{t}_2 \in G_{i_2}^2$, the nodes $m_{j_{t_1}}$ and $m_{j_{t_2}}$ are incomparable in $\mathcal{W}_\xi$.*

Before we are able to prove this Lemma it is necessary to introduce the notion of successor limits of measures. We find this limit notion to be of independent interest and therefore we use broader terminology to define it and prove its properties.

Notation 4.7. Let $\mathcal{T}$ be a countably branching well founded tree. For each $t \in \mathcal{T}$ we denote $\mathrm{succ}_{\mathcal{T}}(t)$ the set of immediate successors of t. In particular, if t is maximal then $\mathrm{succ}_{\mathcal{T}}(t)$ is empty. For $t \in \mathcal{T}$ we denote $V_t = \{s \in \mathcal{T} : t \leq s\}$. We view $\mathcal{T}$ as topological space with the topology generated by the sets V_t and $\mathcal{T} \setminus V_t$, $t \in \mathcal{T}$. This is a compact metric topology for which the sets of the form $V_t \setminus (\cup_{s \in F} V_s)$, $t \in \mathcal{T}$ and $F \subset \mathrm{succ}_{\mathcal{T}}(t)$ finite, form a base of clopen sets. We denote by $\mathcal{M}_+(\mathcal{T})$ the cone of all bounded positive measures $\mu : \mathcal{P}(\mathcal{T}) \to [0, +\infty)$. For $\mu \in \mathcal{M}_+(\mathcal{T})$ we define the support of μ to be the set $\mathrm{supp}(\mu) = \{t \in \mathcal{T} : \mu(\{t\}) > 0\}$. A set A in $\mathcal{M}_+(\mathcal{T})$ is called bounded if $\sup_{\mu \in \mathcal{A}} \mu(\mathcal{T}) < \infty$.

Recall that a sequence (μ_i) in $\mathcal{M}_+(\mathcal{T})$ converges in the w^*-topology to a $\mu \in \mathcal{M}_+(\mathcal{T})$ if and only if for all clopen sets $V \subset \mathcal{T}$ we have $\lim_i \mu_i(V) = \mu(V)$ if and only if for all $t \in \mathcal{T}$ we have $\lim_i \mu_i(V_t) = \mu(V_t)$.

Definition 4.8. Let $\mathcal{T}$ be a countably branching well founded tree, (μ_i) be a disjointly supported sequence in $\mathcal{M}_+(\mathcal{T})$ and $\nu \in \mathcal{M}_+(\mathcal{T})$. We say that ν is the successor-determined limit of (μ_i) if for all $t \in \mathcal{T}$ we have $\mu(\{t\}) = \lim_i \mu_i(\mathrm{succ}_{\mathcal{T}}(t))$. In this case we write $\nu = \text{succ-lim}_i\, \mu_i$.

Remark 4.9. It is possible for a disjointly supported and bounded sequence $(\mu_i) \in \mathcal{M}_+(\mathcal{T})$ to satisfy $w^*\text{-}\lim_i \mu_i \neq \text{succ-}\lim_i \mu_i$. Take for example $\mathcal{T} = [\mathbb{N}]^{\leq 2}$ (all subsets of $\mathbb{N}$ with at most two elements with the partial order of initial segments). Define $\mu_i = \delta_{\{i,i\}}$. Then, $w^*\text{-}\lim_i \mu_i = \delta_\emptyset$ whereas $\text{succ-}\lim_i \mu_i = 0$.

Although these limits are not the necessarily the same, there is an explicit formula relating $\text{succ-}\lim_i \mu_i$ to $w^*\text{-}\lim_i \mu_i$.

Lemma 4.10. *Let $\mathcal{T}$ be a countable well founded tree, (μ_i) be a bounded and disjointly supported sequence in $\mathcal{M}_+(\mathcal{T})$ so that $w^*\text{-}\lim_i \mu_i = \mu$ exists and for all $t \in \mathcal{T}$ the limit $\nu(\{t\}) = \lim_i \mu_i(\text{succ}_\mathcal{T}(t))$ exists as well. Then, for every $t \in \mathcal{T}$ and enumeration (t_j) of $\text{succ}_\mathcal{T}(t)$ we have*

$$\mu(\{t\}) = \nu(\{t\}) + \lim_j \lim_i \mu_i\Big(\cup_{k\geq j} (V_{t_k} \setminus \{t_k\})\Big). \tag{4}$$

In particular, $\mu(\{t\}) = \nu(\{t\})$ if and only if the double limit in (4) is zero.

Proof. For $j \in \mathbb{N}$ we have $\{t\} \cup (\cup_{k\geq j} V_{t_k}) = V_t \setminus (\cup_{k<j} V_{t_k})$ which is clopen and thus

$$\lim_i \mu_i\big(\{t\} \cup (\cup_{k\geq j} V_{t_k})\big) = \mu\big(\{t\} \cup (\cup_{k\geq j} V_{t_k})\big). \tag{5}$$

Because (μ_i) is disjointly supported we observe that for all $j \in \mathbb{N}$

$$\lim_i \mu_i(\{t_k : k \geq j\}) = \lim_i \mu_i(\text{succ}_\mathcal{T}(t)) = \nu(\{t\}). \tag{6}$$

We calculate

$$\begin{aligned}
\mu(\{t\}) &= \lim_{j\to\infty} \mu\Big(\{t\} \cup (\cup_{k\geq j} V_{t_j})\Big) \stackrel{(5)}{=} \lim_j \lim_i \mu_i\Big(\{t\} \cup (\cup_{k\geq j} V_{t_j})\Big) \\
&= \lim_j \lim_i \mu_i(\cup_{k\geq j} V_{t_j}) = \lim_j \lim_i \mu_i(\{t_k : k \geq j\} \cup (\cup_{k\geq j} (V_{t_k} \setminus \{t_k\}))) \\
&= \lim_j \lim_i \mu_i(\{t_k : k \geq j\}) + \lim_j \lim_i \mu_i\Big(\cup_{k\geq j} (V_{t_k} \setminus \{t_k\})\Big).
\end{aligned}$$

Thus, (6) yields the conclusion. □

Corollary 4.11. *Let $\mathcal{T}$ be a countable well founded tree and (μ_i) be a bounded and disjointly supported sequence in $\mathcal{M}_+(\mathcal{T})$. Then, there exist a subsequence (μ_{i_n}) of (μ_i) and $\nu \in \mathcal{M}_+(\mathcal{T})$ with $\nu = \text{succ-}\lim_n \mu_{i_n}$.*

Proof. By passing to a subsequence, $\mu = w^*\text{-}\lim_i \mu_i$ exists and for all $t \in \mathcal{T}$ the limit $\nu(\{t\}) = \lim_i \mu_i(\text{succ}_\mathcal{T}(t))$ exists as well. By (4) for all $t \in \mathcal{T}$ we have $\nu(\{t\}) \leq \mu(\{t\})$. Thus $\sum_{t\in\mathcal{T}} \nu(\{t\}) \leq \mu(\mathcal{T})$, i.e., ν defines a bounded positive measure □

