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INEQUALITIES FOR THE VARIATION OPERATOR

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Abstract. Let f be a measurable function defined on $\mathbb{R}$. For each $n \in \mathbb{Z}$ define the operator A_n by

$$A_nf(x) = \frac{1}{2^n} \int_x^{x+2^n} f(y)\, dy.$$

Consider the variation operator

$$\mathcal{V}f(x) = \left(\sum_{n=-\infty}^{\infty} |A_nf(x) - A_{n-1}f(x)|^s \right)^{1/s}$$

for $2 \leq s < \infty$.
It has been proved in [1] that $\mathcal{V}$ is of strong type (p,p) for $1 < p < \infty$ and is of weak type $(1,1)$, it maps L^∞ to BMO. We first provide a completely different proof for these known results and in addition we prove that $\mathcal{V}$ maps H^1 to L^1. Furthermore, we prove that it satisfies vector-valued weighted strong type and weak type inequalities. As a special case it follows that $\mathcal{V}$ satisfies weighted strong type and weak type inequalities.

Let f be a measurable function defined on $\mathbb{R}$. For each $n \in \mathbb{Z}$ define the operator A_n by

$$A_nf(x) = \frac{1}{2^n} \int_x^{x+2^n} f(y)\, dy.$$

It is a well known problem to study the different kinds of convergence of the sequence $\{A_nf\}_n$ when $f \in L^p(\mathbb{R})$ for some $1 \leq p < \infty$.
Consider the variation operator

$$\mathcal{V}f(x) = \left(\sum_{n=-\infty}^{\infty} |A_nf(x) - A_{n-1}f(x)|^s \right)^{1/s}$$

for $2 \leq s < \infty$.
Analyzing the boundedness of the variation operator $\mathcal{V}f$ is a method of measuring the speed of convergence of the sequence $\{A_nf\}$.

If a positive function $w \in L^1_{\text{loc}}(\mathbb{R})$ satisfies the following condition we say that w is an A_p weight for some $1 < p < \infty$:

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$$\sup_I \left(\frac{1}{|I|} \int_I w(x)\,dx \right) \left(\frac{1}{|I|} \int_I w(x)^{-\frac{1}{p-1}}\,dx \right)^{p-1} < \infty,$$

where the supremum is taken over all intervals I in $\mathbb{R}$.
The function w is an A_∞ weight if there exist $\delta > 0$ and $\epsilon > 0$ such that given an interval I in $\mathbb{R}$, for any measurable $E \subset I$,

$$|E| < \delta \cdot |I| \implies w(E) < (1-\epsilon) \cdot w(I).$$

Here

$$w(E) = \int_E w.$$

It is well known and easy to see that $w \in A_p \implies w \in A_\infty$ if $1 < p < \infty$.
We say that $w \in A_1$ if given an interval I in $\mathbb{R}$ there is a positive constant C such that

$$\frac{1}{|I|} \int_I w(y)\,dy \le Cw(x)$$

for a.e. $x \in I$.
We say that an operator $T : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is of strong type (p, p) if there exists a positive constant C such that

$$\|Tf\|_p \le C\|f\|_p$$

for all $f \in L^p(\mathbb{R})$. We say that T is of weak type $(1, 1)$ (or satisfies a weak type $(1, 1)$ inequality) if there exists a positive constant C such that

$$|\{x : |Tf(x)| > \lambda\}| \le \frac{C}{\lambda}\|f\|_1$$

for all $f \in L^1(\mathbb{R})$. We say that an operator T maps $L^p(w)$ to itself if there is a positive constant C such that

$$\int_{\mathbb{R}} |Tf(x)|^p w(x)\,dx \le C \int_{\mathbb{R}} |f(x)|^p w(x)\,dx$$

for all $f \in L^p(w)$.
We say that the kernel $K : (-\infty, 0) \to l^s$ satisfies the D_r condition for $1 \le r < \infty$, and write $K \in D_r$, if there exists a sequence $\{c_l\}_{l=2}^{\infty}$ of positive numbers $\sum_l c_l < \infty$ such that

$$\left(\int_{S_l(x)} \|K(x-y) - K(-y)\|_{l^s}^r\,dy \right)^{1/r} \le c_l |S_l(x)|^{-1/r'},$$

for all $l \ge 2$ and all $x > 0$, where $S_l(x) = (2^l x, 2^{l+1} x)$ and $\frac{1}{r} + \frac{1}{r'} = 1$.
When $K \in D_1$ we have the Hörmander condition:

$$\int_{\{y > 4x\}} \|K(x-y) - K(-y)\|_{l^s}\,dy \le C$$

where C is a positive constant which does not depend on $x > 0$.
A linear operator T mapping $\mathbb{R}$-valued functions into l^s-valued functions is called a singular integral operator of convolution type if the following conditions are satisfied:

