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Inhomogeneously broadened Maxwell-Bloch equations

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Abstract

We study a reduction of a Maxwell-Bloch system that includes the effects of both a permanent dipole and inhomogeneous broadening. We present the Lax pair and use an appropriate Bäcklund transformation to solve the equations exactly and produce a family of soliton solutions.

1. Introduction

A nonlinear interaction occurs when high intensity, nearly resonant light propagates in a dielectric, mainly due to saturable quantum mechanical resonances between the frequency of the electromagnetic light waves and natural oscillatory modes of the medium [9]. Equations that capture the relevant phenomenology are the classical Maxwell wave equation coupled with the quantum mechanical Bloch equations.

Under certain physically motivated assumptions the Maxwell-Bloch (MB) system reduces to several models, each one derived at a different level of approximation. Two physical phenomena closely related to the interaction of light and matter are the effects of: (i) inhomogeneous broadening, and (ii) a permanent dipole. In reducing the MB equations either the first and/or the second phenomena were neglected. Lamb in [7, 8], Eilbeck et al. in [3] incorporated the effects of inhomogeneous broadening in the absence of a permanent dipole. More recently, in [1], the effect of a permanent dipole was retained in the absence of inhomogeneous broadening. In this paper we study a set of reduced MB equations that retain the properties of *both* events.

Inhomogeneous broadening describes the phenomenon that occurs when the atoms that constitute a medium possess different resonant frequencies due to microscopic interactions between them. Each atom is assumed to be at a different resonant frequency, ω, with probability given by a function $g(\omega)$. $g(\omega)$ is a function strongly peaked at ω_0, the frequency of an *isolated* atom, and describes the spread of the atomic frequencies around ω_0. In the equations that follow, (1)-(2), we include the inhomogeneous broadening effect by using the average of the relevant functions over all possible atomic frequencies in the form of an integral. The scenario that neglects the effects of inhomogeneous broadening (sharp line case) is equivalent to the situation when all the atoms of the medium oscillate with the exact same frequency, ω_0, and the probability function, $g(\omega)$, takes the form of a delta-function around ω_0 [3].

The characteristics of the presence of a permanent dipole in the system include the way in which the internal polarization field is created, and how the externally applied electric field influences the states of the medium through the interaction.

In this paper we introduce a set of reduced Maxwell-Bloch equations that incorporate *both* the effects of a permanent dipole and inhomogeneous broadening. We remark that the derivation of the relevant equations, (1)-(2) does not involve the slowly varying envelope approximation. We demonstrate the integrability of this new, more general reduction of the MB system by providing a Lax pair. The Lax pair found for this new set of equations, which we call the inhomogeneously broadened reduced Maxwell-Bloch equations (ib-rMB), is not a straight forward generalization of the one found for the reduction of the MB equations studied in [1] by the authors. The method for constructing the new Lax pair involves the use of a pseudo-potential [10, 4, 5, 6]. In section 2 we present the ib-rMB equations in matrix and scalar form, and in section 3 we show how one can reduce these equations in the sharp line case, to the rMB model introduced in [1] using an appropriate change of basis. In section 4 the Lax pair is presented and in section 5 we construct a Bäcklund transformation that enables us to obtain the soliton solutions family in section 6. In section 7 we graphically investigate the effects of the permanent dipole and inhomogeneous broadening on the solutions.

2. Inhomogeneously broadened model

We begin by introducing a set of equations that recount the interaction between light and matter as described in the introduction. These equations include the effects of a permanent dipole and inhomogeneous broadening.