Lemma 4.12. *Let $\mathcal{T}$ be a countable well founded tree and (μ_i) be a bounded and disjointly supported sequence in $\mathcal{M}_+(\mathcal{T})$ so that $\text{succ-}\lim_i \mu_i = \nu$ exists. Then, there exist an infinite $L \subset \mathbb{N}$ and partitions A_i, B_i of $\text{supp}(\mu_i)$, $i \in L$, so that the following are satisfied.*

(i) *If for all $i \in L$ we define the measure μ_i^1 given by $\mu_i^1(C) = \mu_i(C \cap A_i)$, then $\nu = w^*\text{-}\lim_{i\in L} \mu_i^1 = \text{succ-}\lim_{i\in L} \mu_i^1$.*
(ii) *If for all $i \in L$ we define the measure μ_i^2 given by $\mu_i^2(C) = \mu_i(C \cap B_i)$ then for all $t \in \mathcal{T}$ the sequence $(\mu_i^2(\text{succ}_\mathcal{T}(t)))_i$ is eventually zero. In particular, $\text{succ-}\lim_{i\in L} \mu_i^2 = 0$.*

Proof. Enumerate $\mathcal{T} = \{s_n : n \in \mathbb{N}\}$ and assume, passing if necessary to a subsequence, that for all $n \in \mathbb{N}$ and $i > n$ we have

$$|\mu_i(\mathrm{succ}_{\mathcal{T}}(s_n)) - \nu(s_n)| < \frac{1}{2^n}. \tag{7}$$

Let us point out that for $m \neq n$ the sets $\mathrm{succ}_{\mathcal{T}}(s_m)$ and $\mathrm{succ}_{\mathcal{T}}(s_n)$ are disjoint and $\cup_n \mathrm{succ}_{\mathcal{T}}(s_n) = \mathcal{T} \setminus \{t_0\}$, where t_0 denotes the root of the tree $\mathcal{T}$. We may, and will, assume that for all $i \in \mathbb{N}$, $t_0 \notin \mathrm{supp}(\mu_i)$. Define for each $i \in \mathbb{N}$ the sets

$$A_i = \mathrm{supp}(\mu_i) \cap \Big(\cup_{n=1}^{i} \mathrm{succ}_{\mathcal{T}}(s_n)\Big) \text{ and } B_i = \mathrm{supp}(\mu_i) \cap \Big(\cup_{n=i+1}^{\infty} \mathrm{succ}_{\mathcal{T}}(s_n)\Big).$$

We point out that for all $i \in \mathbb{N}$, A_i, B_i forms a partition of $\mathrm{supp}(\mu_i)$ and we will show that it has the desired properties.

Statement (ii) follows directly from the fact that for every $t \in \mathcal{T}$ the sequence of sets $(B_i \cap \mathrm{succ}_{\mathcal{T}}(t))_i$ is eventually empty. To show that (i) holds we fix $t \in \mathcal{T}$ and let (t_j) be an enumeration of $\mathrm{succ}_{\mathcal{T}}(t)$. Define $L_j = \cup_{k=j}^{\infty}\{n \in \mathbb{N} : t_k \leq s_n\}$, for each $j \in \mathbb{N}$, and observe that $\cap_j L_j = \emptyset$. Also observe that for all $j \in \mathbb{N}$ we have $\cup_{k\geq j}(V_{t_k} \setminus \{t_k\}) = \cup_{n\in L_j}\mathrm{succ}_{\mathcal{T}}(s_n)$. Therefore we have

$$\begin{aligned}
&\mu_i\Big(A_i \cap \big(\cup_{k\geq j}(V_{t_k} \setminus \{t_k\})\big)\Big) = \mu_i\Big(\big(\cup_{n=1}^{i}\mathrm{succ}_{\mathcal{T}}(s_n)\big) \cap \big(\cup_{n\in L_j}\mathrm{succ}_{\mathcal{T}}(s_n)\big)\Big)\\
&= \mu_i\Big(\cup_{n\in L_j\cap[1,i]}\mathrm{succ}_{\mathcal{T}}(s_n)\Big) = \sum_{n\in L_j\cap[1,i]} \mu_i\big(\mathrm{succ}_{\mathcal{T}}(s_n)\big)\\
&\overset{(7)}{\leq} \sum_{n\in L_j\cap[1,i]}\nu(s_n) + \sum_{n\in L_j\cap[1,i]}\frac{1}{2^n} \leq \nu(\{s_n : n \in L_j\}) + 2^{-\min(L_j)+1}\\
&= \nu(\cup_{k\geq j}V_{t_k}) + 2^{-\min(L_j)+1}.
\end{aligned}$$

Therefore, $\lim_j \sup_i \mu_i\Big(A_i \cap \big(\cup_{k\geq j}(V_{t_k} \setminus \{t_k\})\big)\Big) = 0$ and by Lemma 4.10, (i) is satisfied. □

Proof of Lemma 4.6. Apply Lemma 4.12 so that, by passing to a subsequence of (μ_i), there are, for each $i \in \mathbb{N}$, partitions A_i, B_i of $\mathrm{supp}(\mu_i)$ so that the conclusion of that Lemma it satisfied. Define, for each $i \in \mathbb{N}$, the measures μ_i^1, μ_i^2 given by $\mu_i^1(C) = \mu_i(A_i \cap C)$ and $\mu_i^2(C) = \mu_i(B_i \cap C)$. Let $\nu = w^*\text{-}\lim_i \mu_i^1 = \text{succ-}\lim_i \mu_i^1$. Pick a finite subset F of $\widetilde{\mathcal{T}}_\xi$ so that $\nu(\widetilde{\mathcal{T}}_\xi \setminus F) < \varepsilon/2$. Then, because $\nu = w^*\text{-}\lim_i \mu_i^1$ we have $\lim_i \mu_i^1(\widetilde{\mathcal{T}}_\xi) = \nu(\widetilde{\mathcal{T}}_\xi)$ and because $\nu = \text{succ-}\lim_i \mu_i^1$

$$\lim_i \Big|\mu_i^1(\widetilde{\mathcal{T}}_\xi) - \mu_i^1(\cup_{\tilde{t}\in F}\mathrm{succ}(\tilde{t}))\Big| = \Big|\nu(\widetilde{\mathcal{T}}_\xi) - \lim_i \sum_{\tilde{t}\in F}\mu_i^1(\mathrm{succ}(\tilde{t}))\Big| = \nu(\widetilde{\mathcal{T}}_\xi \setminus F) < \frac{\varepsilon}{2}.$$

We can find $i_0 \in \mathbb{N}$ so that for all $i \geq i_0$ we have

$$\Big|\mu_i(A_i) - \mu_i\Big(A_i \cap \big(\cup_{\tilde{t}\in F}\mathrm{succ}(\tilde{t})\big)\Big)\Big| = \Big|\mu_i^1(\widetilde{\mathcal{T}}_\xi) - \mu_i^1(\cup_{\tilde{t}\in F}\mathrm{succ}(\tilde{t}))\Big| < \frac{\varepsilon}{2}. \tag{8}$$