(i) T is a bounded operator from $L^q(\mathbb{R})$ to $L^q_{l^s}(\mathbb{R})$ for some q, $1 \le q \le \infty$.
(ii) There exists a kernel $K \in D_1$ such that

$$Tf(x) = \int K(x-y)\cdot f(y)\,dy$$

for every $f \in L^q(\mathbb{R})$ with compact support and for a.e. $x \notin \operatorname{supp}(f)$.

Given a locally integrable function f we define the sequence-valued operator T as follows:

$$\begin{aligned}
Tf(x) &= \{A_n f(x) - A_{n-1}f(x)\}_{n\in\mathbb{Z}} \\
&= \left\{ \int_{\mathbb{R}} \left(\frac{1}{2^n}\chi_{(-2^n,0)}(x-y) - \frac{1}{2^{n-1}}\chi_{(-2^{n-1},0)}(x-y) \right) f(y)\,dy \right\}_{n\in\mathbb{Z}} \\
&= \int_{\mathbb{R}} K(x-y)\cdot f(y)\,dy,
\end{aligned}$$

where K is the sequence-valued function

$$K(x) = \left\{ \frac{1}{2^n}\chi_{(-2^n,0)}(x) - \frac{1}{2^{n-1}}\chi_{(-2^{n-1},0)}(x) \right\}_{n\in\mathbb{Z}} = \{K_n(x)\}_{n\in\mathbb{Z}}\,.$$

It is clear that

$$\|Tf(x)\|_{l^s} = \mathcal{V}f(x).$$

Lemma 1. *The kernel operator*

$$K(x) = \left\{ \frac{1}{2^n}\chi_{(-2^n,0)}(x) - \frac{1}{2^{n-1}}\chi_{(-2^{n-1},0)}(x) \right\}_{n\in\mathbb{Z}}$$

satisfies the D_r condition for $r \ge 1$.

Proof. Let $x_0 \in \mathbb{R}$ and $i \in \mathbb{Z}$ be given, consider x and y in $\mathbb{R}$ such that $x_0 < x \le x_0 + 2^i$ and $x_0 + 2^j < y < x_0 + 2^{j+1}$ with $j > i$. Let $\phi_n(y) = \chi_{(-2^n,0)}(y)$. Then $\phi_n(x-y) - \phi_n(x_0-y) = 0$ unless $n = j$ in which case

$$\phi_j(x-y) - \phi_j(x_0-y) = \chi_{(x_0+2^j,x+2^j)}(y).$$

To see this first it is clear that $\phi_n(x-y) = \chi_{(x,x+2^n)}(y)$. Now if $n < i$ then

$$x + 2^n < x - x_0 + x_0 + 2^i \le x_0 + 2^j < y.$$

Thus $\phi_n(x-y) = 0$. Clearly, the same holds for $\phi_n(x_0-y)$. If $i \le n < j$ then