The ib-rMB matrix equations are given by:

$$\frac{\partial E}{\partial x} = -\frac{1}{2}\left\langle \frac{\partial}{\partial t}\mathrm{Tr}\left(P_\omega d\right)\right\rangle_g \tag{1}$$

$$\frac{\partial P_\omega}{\partial t} = -i\left[H_\omega - Ed, P_\omega\right]\ , \tag{2}$$

where

$$P_\omega = \begin{pmatrix} \frac{1}{2}U_\omega & R_\omega + iS_\omega \\ R_\omega - iS_\omega & -\frac{1}{2}U_\omega \end{pmatrix}\ ,\ d = \begin{pmatrix} \frac{1}{2}\Delta d & 0 \\ 0 & -\frac{1}{2}\Delta d \end{pmatrix}\ \ ,\ H_\omega = \begin{pmatrix} \frac{1}{2}\Delta h & \frac{1}{2}\omega \\ \frac{1}{2}\omega & -\frac{1}{2}\Delta h \end{pmatrix}\ . \tag{3}$$

The dynamical variables are the electric field, E, and the elements $(R_\omega, S_\omega, U_\omega)$ of the matrix P_ω (which represent, as we will see in section 3, linear combinations of the elements of the polarization matrix). The subscript ω for $(R_\omega, S_\omega, U_\omega)$ is meant to indicate the dependence of those quantities on the varying parameter ω, which portrays the different oscillation frequencies of the atoms of the medium. The function that gives the spread of the frequencies around a specific resonant frequency, ω_0, is $g(\omega)$ and in the sharp line case $g(\omega) = \delta(\omega - \omega_0)$. For a function F of ω, $< F(\omega) >_g$ is the average of the function F over all possible frequencies: $< F(\omega) >_g = \int_{-\infty}^{\infty} F(\omega)g(\omega)d\omega$. Time and space are represented by t and x. The permanent dipole effect is encoded in the parameters Δd and Δh, and we make this statement precise in the following section.

Using matrices (3) in equations (1)-(2) yields the scalar ib-rMB model which we shall study in this paper:

$$\frac{\partial E}{\partial x} = \frac{\Delta d}{2}\left\langle \omega S_\omega \right\rangle_g \tag{4}$$

$$\frac{\partial R_\omega}{\partial t} = (\Delta h - \Delta dE)\omega S_\omega \tag{5}$$
$$\frac{\partial S_\omega}{\partial t} = -(\Delta h - \Delta dE)R_\omega + \frac{1}{2}\omega U_\omega \tag{6}$$
$$\frac{\partial U_\omega}{\partial t} = -2\omega S_\omega\ . \tag{7}$$

3. Relation to rMB model

In this section we present the rMB model studied in [1] and explain how we can reduce the ib-rMB system to the rMB model in the sharp line case. This will provide essential information on how the permanent dipole in being represented in the new system and the precise effects it has on the solutions. The rMB matrix equations (that do not include the effect of inhomogeneous broadening) are:

$$\frac{\partial E}{\partial t} + \frac{c}{\sqrt{\epsilon_\infty}}\frac{\partial E}{\partial x} = -\frac{\mathcal{N}}{2\epsilon_0\epsilon_\infty}\frac{\partial \mathrm{Tr}(\mu\rho)}{\partial t} \tag{8}$$
$$\frac{\partial \rho}{\partial t} = -\frac{i}{h}[H_0 - \mu E, \rho]\ , \tag{9}$$

where E is the electric field, $\mu = \begin{pmatrix}\mu_{11} & \mu_{12}\\ \mu_{12}^* & \mu_{22}\end{pmatrix}$ is the constant dipole matrix , $\rho = \begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{12}^* & \rho_{22}\end{pmatrix}$ is the dynamical polarization matrix and $H_0 = \begin{pmatrix}h\omega_1 & 0\\ 0 & h\omega_2\end{pmatrix}$ is the constant hamiltonian matrix that has as eigenvalues the energy values at the two levels of the medium. We remark that $\mathrm{Tr}(\rho) = 1$. Using these matrices in equations (8)-(9) produces the scalar form of the rMB equations (after effective scaling):