We may, using the fact that the sets $\cup\{\mathrm{supp} g_t : \tilde{t} \in \mathrm{supp}\mu_i\}$ for $i \in \mathbb{N}$ are disjoint, find $j_0 \geq i_0 \in \mathbb{N}$ such that

$$\cup_{\tilde{s}\in F}\mathrm{supp} g_s < \mathrm{supp} g_t \text{ for every } \tilde{t} \in \cup_{i\geq j_0}\mathrm{supp}(\mu_i^1). \tag{9}$$

We define $G_i^1 = A_i \cap (\cup_{\tilde{t}\in F}\mathrm{succ}(\tilde{t}))$, $i \geq j_0$. By (8) we have that for all $i \geq j_0$, $|\mu_i(A_i) - \mu_i(G_i^1)| < \varepsilon/2$. Additionally, $\{t \in \widetilde{\mathcal{T}}_\xi : \tilde{t} \in \cup_{i\geq j_0} G_1^i\}$ is essentially incomparable. Indeed, let $\tilde{s}_1, \tilde{s}_2 \in \cup_{i\geq j_0} G_i^1$ with $m_{j_{s_1}} <_{\mathcal{W}} m_{j_{s_2}}$ and $(h, m_{j_{s_1}}) \in \widetilde{\mathcal{T}}_\xi$ be such that and $(h, m_{j_{s_1}}) \leq_{\widetilde{\mathcal{T}}_\xi} s_2$. Then (9) implies that $\mathrm{supp} h < \mathrm{supp} g_{s_1}$.

For the remaining part of the proof, since for all $i \in \mathbb{N}$ the set $B_i = \text{supp}\mu_i^2$ is finite (as a subset of the finite support of μ_i) and for each $\tilde{t} \in \widetilde{\mathcal{T}}_\xi$ the sequence $(\mu_i^2(\text{succ}(\tilde{t})))_i$ is eventually zero, we may pass to a subsequence so that for all $i < j$ we have $\{m_{j_t} : \tilde{t} \in \text{supp}\mu_i^2\} \cap \{m_{j_t} : \tilde{t} \in \text{supp}\mu_j^2\} = \emptyset$. We can therefore define the bounded sequence of disjointly supported measures (ν_i) on $\widetilde{\mathcal{W}}_\xi$ with $\nu_i(\{(w_1, \ldots, w_k)\}) = \mu_i^2(\{\tilde{t} \in \widetilde{\mathcal{T}}_\xi : m_{j_t} = w_k\})$. Hence, applying Proposition 3.1 and passing to a subsequence, we obtain a subset E_i of $\text{supp}\nu_i$ such that $\nu_i(\widetilde{\mathcal{W}}_\xi \setminus E_i) < \varepsilon/2$ and the sets E_i, $i \in \mathbb{N}$, are pairwise incomparable. It is easy to verify that $G_i^2 = \{\tilde{t} \in B_i : m_{j_t} \in 2_i\}$, $i \in \mathbb{N}$, are pairwise incomparable and $|\mu_i(B_i) - \mu_i(G_i^2)| = \mu_i^2(\widetilde{\mathcal{T}}_\xi \setminus G_i^2) < \varepsilon/2$ for every $i \in \mathbb{N}$. □

We now define the space $T^\xi_{\textit{ess-inc}}$ in a similar way to T^ξ_{inc}, that is, using the notion of the Tsirelson extension W_G of a ground set G.

Definition 4.13. Define the following norming sets on $c_{00}(\mathbb{N})$.

$$G_0 = \{ \pm e_n^* : n \in \mathbb{N}\}$$

$$G_1 = \Big\{\frac{1}{m_j}\sum_{n\in\mathbb{N}} g(n)e_n^* : j \in \mathbb{N} \text{ and } g : \mathbb{N} \to \{-1, 0, 1\} \text{ with supp} g \in \mathcal{S}_{n_j}\Big\}.$$

For each $f = m_j^{-1}\sum_{n\in\mathbb{N}} g(n)e_n^*$ in G_1, set $t_f = (g, m_j)$. Moreover, if $G = G_1 \cup G_0$ and f is in W_G with a tree analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$, define

$$\mathcal{M}_f^1 = \{\alpha : \alpha \text{ is a maximal node of } \mathcal{A} \text{ and } f_a \in G_1\}.$$

Let W be the subset of W_G containing all functionals f such that $\{t_{f_\alpha} : \alpha \in \mathcal{M}_f^1\}$ is an essentially incomparable subset of $\mathcal{T}_\xi$. Denote by $T^\xi_{\textit{ess-inc}}$ the completion of $c_{00}(\mathbb{N})$ with respect to the norm $\|\cdot\|_W$ induced by W.

Remark 4.14.
(i) The standard basis $(e_j)_j$ of $c_{00}(\mathbb{N})$ forms a 1-unconditional basis for the space $T^\xi_{\textit{ess-inc}}$ and it is also boundedly complete since $T^\xi_{\textit{ess-inc}}$ admits a uniformly unique ℓ_1 spreading model as shown in Proposition 4.15.
(ii) If $f \in W_G$ with a tree analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$ and $m_{j_{t_{f_\alpha}}}$, for $\alpha \in \mathcal{M}_f^1$, are pairwise incomparable nodes in $\mathcal{W}_\xi$, then $f \in W$.
(iii) The norming set of $T^\xi_{\textit{ess-inc}}$ contains the norming set of Tsirelson's original space, i.e., the Tsirelson extension of G_0.

Proposition 4.15. *The space $T^\xi_{\textit{ess-inc}}$ admits ℓ_1 a uniformly unique joint spreading model with respect to $\mathscr{F}_b(T^\xi_{\textit{ess-inc}})$.*

Proof. Let $(x_j^i)_j$, $1 \le i \le l$, be an array of normalized block sequences in $T^\xi_{\textit{ess-inc}}$ and fix $\varepsilon > 0$. Passing to a subsequence, we may assume that $\text{supp}x_j^{i_1} < \text{supp}x_{j+1}^{i_2}$ for all $i_1, i_2 = 1, \ldots, l$ and $j \in \mathbb{N}$. For each $i = 1, \ldots, l$ and $j \in \mathbb{N}$, pick a functional $f_j^i = \sum_{\alpha\in\mathcal{M}_j^i} f_{j,\alpha}^i/2^{k_{j,\alpha}^i}$ in W with $f_j^i(x_j^i) \ge 1 - \varepsilon$ and $f_{j,\alpha}^i(x_j^i) > 0$ for every $\alpha \in \mathcal{M}_j^i$, where $\mathcal{M}_j^i$ denotes the set of all maximal nodes of a fixed tree analysis of f_j^i. Moreover, for each $\alpha \in \mathcal{M}^1_{f_j^i} = \{\alpha \in \mathcal{M}_j^i : f_{j,\alpha}^i \in G_1\}$, define $t_{j,\alpha}^i = t_{f_{j,\alpha}^i}$ and, for each $j \in \mathbb{N}$, the measure μ_j as follows:

$$\mu_j = \sum_i \sum_{\alpha\in\mathcal{M}^1_{f_j^i}} \frac{f_{j,\alpha}^i(x_j^i)}{2^{k_{j,\alpha}^i}}\delta_{\tilde{t}_{j,\alpha}^i}.$$