$$x + 2^n \le x_0 + 2^i + 2^n \le x_0 + 2\cdot 2^n \le x_0 + 2^j,$$

and $\phi_n(x-y) = \phi_n(x_0-y) = 0$. If $n > j$ then

$$x + 2^n > x_0 + 2^n \ge x_0 + 2^{j+1} \ge y,$$

and since $y > x > x_0$ we have

$$\phi_n(x-y) - \phi_n(x_0-y) = 1 - 1 = 0.$$

Finally, if $n = j$ then

$$\phi_j(x_0-y) = \chi_{(x_0,x_0+2^j)}(y) = 0,$$

while

$$\phi_j(x-y) = \chi_{(x,x+2^j)}(y) = 1$$

whenever

$$x_0 + 2^j \le y < x + 2^j.$$

We now have

$$\begin{aligned}
\|K(x-y) - K(x_0-y)\|_{l^s}^s &= \sum_n \Bigg| \frac{1}{2^n}\phi_n(x-y) - \frac{1}{2^{n-1}}\phi_{n-1}(x-y) \\
&\qquad - \left(\frac{1}{2^n}\phi_n(x_0-y) - \frac{1}{2^{n-1}}\phi_{n-1}(x_0-y)\right)\Bigg|^s \\
&= \sum_n \Bigg| \frac{1}{2^n}\phi_n(x-y) - \frac{1}{2^n}\phi_n(x_0-y) \\
&\qquad - \left(\frac{1}{2^{n-1}}\phi_{n-1}(x-y) - \frac{1}{2^{n-1}}\phi_{n-1}(x_0-y)\right)\Bigg|^s \\
&= 2\left|\frac{1}{2^j}\phi_j(x-y) - \frac{1}{2^j}\phi_j(x_0-y)\right|^s \\
&= 2\left|\frac{1}{2^j}\chi_{(x_0+2^j, x+2^j)}(y)\right|^s.
\end{aligned}$$

Thus we get

$$\|K(x-y) - K(x_0-y)\|_{l^s} = 2^{1/s}\left|\frac{1}{2^j}\chi_{(x_0+2^j, x+2^j)}(y)\right|.$$

Given $x > 0$, choose an integer i such that $2^{i-1} < x \le 2^i$. By using our previous observation we obtain

$$\begin{aligned}
\left(\int_{2^l x}^{2^{l+1}x} \|K(x-y) - K(-y)\|_{l^s}^r \, dy\right)^{1/r} &\le \left(\int_{2^{l+i-1}}^{2^{l+i}} \|K(x-y) - K(-y)\|_{l^s}^r \, dy\right)^{1/r} \\
&\quad + \left(\int_{2^{l+i}}^{2^{l+i+1}} \|K(x-y) - K(-y)\|_{l^s}^r \, dy\right)^{1/r} \\
&\le 2^{2/s}\frac{2^{i/r}}{2^{l+i}} \\
&\le C 2^{-l/r} |S_l(x)|^{-1/r'}
\end{aligned}$$

and this completes our proof. □

It is easy to check that the Fourier transform of the kernel K of our vector-valued operator T is in $L^\infty_{l^2}$, i.e. that

$$\sum_{n\in\mathbb{Z}} |\widehat{K_n}(\xi)|^2 \le C$$

for all $\xi \in \mathbb{R}$. This, together with $s \ge 2$, shows that T maps $L^2(\mathbb{R})$ into $L^2_{l^s}(\mathbb{R})$. Since we also have the D_1 condition, we deduce that T is a singular integral operator of convolution type.

Lemma 2. *A singular integral operator T mapping A-valued functions into B-valued functions can be extended to an operator defined in all L^p_A, $1 \le p < \infty$, and satisfying*
(a) $\|Tf\|_{L^p_B} \le C_p \|f\|_{L^p_A}, \quad 1 < p < \infty,$

(b) $\|Tf\|_{WL^1_B} \leq C_1 \|f\|_{L^1_A}$,
(c) $\|Tf\|_{L^1_B} \leq C_2 \|f\|_{H^1_A}$,
(d) $\|Tf\|_{\mathrm{BMO}(B)} \leq C_3 \|f\|_{L^\infty(A)}, \quad f \in L^\infty_c(A)$,
where $C_p, C_1, C_2, C_3 > 0$.

Proof. See J. L. Rubio de Francia *et al* [2]. □

Our first result is the following:

Theorem 1. *The variation operator* $\mathcal{V}f$ *maps* H^1 *to* L^1 *for* $2 \leq s < \infty$.

Proof. We have proved that the operator

$$\begin{aligned} Tf(x) &= \left\{ \int_{\mathbb{R}} \left(\frac{1}{2^n} \chi_{(-2^n,0)}(x-y) - \frac{1}{2^{n-1}} \chi_{(-2^{n-1},0)}(x-y) \right) f(y)\, dy \right\}_{n \in \mathbb{Z}} \\ &= \int_{\mathbb{R}} K(x-y) \cdot f(y)\, dy \end{aligned}$$

is a singular integral operator. Since also $\|Tf(x)\|_{l^s} = \mathcal{V}f(x)$ applying Lemma 2 (c) to our operator T shows that there exists a constant $C > 0$ such that

$$\|\mathcal{V}f\|_1 \leq C \|f\|_{H^1}$$

for all $f \in H^1$. □

Remark 1. It is clear that our argument also provides a different proof for the following known facts (see [1]):
(a) $\|\mathcal{V}f\|_p \leq C_p \|f\|_p, \quad 1 < p < \infty$,
(b) $\|\mathcal{V}f\|_{WL^1} \leq C_1 \|f\|_1$,
(c) $\|\mathcal{V}f\|_{\mathrm{BMO}} \leq C_2 \|f\|_\infty, \quad f \in L^\infty_c(\mathbb{R})$,
where $C_p, C_1, C_2 > 0$.