$$\frac{\partial e}{\partial t} + \frac{\partial e}{\partial x} = \frac{i}{2}(\rho_{12} - \rho_{12}^*) \tag{10}$$
$$\frac{\partial \mathrm{Re}\rho_{12}}{\partial t} = [1 - 2\Omega\sin(\theta)e]\,\mathrm{Im}\rho_{12} \tag{11}$$
$$\frac{\partial \mathrm{Im}\rho_{12}}{\partial t} = -[1 - 2\Omega\sin(\theta)e]\,\mathrm{Re}\rho_{12} - \Omega\cos(\theta)e\Delta\rho \tag{12}$$
$$\frac{\partial \Delta\rho}{\partial t} = -2i\Omega\cos(\theta)e(\rho_{12} - \rho_{12}^*). \tag{13}$$

The dynamical variables are $e, \Delta\rho$, and $\rho_{12} := \mathrm{Re}\rho_{12} + i\mathrm{Im}\rho_{12}$. The real electric field is represented by e. $\Delta\rho = \rho_{11} - \rho_{22}$ is also real and represents the difference of the diagonal elements of the electronic density matrix ρ. The off-diagonal elements of ρ, ρ_{12} and ρ_{12}^* correspond to a polarization induced by a transition between the lower and the upper energy levels [9]. The scaled temporal and spatial variables are t and x. Ω and θ are physical parameters that encode the effects of a permanent dipole in the system. There is no permanent dipole when the diagonal entries of μ are both zero which is equivalent to $\theta = 0$ [1] .

Having presented both the ib-rMB and rMB equations, we now show why the two models are equivalent in the sharp line case. We begin by summarizing the procedure: (1) We place the ib-rMB equations in a traveling wave frame, (2) We impose the sharp line condition, $g(\omega) = \delta(\omega - \omega_0)$. Namely, all the atoms share the same transition frequency, ω_0, (3) We change the basis, by conjugating the basis matrices. Performing steps (1)-(3) to equations (4)-(7), transforms them in equations (10)-(13) up to multiplicative constants.

We first place the ib-rMB equations (4)-(7) in a traveling wave frame by a change of variables: $x \mapsto x, t \mapsto t + x$, and take $g(\omega) = \delta(\omega - \omega_0)$. For the rest of this section $\omega = \omega_0$. For simplification reasons we use the symbol ω in the following equations but it is to be kept in mind that we have chosen a specific ω and that is ω_0. This produces the following set of equations:

$$\frac{\partial E}{\partial t} + \frac{\partial E}{\partial x} = \frac{\Delta d}{2}\omega S \tag{14}$$

$$\frac{\partial R}{\partial t} = (\Delta h - \Delta d E)\omega S \tag{15}$$

$$\frac{\partial S}{\partial t} = -(\Delta h - \Delta d E)R + \frac{1}{2}\omega U \tag{16}$$

$$\frac{\partial U}{\partial t} = -2\omega S\ . \tag{17}$$

There exists a change of variables for (R, S, U) that transforms equations (14)-(17) to the rMB equations (10)-(13). We first notice that the matrix equations (8)-(9) remain the same if we substitute the matrices μ, ρ and H_0 by the traceless matrices $\tilde{\mu} = \mu - \frac{1}{2}\mathrm{Tr}(\mu)$, $\tilde{\rho} = \rho - \frac{1}{2}\mathrm{Tr}(\rho)$, and $\tilde{H}_0 = H_0 - \frac{1}{2}\mathrm{Tr}(H_0)$ respectively. The traceless matrices take the form

$$\tilde{\mu} = \begin{pmatrix} \frac{1}{2}\Delta\mu & \mu_{12} \\ \mu_{12}^* & -\frac{1}{2}\Delta\mu \end{pmatrix}\ ,\ \tilde{\rho} = \begin{pmatrix} \frac{1}{2}\Delta\rho & \rho_{12} \\ \rho_{12}^* & -\frac{1}{2}\Delta\rho \end{pmatrix}\ ,\ \tilde{H}_0 = \begin{pmatrix} \frac{1}{2}\Delta h & 0 \\ 0 & -\frac{1}{2}\Delta h \end{pmatrix},$$

where $\Delta\mu = \mu_{11} - \mu_{22} = 2\mu_{11}$, $\Delta\rho = \rho_{11} - \rho_{22}$ and $\Delta h = h(\omega_1 - \omega_2)$.