Passing to a subsequence assume that $\lim_j \mu_j(\widetilde{\mathcal{T}}_\xi) = c$. If $c = 0$, then we may assume that $f^i_{j,\alpha} \in G_0$ for every $i = 1, \ldots, l$, $j \in \mathbb{N}$ and $\alpha \in \mathcal{M}^i_j$, in which case the desired result is immediate. Hence, if $c > 0$, applying Lemma 4.6 and passing to a subsequence, we obtain $(G^1_j)_j$, $(G^2_j)_j$ satisfying items (i) - (iv) with $\mu_j(\widetilde{\mathcal{T}}_\xi \setminus G^1_j \cup G^2_j) < 1/8$. Then, for each pair (i,j), set

$$\mathcal{M}^1_{i,j} = \{\alpha \in \mathcal{M}^i_{f^i_j} : t^i_{j,\alpha} \in G^1_j\} \quad \text{and} \quad \mathcal{M}^2_{i,j} = \mathcal{M}^j_i \setminus \mathcal{M}^1_{i,j}$$

and

$$f^k_{i,j} = \sum_{\alpha \in \mathcal{M}^k_{i,j}} f^i_{j,\alpha}/2^{k^i_{j,\alpha}}, \quad k = 1, 2.$$

Note in particular that, for every pair (i,j), the fact that $\mu_j(\widetilde{\mathcal{T}}_\xi \setminus G^1_j \cup G^2_j) < 1/8$ implies that $|f^i_j(x^i_j) - (f^1_{i,j}(x^i_j) + f^2_{i,j}(x^i_j))| < 1/8$ and hence that there exists $k = 1, 2$ such that $f^k_{i,j}(x^i_j) \geq (7 - \varepsilon)/16$. Set

$$A_k = \{(i,j) : f^k_{i,j}(x^i_j) \geq (7 - 8\varepsilon)/16\}, \quad k = 1, 2.$$

Let $n \in \mathbb{N}$, $\{\lambda_{ij}\}_{i=1,j=1}^{l,n} \subset [-1, 1]$ with $\sum_{i,j} |\lambda_{ij}| = 1$ and $s = (s_i)_{i=1}^l \in S\text{-}Plm_l([\mathbb{N}]^k)$ with $ln \leq \min \operatorname{supp} x^1_{s_1(1)}$. Then let $k = 1, 2$ be such that $\sum_{(i,s_i(j)) \in A_k} |\lambda_{ij}| \geq 1/2$ and observe that $f = 1/2 \sum_{(i,s_i(j)) \in A_k} f^k_{i,s_i(j)}$ is in W. Hence, we calculate

$$\Big\| \sum_{i=1}^{l} \sum_{j=1}^{n} |\lambda_{ij}| x^i_{s_i(j)} \Big\| \geq f\Big(\sum_{i=1}^{l} \sum_{j=1}^{n} |\lambda_{ij}| x^i_{s_i(j)} \Big) = \frac{1}{2} \sum_{(i,s_i(j)) \in A_k} |\lambda_{ij}| f^k_{i,s_i(j)}\big(x^i_{s_i(j)}\big) \geq \frac{7 - 8\varepsilon}{32}$$

and due to unconditionality this yields that

$$\Big\| \sum_{i=1}^{l} \sum_{j=1}^{n} \lambda_{ij} x^i_{s_i(j)} \Big\| \geq \frac{7 - 8\varepsilon}{32}.$$

□

It remains to show that for every $M \in [\mathbb{N}]^\infty$, the space $T^\xi_{ess\text{-}inc}$ contains a c_0-tree of height ω^ξ supported by $(e_j)_{j \in M}$. To this end, let us recall the following definition.

Definition 4.16. Let $n \in \mathbb{N}$ and $\varepsilon > 0$. We say that a convex combination $x = \sum_{i \in \Delta} \lambda_i e_i$ in $c_{00}(\mathbb{N})$ is an (n, ε)-special convex combination if

(i) $\Delta \in \mathcal{S}_n$ and
(ii) $\sum_{i \in \Delta'} \lambda_i < \varepsilon$ for every $\Delta' \in \mathcal{S}_m$ with $m < n$.

The main ingredient in the proof of the following proposition is the notion of repeated averages, first defined by Argyros, Mercourakis, and Tsarpalias. in [AMT]. We refer the reader to [AT, Chapter 2] for further details.

Proposition 4.17. *For every $n \in \mathbb{N}$ and $\varepsilon > 0$, there is a $k \in \mathbb{N}$ such that, for every maximal subset F of $\mathcal{S}_n$ with $k < F$, there exists an (n, ε)-special convex combination x in $c_{00}(\mathbb{N})$ with* $\operatorname{supp} x = F$.

For a functional f in W with tree analysis $(f_\alpha)_{\alpha \in \mathcal{A}}$, we define the height of f, denoted by $h(f)$, as the maximum of $|a|$ over all maximal nodes $\alpha \in \mathcal{A}$. Moreover, if $f = m_j^{-1} \sum_{n \in \mathbb{N}} g(n) e^*_n$ is in W, we say that f is a weighted functional and define the weight of f as $w(f) = m_j$.

Lemma 4.18. *Let $j \in \mathbb{N}$ and f be a functional in W with a tree analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$ such that $w(f_\alpha) < m_j$ for every $\alpha \in \mathcal{M}_f^1$. Then* $\text{supp} f \in \mathcal{S}_k$*, where $k \leq n_{j-1} + h(f)$.*

Proof. For each $\alpha \in \mathcal{A}$, let $k_\alpha \in \mathbb{N}$ be such that $\text{supp} f_\alpha \in \mathcal{S}_{k_\alpha}$. Then, since $w(f_\alpha) < m_j$, we have that $k_\alpha \leq n_{j-1}$ for every $\alpha \in \mathcal{M}_f^1$. Note then that, as follows from the definition of W_G, this implies that $k_\alpha \leq n_{j-1} + 1$ for every $\alpha \in \mathcal{A}$ with $|\alpha| = h(f) - 1$. In particular, a finite induction argument yields that $k_\alpha \leq n_{j-1} + i$ whenever $|\alpha| = h(f) - i$ and this proves the desired result. □

Proposition 4.19. *Let $j \in \mathbb{N}$ and $x = \sum_{i\in\Delta} \lambda_i e_i$ be an (n_j, m_j^{-2})-special convex combination, then*

$$\frac{1}{m_j} \leq \|x\|_W \leq \frac{1}{m_j} + \frac{1}{m_j^2}.$$

Proof. Pick an f in W and define $\Delta_1 = \{i \in \Delta : |f(e_i)| > m_j^{-1}\}$ and $\Delta_2 = \Delta \setminus \Delta_1$. Consider the tree analysis $(f_\alpha^1)_{\alpha\in\mathcal{A}}$ of $f_1 = f|_{\Delta_1}$ and note that $w(f_\alpha^1) < m_j$ for every $\alpha \in \mathcal{M}_{f_1}^1$. Indeed, if $w(f_\alpha^1) = m_{j'} \geq m_j$ for some α, then for any $i \in \text{supp} f_\alpha^1$ we have that $|f(e_i)| \leq m_{j'}^{-1}$ and this is a contradiction. Moreover, the fact that $|f_1(e_i)| > m_j^{-1}$ for every $i \in \Delta_1 = \text{supp} f_1$ implies that $h(f_1) < \log_2 m_{j-1}$ and hence the previous proposition yields that $\text{supp} f_1 \in \mathcal{S}_l$, where $l \leq \log_2 m_j + n_{j-1} < n_j$. Therefore, since $x = \sum_{i\in\Delta} \lambda_i e_i$ is an (n_j, m_j^{-2})-special convex combination, we have that

$$|f|_{\Delta_1}(\sum_{i\in\Delta} \lambda_i e_i)| \leq \sum_{i\in\Delta_1} \lambda_i < \frac{1}{m_j^2}.$$