Lemma 3. *Let* T *be a singular integral operator with kernel* $K \in D_r$, *where* $1 < r < \infty$. *Then, for all* $1 < \rho < \infty$, *the weighted inequalities*

$$\left\| \left(\sum_j \|Tf_j\|_B^\rho \right)^{1/\rho} \right\|_{L^p(w)} \leq C_{p,\rho}(w) \left\| \left(\sum_j \|f_j\|_A^\rho \right)^{1/\rho} \right\|_{L^p(w)}$$

hold if $w \in A_{p/r'}$ *and* $r' \leq p < \infty$, *or if* $w \in A_p^{r'}$ *and* $1 < p \leq r'$. *Likewise, if* $w(x)^{r'} \in A_1$, *then the weak type inequality*

$$w\left(\left\{ x : \left(\sum_j \|Tf_j(x)\|_B^\rho \right)^{1/\rho} > \lambda \right\} \right) \leq C_\rho(w) \frac{1}{\lambda} \int \left(\sum_j \|f_j(x)\|_A^\rho \right)^{1/\rho} w(x)\, dx$$

holds.

Proof. See J. L. Rubio de Francia *et al* [2]. □

Our next result is the following:

Theorem 2. *Let $2 \leq s < \infty$. Then, for all $1 < \rho < \infty$, the weighted inequalities*

$$\left\| \left(\sum_j (\mathcal{V} f_j)^\rho \right)^{1/\rho} \right\|_{L^p(w)} \leq C_{p,\rho}(w) \left\| \left(\sum_j |f_j|^\rho \right)^{1/\rho} \right\|_{L^p(w)}$$

hold if $w \in A_{p/r'}$ and $r' \leq p < \infty$, or if $w \in A_p^{r'}$ and $1 < p \leq r'$. Likewise, if $w(x)^{r'} \in A_1$, then the weak type inequality

$$w\left(\left\{ x : \left(\sum_j (\mathcal{V} f_j(x))^\rho \right)^{1/\rho} > \lambda \right\} \right) \leq C_\rho(w) \frac{1}{\lambda} \int \left(\sum_j |f_j(x)|^\rho \right)^{1/\rho} w(x)\, dx$$

holds.

Proof. We have proved that the operator

$$\begin{aligned} Tf(x) &= \left\{ \int_{\mathbb{R}} \left(\frac{1}{2^n} \chi_{(-2^n,0)}(x-y) - \frac{1}{2^{n-1}} \chi_{(-2^{n-1},0)}(x-y) \right) f(y)\, dy \right\}_{n \in \mathbb{Z}} \\ &= \int_{\mathbb{R}} K(x-y) \cdot f(y)\, dy, \end{aligned}$$

is a singular integral operator and its kernel operator K satisfies the D_r condition for $1 \leq r < \infty$. Since also $\|Tf(x)\|_{l^s} = \mathcal{V} f(x)$, applying Lemma 3 to our operator T gives the result of our theorem. □

In particular we have the following corollary:

Corollary 3. *Let $2 \leq s < \infty$. Then the weighted inequalities*

$$\|\mathcal{V} f\|_{L^p(w)} \leq C_{p,\rho}(w) \|f\|_{L^p(w)}$$

hold if $w \in A_{p/r'}$ and $r' \leq p < \infty$, or if $w \in A_p^{r'}$ and $1 < p \leq r'$. Likewise, if $w(x)^{r'} \in A_1$, then the weak type inequality

$$w\left(\{ x : \mathcal{V} f(x) > \lambda \} \right) \leq C_\rho(w) \frac{1}{\lambda} \int |f(x)| w(x)\, dx$$

holds.

References

[1] R. Jones, R. Kaufman J. M. Rosenblatt and M. Wierdl, *Oscillation in ergodic thery*, Ergod. Th. & Dynam. Sys. **18** (1998) 889-935.

[2] J. L. Rubio de Francia, F. J. Ruiz and J. L. Torrea, *Calderón-Zygmund theory for operator-valued kernels*, Adv. Math. **62** (1986) 7-48.

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