We then observe that conjugation does not essentially alter equations (8)-(9), but it will formulate them in a new basis. We conjugate $\tilde{\mu}, \tilde{\rho}$ and $\tilde{H}_0$ with the matrix that diagonalizes $\tilde{\mu}$.

The eigenvalues of the matrix $\tilde{\mu}$ (with $\mu_{12} = \mu e^{i\theta}$), are given by $\lambda_{1,2} = \pm\frac{1}{2}\sqrt{4\mu^2 + \Delta\mu^2}$ and the corresponding eigenvectors are $v_1 = (\mu_{12}, -(\lambda_2 + \mu_{11}))^t$ and $v_2 = (\lambda_2 + \mu_{11}, \mu_{12}^*)^t$. One can then check that

$$\tilde{\mu} = BdB^{-1},$$

where B is the matrix with columns the eigenvectors, and d is the diagonal matrix with the eigenvalues down the diagonal:

$$B = \begin{pmatrix} \lambda_2 + \mu_{11} & \mu_{12} \\ \mu_{12}^* & -(\lambda_2 + \mu_{11}) \end{pmatrix}\ , d = \begin{pmatrix} \lambda_2 & 0 \\ 0 & -\lambda_2 \end{pmatrix}.$$

Diagrammatically we have:

$$\begin{array}{lllll} \mu & \mapsto & \tilde{\mu} & \mapsto & B^{-1}\tilde{\mu}B := d \\ \rho & \mapsto & \tilde{\rho} & \mapsto & B^{-1}\tilde{\rho}B := P \\ H_0 & \mapsto & \tilde{H}_0 & \mapsto & B^{-1}\tilde{H}_0 B := H\ . \end{array}$$

We restrict the form of H to be a symmetric matrix (instead of hermitian) which is equivalent to assuming $\mu_{12}^* = \mu_{12} \Leftrightarrow \mu_{12} = \mu e^{i\theta}$, $\theta = 0$. The degree of freedom represented by θ was compromised to ensure the integrability of the ib-rMB model. Conjugation produces

$$d = \begin{pmatrix} \frac{1}{2}\Delta d & 0 \\ 0 & -\frac{1}{2}\Delta d \end{pmatrix} \ , \ P = \begin{pmatrix} \frac{1}{2}U & R+iS \\ -iS & -\frac{1}{2}U \end{pmatrix} \ , \ H = \begin{pmatrix} \frac{1}{2}\Delta h & \frac{1}{2}\omega \\ \frac{1}{2}\omega & -\frac{1}{2}\Delta h \end{pmatrix} \ ,$$

where the dynamical variables are:

$$U = \frac{-1}{(\mu^2+\alpha^2)} \left\{ (\mu^2-\alpha^2)\Delta\rho - 4\alpha\mu \mathrm{Re}\rho_{12} \right\}$$
$$R = \frac{1}{(\mu^2+\alpha^2)} \left\{ (\mu^2-\alpha^2)\mathrm{Re}\rho_{12} + \alpha\mu\Delta\rho \right\}$$
$$S = -\mathrm{Im}\rho_{12}$$

and the parameters are given by:

$$\Delta d = 2\lambda_2 = -\sqrt{4\mu^2+\Delta\mu^2} \qquad \Delta h = \tfrac{-\Delta\omega(\mu^2-\alpha^2)}{(\mu^2+\alpha^2)}$$
$$\Delta\omega = h(\omega_1-\omega_2) \qquad \omega = \tfrac{2\mu\alpha\Delta\omega}{(\mu^2+\alpha^2)}$$
$$\alpha = \lambda_2 + \tfrac{1}{2}\Delta\mu \ .$$