We also calculate

$$|f|_{\Delta_2}(\sum_{i\in\Delta} \lambda_i e_i)| \leq \frac{1}{m_j} \sum_{i\in\Delta_2} \lambda_i \leq \frac{1}{m_j}$$

and conclude that $\|x\|_W \leq m_j^{-1} + m_j^{-2}$. For the remaining part notice that the functional $f = m_j^{-1} \sum_{i\in\Delta} e_i^*$ is in W. □

Proposition 4.20. *Let $j \in \mathbb{N}$ and $x = \sum_{i\in\Delta} \lambda_i e_i$ be an (n_j, m_j^{-2})-special convex combination. Then $|f(x)| < 2m_j^{-2}$, for every $f \in W$ with a tree analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$ such that $w(f_\alpha) \neq m_j$ for all $\alpha \in \mathcal{M}_f^1$.*

Proof. Define $\Delta_1 = \{i \in \Delta : |f(e_i)| > m_j^{-2}\}$ and $\Delta_2 = \Delta \setminus \Delta_1$ and let $(f_\alpha^1)_{\alpha\in\mathcal{A}_1}$ be the tree analysis of $f_1 = f|_{\Delta_1}$. Similar arguments as in the previous proof yield that $w(f_\alpha^1) < m_j^2 < m_{j+1}$ and hence $w(f_\alpha^1) < m_j$ for all $\alpha \in \mathcal{M}_{f_1}^1$, since $w(f_\alpha) \neq m_j$ for all $\alpha \in M_f^1$. Moreover, since $|f_1(e_i)| > m_j^2$ for all $i \in \text{supp} f_1$, we have that $h(f_1) < \log_2 m_j^2$ and therefore Proposition 4.18 yields that $\Delta_1 = \text{supp} f_1 \in \mathcal{S}_l$ with $l \leq \log_2 m_j^2 + n_{j-1} < n_j$. The fact that $x = \sum_{i\in\Delta} \lambda_i e_i$ is an (n_j, m_j^{-2})-special convex combination implies that

$$|f_1(\sum_{i\in\Delta} \lambda_i e_i)| \leq \sum_{i\in\Delta_1} \lambda_i < \frac{1}{m_j^2}.$$

We also calculate

$$|f|_{\Delta_2}(\sum_{i\in\Delta} \lambda_i e_i)| \leq \frac{1}{m_j^2} \sum_{i\in\Delta_2} \lambda_i \leq \frac{1}{m_j^2}$$

and this completes the proof. □

Let M be an infinite subset of the naturals and consider the collection $T_\xi(M)$ of all finite sequences $(x_1,\ldots,x_k)$ of vectors in $c_{00}(\mathbb{N})$ such that

(i) $x_l = m_{j_l}\sum_{i\in\Delta_l}\lambda_i e_i$, where $\sum_{i\in\Delta_l}\lambda_i e_i$ is an $(n_{j_l}, m_{j_l}^{-2})$-special convex combination for every $l=1,\ldots,k$,

(ii) Δ_l is a subset of M for every $l=1,\ldots,k$ and

(iii) $((\chi_{\Delta_1}, m_{j_1}),\ldots,(\chi_{\Delta_k}, m_{j_k}))\in\widetilde{\mathcal{T}}_\xi$.

Note that $T_\xi(M)$, equipped with the initial segment order, is a well-founded infinite branching tree of height ω^ξ.

Proposition 4.21. *Let M be an infinite subset of the naturals and $(x_1,\ldots,x_k)$ be any node of $T_\xi(M)$, then $\|x_1+\ldots+x_k\|_W\le 3$.*

Proof. Let $f\in W$ with a tree analysis $(f_\alpha)_{\alpha\in\mathcal{A}}$. Observe that there exists at most one $1\le l_0\le k$ such that there is an $\alpha\in\mathcal{M}_f^1$ with $w(f_\alpha)=m_{j_{l_0}}$ and $\mathrm{supp} f_a\cap\Delta_{l_0}$ is non-empty. Indeed, suppose that there exist $1\le l_1<l_2\le k$ and $\alpha_1,\alpha_2\in\mathcal{M}_f^1$ with $w(f_{\alpha_1})=m_{j_{l_1}}$, $w(f_{\alpha_2})=m_{j_{l_2}}$, $\mathrm{supp} f_{\alpha_1}\cap\Delta_{l_1}\neq\emptyset$ and $\mathrm{supp} f_{\alpha_2}\cap\Delta_{l_2}\neq\emptyset$. Then since $\{t_{f_\alpha}:\alpha\in\mathcal{M}_f^1\}$ is essentially incomparable and $m_{j_{l_1}}<_{\mathcal{W}_\xi} m_{j_{l_2}}$ we have that $\Delta_{l_1}<\Delta_{t_{f_{\alpha_1}}}=\mathrm{supp} f_{\alpha_1}$ which is a contradiction.

Therefore, for any $l\neq l_0$, we have that $w(f_\alpha)\neq m_{j_l}$ for every $\alpha\in\mathcal{M}_f^1$ and the previous proposition yields that $|f(x_l)|<2m_{j_l}^{-1}$. Moreover, Proposition 4.19 yields that $|f(x_{l_0})|\le 1+m_{j_{l_0}}^{-1}$ and hence we conclude that

$$|f(x_1+\cdots+x_k)|\le 1+2\sum_{l=1}^{k}\frac{1}{m_{j_l}}\le 3.$$

□

The previous proposition and the fact that the tree $T_\xi(M)$ is of height ω^ξ yield the following result.

Theorem 4.22. *For every $M\in[\mathbb{N}]^\infty$, the space $T^\xi_{ess\text{-}inc}$ contains a c_0-tree of height ω^ξ, supported by $(e_j)_{j\in M}$. In particular, the space generated by $(e_j)_{j\in M}$ is not Asymptotic ℓ_1.*

Remark 4.23. There exist modifications of the ground set G that yield, for any $1<p<\infty$, a space, as in the previous theorem, that contains ℓ_p-trees instead of c_0-trees.

Theorem 4.24. *The space $T^\xi_{ess\text{-}inc}$ is reflexive.*

Proof. Note that since $\mathcal{T}_\xi$ is a countable compact space with respect to the pointwise convergence topology, the completion of $c_{00}(\mathbb{N})$ with respect to $\|\cdot\|_G$ is embedded in $C[\mathcal{T}_\xi]$, i.e., the space of all continuous real functions on $\mathcal{T}_\xi$, and hence is c_0-saturated. Furthermore, $T^\xi_{ess\text{-}inc}$ admits a boundedly complete basis and therefore does not contain c_0. The above imply that the identity operator $Id:(c_{00}(\mathbb{N}),\|\cdot\|_W)\to(c_{00}(\mathbb{N}),\|\cdot\|_G)$ is strictly singular and hence for any normalized block sequence $(x_j)_j$ in $T^\xi_{ess\text{-}inc}$ there exists a subsequence $(x_j)_{j\in M}$ such that $\lim_{j\in M}\|x_j\|_G=0$. The remainder of the proof is identical to the last paragraph of Proposition 3.13. □

5. More non-asymptotic ℓ_p spaces with uniformly unique ℓ_p joint spreading models

In this final section we show that, for every $1<p<\infty$, there is a reflexive Banach space that admits a uniformly unique ℓ_p asymptotic model whereas it is not Asymptotic ℓ_p. This was also observed in [BLMS, Section 7.2] for a slightly different type of spaces. We show this for a class of spaces very similar to those defined in [OS, Example 4.3].

Definition 5.1. Let $1 < p < \infty$ and denote its conjugate by q, i.e., $p^{-1} + q^{-1} = 1$. Fix a countable ordinal ξ and define the following norming sets on $c_{00}(\mathbb{N})$.

$$G_1^\xi = \Big\{ \sum_{i\in S} \epsilon_i e_i^* : S \text{ is a segment of } \mathcal{T}_\xi \text{ and } \epsilon_i = \pm 1 \Big\}$$

$$G_{1,p}^\xi = \Big\{ \sum_{i=1}^m b_i f_i : m \in \mathbb{N},\ \sum_{i=1}^m |b_i|^q \leq 1,\ f_i \in G_1^\xi \text{ for } i = 1,\dots,m \text{ and } \text{supp} f_1,\dots,\text{supp} f_m \text{ are pairwise disjoint} \Big\}.$$

Denote by $JT_{1,p}^\xi$ the completion of $c_{00}(\mathbb{N})$ with respect to the norm induced by the norming set $G_{1,p}^\xi$.

We start with some necessary remarks on the above norming sets and a Ramsey type result.

Remark 5.2. Let $(f_j)_j$ be a sequence in G_1^ξ with $f_j = \sum_{i\in S_j} \epsilon_i^j e_i^*$, $j \in \mathbb{N}$, and for each $i, j \in \mathbb{N}$, define $\epsilon_j(i) = \epsilon_j^i$ if $i \in S_j$ and $\epsilon_j(i) = 0$ otherwise. Passing to a subsequence, we may assume that $(S_j)_j$ converges pointwise to a segment S, since $\mathcal{T}_\xi$ is well-founded, and that $(\epsilon_j)_j$ also converges to some ϵ in $\{-1,1\}^{\mathbb{N}}$. Then, clearly, $(f_j)_j$ converges pointwise to $f = \sum_{i\in S} \epsilon(i) e_i^*$ and f is in G_1^ξ.

Remark 5.3. Let x be a normalized vector in $JT_{1,p}^\xi$ with finite support.

(i) If for some $\varepsilon > 0$ there is a family $\{f_i\}_{i\in I}$ in G_1^ξ whose members have pairwise disjoint supports and $|f_i(x)| \geq \varepsilon$ for all $i \in I$, then $\#I \leq \varepsilon^{-p}$.
(ii) Let $f_1,\dots,f_m \in G_1^\xi$ have pairwise disjoint supports and $\text{supp} f_i \subset \text{range}(x)$ for $i = 1,\dots,m$. Then, for any choice of scalars $b_1,\dots,b_m$, we have that

$$\Big| \sum_{i=1}^m b_i f_i(x) \Big|^q \leq \sum_{i=1}^m |b_i|^q.$$

Definition 5.4. We call a family $(F_j)_j$ of finite subsets of $JT_{1,p}^\xi$ a normalized block family if for any choice of $x_j \in F_j$, $j \in \mathbb{N}$, the sequence $(x_j)_j$ is block and $\|x\| = 1$ for any $x \in F_j$ and $j \in \mathbb{N}$. Moreover, for such a family, define $M(F_j) = \max\{\text{supp} x : x \in F_j\}$ and $r(F_j) = \#(M(F_{j-1}), M(F_j)]$, where $M(F_0) = 0$.

Lemma 5.5. *Let $(F_j)_j$ be a normalized block family in $JT_{1,p}^\xi$ with $\sup_j \#F_j < \infty$. Then, for every $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, there is an $L \in [\mathbb{N}]^\infty$ such that, for any segment S of $\mathcal{T}_\xi$ with $\min S \leq n_0$ and any $f \in G_1^\xi$ with $\text{supp} f = S$, there is at most one $j \in L$ with the property that $|f(x)| \geq \varepsilon$ for some $x \in F_j$.*

Proof. For a segment S of $\mathcal{T}_\xi$, let G_S denote the set of all $f \in G_1^\xi$ with $\text{supp} f = S$. If the conclusion is false for some $\varepsilon > 0$ and $n_0 \in \mathbb{N}$, then using Ramsey Theorem from [Ra], there exists an $L \in [\mathbb{N}]^\infty$ such that, for any $i < j$ in L, there is a segment S_{ij} with $\min S_{ij} \leq n_0$, a functional $f_{ij} \in G_{S_{ij}}$ and $x_{ij} \in F_i$, $y_{ij} \in F_j$ such that $|f_{ij}(x_{ij})| \geq \varepsilon$ and $|f_{ij}(y_{ij})| \geq \varepsilon$. Assume for convenience that $L = \mathbb{N}$. Since $\#\sup_j F_j < \infty$, using the pigeon hole principle and a diagonal argument we may assume that there exist sequences $(x_j)_j$, $(y_j)_j$ such that $x_j, y_j \in F_j$ and, for every $i < j \in \mathbb{N}$, a segment S_{ij} of $\mathcal{T}_\xi$ with $\min S_{ij} \leq n_0$ and an $f_{ij} \in G_{S_{ij}}$ such that $|f_{ij}(x_i)| \geq \varepsilon$ and $|f_{ij}(y_j)| \geq \varepsilon$.

For each $i < j < k$ in $\mathbb{N}$, define $S_{ijk} = S_{ik} \cap S_{jk} \cap \mathrm{range}(y_k)$. Once more, using Ramsey theorem and passing to a further infinite subset, we may assume that S_{ijk} is either always empty or always non-empty for every $i < j < k$ in $\mathbb{N}$. Item (i) of Remark 5.3 and the fact that $\|y_k\| = 1$ for all $k \in \mathbb{N}$ rules out the first case and hence $S_{ijk} \neq \emptyset$ for all $i < j < k$ in $\mathbb{N}$. This in particular implies that if we fix $i < j_1 < k$ and $i < j_2 < k$, then $S_{ij_1k}|_{[n_0,m(x_k))} = S_{ij_2k}|_{[n_0,m(x_k))}$. For any $j \in \mathbb{N}$, take an arbitrary i with $1 < i < j$ and set $S_j = S_{1ij}|_{[n_0,m(x_j))}$. Then we conclude that, for any $j \in \mathbb{N}$, there is an $f_j \in G_{S_j}$ such that $|f_j(x_i)| \geq \varepsilon$ for all $i < j$, where $\min S_j \leq n_0$. This is a contradiction, since Remark 5.2 implies that there exists an $f \in G_1^\xi$ with the property that $|f(x_j)| \geq \varepsilon$ for all $j \in \mathbb{N}$, whereas supp$f$ is finite since $\mathcal{T}_\xi$ is well-founded. □

Lemma 5.6. *Let $\varepsilon > 0$ and $(F_j)_j$ be a normalized block family in $JT^\xi_{1,p}$ with $\sup_j \#F_j < \infty$. Then there exists a strictly increasing sequence $(n_j)_j$ of naturals and a decreasing sequence $(\varepsilon_j)_j$ of positive reals such that*

(i) for every $j \in \mathbb{N}$, every segment S of $\mathcal{T}_\xi$ with $\min S \leq M(F_{n_j})$ and $f \in G_1^\xi$ with supp$f = S$, there exists at most one $j' > j$ such that $|f(x)| \geq \varepsilon_j$ for some $x \in F_{n_{j'}}$ and

(ii) $\sum_{j=1}^\infty r(F_{n_j}) \sum_{i=j}^\infty (i+1)\varepsilon_i < \varepsilon$.

Proof. Let $(\delta_j)_j$ be a sequence of positive reals such that $\sum_{j=1}^\infty \delta_j < \varepsilon$. We will construct $(n_j)_j$ and $(\varepsilon_j)_j$ by induction, along with a decreasing sequence $(L_j)_j$ of infinite subsets of $\mathbb{N}$. Set $n_1 = 1$ and $L_1 = \mathbb{N}$ and choose $\varepsilon_1 > 0$ such that $2r(F_1)\varepsilon_1 < \delta_1$. Suppose that $n_1, \ldots, n_j$, $\varepsilon_1, \ldots, \varepsilon_j$ and $L_1, \ldots, L_j$ have been chosen for some j in $\mathbb{N}$. Then, the previous lemma yields an $L_{j+1} \in [L_j]^\infty$ such that, for every segment S of $\mathcal{T}_\xi$ with $\min S \leq M(F_{n_j})$ and every $f \in G_1^\xi$ with supp$f = S$, there is at most one $j' > j$ such that $|f(x)| \geq \varepsilon_j$ for some $x \in F_{n_{j'}}$. Choose $n_{j+1} \in L_{j+1}$ with $n_{j+1} > n_j$ and $\varepsilon_{j+1} < \varepsilon_j$ such that

(a) $r(F_{n_{j+1}})(j+2)\varepsilon_{j+1} < \delta_{j+1}$ and

(b) $r(F_{n_k}) \sum_{i=k}^{j+1} (i+1)\varepsilon_i < \delta_k$ for all $k \leq j$.

It follows quite easily that $(n_j)_j$ and $(\varepsilon_j)_j$ are as desired. □

Proposition 5.7. *Let $\varepsilon > 0$ and $(F_j)_j$ be a normalized block family in $JT^\xi_{1,p}$ with $\sup_j \#F_j < \infty$ satisfying the following.*

(i) For every $j \in \mathbb{N}$, every segment S of $\mathcal{T}_\xi$ with $\min S \leq M(F_n)$ and $f \in G_1^\xi$ with supp$f = S$, there exists at most one $j' > j$ such that $|f(x)| \geq \varepsilon_j$ for some $x \in F_{j'}$ and

(ii) $\sum_{j=1}^\infty r(F_j) \sum_{i=j}^\infty (i+1)\varepsilon_i < \varepsilon$.

Then, for every $n \in \mathbb{N}$, every choice of $x_1, \ldots, x_n$ with $x_j \in F_j$ and scalars $a_1, \ldots, a_n$, we have that

$$\Big(\sum_{j=1}^n |a_j|^p\Big)^{\frac{1}{p}} \leq \Big\|\sum_{j=1}^n a_j x_j\Big\| \leq (2^{\frac{1}{q}} + \varepsilon)\Big(\sum_{j=1}^n |a_j|^p\Big)^{\frac{1}{p}}.$$

Proof. The lower inequality follows easily from the definition of $G^\xi_{1,p}$. Let us first observe that if $(x_j)_j$ is a sequence with each $x_j \in F_j$, then, for any $j \in \mathbb{N}$ and any segment S of $\mathcal{T}_\xi$ with $M(x_{j-1}) < \min S \leq M(x_j)$ and $f \in G_1^\xi$ with supp$f = S$, the following hold due to (i).

(a) $\#\{i > j : |f(x_i)| \geq \varepsilon_j\} \leq 1$.

(b) $\#\{i > j : \varepsilon_{k-1} > |f(x_i)| \geq \varepsilon_k\} \leq k$ for all $k > j$.

Let $f = \sum_{i=1}^m b_i f_i$ be in $G_{1,p}^\xi$ with $\text{supp} f_i = S_i$, for $i = 1, \ldots, m$. For each i, we will denote by $j_{i,1}$ the unique $1 \leq j \leq n$ such that $M(x_{j_{i,1}-1}) < \min S_i \leq M(x_{j_{i,1}})$ and by $j_{i,2}$ the unique, if there exists, $j_{i,1} < j \leq n$ such that $|f_i(x_{j_{i,2}})| \geq \varepsilon_{j_{i,1}}$. Denote by $f_{i,1}$ the restriction of f_i to $\text{range}(x_{j_{i,1}}) \cap \text{range}(x_{j_{i,2}})$ and set $f_{i,2} = f_i - f_{i,1}$ for $i = 1, \ldots, m$, and $I_j = \{i : j = j_{i,1} \text{ or } j = j_{i,2}\}$ for $j = 1, \ldots, n$. Note that, due to (a), each i appears in I_j for at most two j and hence $\sum_{j=1}^n \sum_{i \in I_j} |b_i|^q \leq 2$.
We thus calculate applying item (ii) of Remark 5.3

$$\sum_{i=1}^m b_i f_{i,1}\Big(\sum_{j=1}^n a_j x_j\Big) = \sum_{j=1}^n a_j \sum_{i \in I_j} b_i f_{i,1}(x_j) \leq \Big(\sum_{j=1}^n |a_j|^p\Big)^{\frac{1}{p}} \Big(\sum_{j=1}^n \big|\sum_{i \in I_j} b_i f_{i,1}(x_j)\big|^q\Big)^{\frac{1}{q}}$$
$$\leq \Big(\sum_{j=1}^n |a_j|^p\Big)^{\frac{1}{p}} \Big(\sum_{j=1}^n \sum_{i \in I_j} |b_i|^q\Big)^{\frac{1}{q}} \leq 2^{\frac{1}{q}} \Big(\sum_{j=1}^n |a_j|^p\Big)^{\frac{1}{p}}.$$

Finally, for each $j \in \mathbb{N}$, set $G_j = \{i : M(x_{j_{i,1}-1}) < \min S_i \leq M(x_{j_{i,1}})\}$. Note that, as follows from (b), $\#G_j \leq r(F_j)$ and $|f_{i,2}(\sum_{k=1}^n x_k)| < \sum_{k=i}^\infty (k+1)\varepsilon_k$ for any $i \in G_j$. Hence (ii) yields that $\sum_{i=1}^m |f_{i,2}(\sum_{k=1}^n x_k)| < \varepsilon$ and we conclude that

$$\Big|\sum_{i=1}^m b_i f_{i,2}\Big(\sum_{j=1}^n a_j x_j\Big)\Big| = \Big|\sum_{j=1}^n a_j \sum_{i=1}^m b_i f_{i,2}(x_j)\Big| < \varepsilon \Big(\sum_{j=1}^n |a_j|^p\Big)^{\frac{1}{p}}$$

which along with the above calculation yield the desired result. □

Proposition 5.8. *The space $JT_{1,p}^\xi$ admits a uniformly unique joint spreading model with respect to $\mathscr{F}_b(JT_{1,p}^\xi)$, equivalent to the unit vector basis of ℓ_p.*

Proof. Let $(x_j^1)_j, \ldots, (x_j^l)_j$ be normalized block sequences in $JT_{1,p}^\xi$ and let $\varepsilon > 0$. Applying Lemma 5.6 and passing to a subsequence, we may assume that $F_j = \{x_j^i : i = 1, \ldots, l\}$ is a normalized block family in $JT_{1,p}^\xi$ satisfying items (i) and (ii) of Proposition 5.7. Then, for every $k \in \mathbb{N}$, every $s = (s_i)_{i=1}^l$ in $S\text{-}Plm_l([L]^k)$ and any choice of scalars $(a_{ij})_{i=1,j=1}^{l,k}$, we calculate

$$\Big(\sum_{i=1}^l \sum_{j=1}^k |a_{ij}|^p\Big)^{\frac{1}{p}} \leq \Big\|\sum_{i=1}^l \sum_{j=1}^k a_{ij} x_{s_i(j)}^i\Big\| \leq (2^{\frac{1}{q}} + \varepsilon)\Big(\sum_{i=1}^l \sum_{j=1}^k |a_{ij}|^p\Big)^{\frac{1}{p}}.$$

A diagonal argument then yields that there exists $L \in [\mathbb{N}]^\infty$ such that $((x_j^i)_{j \in L})_{i=1}^l$ generates a joint spreading model $2^{\frac{1}{q}}$-equivalent to the unit vector basis of ℓ_p. □

Proposition 5.9. *The space $JT_{1,p}^\xi$ is reflexive.*

Proof. Note that the unit vector basis of $c_{00}(\mathbb{N})$ forms a boundedly complete unconditional Schauder basis for $JT_{1,p}^\xi$, that is, it does not contain c_0. Moreover, Proposition 5.7 yields that it does not contain ℓ_1 and hence Theorem 2 from [J1] yields the desired result. □

Proposition 5.10. *The space $JT_{1,p}^\xi$ is not Asymptotic ℓ_p.*

Proof. Suppose that $JT_{1,p}^\xi$ is C-Asymptotic ℓ_p and let $n \in \mathbb{N}$ be such that $C \leq n^{\frac{1}{q}}$. Then, following the same arguments as in Proposition 3.14, in the final outcome of $G(n, p, C)$ we, as player (V), have chosen elements of the basis $e_{j_1},\dots,e_{j_n}$ such that $\{j_1,\dots,j_n\}$ is a segment of $\mathcal{T}_\xi$ and hence $\{e_{j_1},\dots,e_{j_n}\}$ is isometric to ℓ_1^n. We then calculate

$$\Big\|n^{-\frac{1}{p}}\sum_{i=1}^n e_{j_i}\Big\| = \Big\|n^{-\frac{1}{p}}\sum_{i=1}^n e_{j_i}\Big\|_{G_1^\xi} = n^{\frac{1}{q}}$$

whereas, since $JT^\xi_{1,p}$ is C-Asymptotic ℓ_p, we have that

$$\Big\|n^{-\frac{1}{p}}\sum_{i=1}^n e_{j_i}\Big\| \leq C$$

and this is a contradiction. □

Remark 5.11. We may also define a conditional version of $JT^\xi_{1,p}$, denoted as JT^ξ_p, by replacing the norming set G_1^ξ with

$$G^\xi_{sum} = \Big\{\sum_{i\in S} e_i^* : S \text{ is a segment of } \mathcal{T}_\xi\Big\}.$$

Note that the above results hold for JT^ξ_p. For the reflexivity part, notice that it suffices to show that $(e_j)_j$ is shrinking for JT^ξ_p. If not, then there is an $x^*\in(JT^\xi_p)^*\setminus\overline{\operatorname{span}}\{e_j^*\}_{j=1}^\infty$ and an $x^{**}\in(JT^\xi_p)^{**}$ with $x^{**}(e_j^*)=0$ for all $j\in\mathbb{N}$ and $x^{**}(x^*)=1$. Then, from Odell-Rosenthal Theorem [OR] and the fact that $x^{**}(e_j^*)=0$, $j\in\mathbb{N}$, we may find a seminormalized block sequence $(x_j)_j$ in JT^ξ_p with w^*-$\lim_j x_j = x^{**}$ and, passing to a subsequence, we may assume that it also satisfies items (i) and (ii) of Proposition 5.7 for some $\varepsilon>0$. Since $x^{**}(x^*)=1$, there exists $n_0\in\mathbb{N}$ such that $x^*(x_n)\geq 1/2$ for all $n\geq n_0$. Then, for $k\in\mathbb{N}$ such that $(2^{\frac{1}{q}}+\varepsilon)k^{-\frac{1}{q}}<1/2$, Proposition 5.7 yields that

$$x^*\Big(\frac{x_{n_0+1}+\ldots+x_{n_0+k}}{k}\Big)\leq (2^{\frac{1}{q}}+\varepsilon)k^{-\frac{1}{q}}$$

which is a contradiction.

Remark 5.12. Note that by replacing the norming set G_1^ξ with

$$G_r^\xi = \Big\{\sum_{i\in S} b_i e_i^* : S \text{ is a segment of } \mathcal{T}_\xi \text{ and } \sum_{i\in S}|b_i|^{r'}\leq 1\Big\}$$

where $r^{-1}+r'^{-1}=1$ and $1<r<p$, we define the spaces $JT^\xi_{r,p}$ whose norm is described in (1). These spaces are also reflexive, admit a unique ℓ_p asymptotic model and are not Asymptotic ℓ_p.

Remark 5.13. The approach used in [BLMS] can be used to show that the spaces $JT^\xi_{r,p}$ and JT^ξ_p have the property that any joint spreading model generated by an array of weakly null sequences is isometrically equivalent to the unit vector basis of ℓ_p. That approach provides less insight and has no potential to apply to cases with a non-isometric result, e.g., the space from Section 3.

Acknowledgement

The authors would like to thank the anonymous referees whose useful remarks helped us identify and correct a mistake in the original proof of Lemma 4.6.

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