More compactly,

$$\begin{pmatrix} S \\ R \\ U \end{pmatrix} = \frac{1}{(\mu^2+\alpha^2)} \begin{pmatrix} -(\mu^2+\alpha^2) & 0 & 0 \\ 0 & (\mu^2-\alpha^2) & \alpha\mu \\ 0 & 4\alpha\mu & -(\mu^2-\alpha^2) \end{pmatrix} \begin{pmatrix} \mathrm{Im}\rho_{12} \\ \mathrm{Re}\rho_{12} \\ \Delta\rho \end{pmatrix} \ . \qquad (18)$$

With the change of basis given by (18) equations (14)-(17) produce the rMB equations (10)-(13) up to rescaling constants. Thus we can regard the ib-rMB equations (4)-(7) as a generalization of the rMB model (10)-(13), when the effects of inhomogeneous broadening are taken into account.

To identify the parameters involved in determining the permanent dipole we recall that there is no permanent dipole when the diagonal elements of the dipole matrix vanish. The traceless dipole matrix we used has the form $\tilde{\mu} = \begin{pmatrix} \frac{1}{2}\Delta\mu & \mu \\ \mu & -\frac{1}{2}\Delta\mu \end{pmatrix}$, thus no permanent dipole corresponds to $\Delta\mu = 0$. This implies that $\lambda_2 = -\mu$ and $\alpha = -\mu$. As far as the ib-rMB equations (4)-(7) are concerned, there is no permanent dipole when:

$$\Delta h = 0 \ , \ \Delta d = -2\mu \ , \ \omega = -h(\omega_1 - \omega_2).$$

4. Lax Pair

The inhomogeneously broadened reduced Maxwell- Bloch equations (4)-(7) are completely integrable. There exists a rational, one-parameter family of pairs of differential operators, that depend on the dynamical variables $E, R_\omega, S_\omega, U_\omega$ and commute in a Lie-bracket sense if and only if $E, R_\omega, S_\omega, U_\omega$ satisfy the ib-rMB equations.

The Lax pair representation is given by,

$$\psi_x = \frac{(\Delta d)^2}{8}\left\{\int_{-\infty}^{\infty}\frac{1}{(\omega^2-\lambda^2)}\left[2\lambda\omega g(\omega)(R_\omega\mathcal{H}+S_\omega\mathcal{F})-\omega^2 g(\omega)U_\omega\mathcal{E}\right]d\omega\right\}\psi$$
$$\psi_t = \left\{\frac{\lambda}{2}\mathcal{H}-\frac{1}{2}(\Delta h-\Delta dE)\mathcal{E}\right\}\psi\ ,$$

where $\mathcal{H},\mathcal{F},\mathcal{E}$ constitute a basis for the Lie algebra $\mathfrak{su}(2)$ and are given as,

$$\mathcal{H}=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix},\quad \mathcal{F}=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix},\quad \mathcal{E}=\begin{pmatrix} 0 & i\\ i & 0\end{pmatrix}. \tag{19}$$

We define two differential operators L and A, such that $[L,A]:=LA-AL=0$ is equivalent to the inhomogeneously broadened rMB equations (4)-(7),

$$\begin{aligned} A &= -\partial_t + Q^{pol}\\ L &= \partial_x + Q\ . \end{aligned} \tag{20}$$

where,

$$Q^{pol}=\lambda(h_0\mathcal{H}+f_0\mathcal{F})+e_0\mathcal{E}$$
$$Q=\int_{-\infty}^{\infty}\frac{1}{(\omega^2-\lambda^2)}[\lambda(h_1\mathcal{H}+f_1\mathcal{F})+e_1\mathcal{E}]d\omega\ ,$$

and

$$\begin{aligned} h_0 &= \tfrac{1}{2}, & h_1 &= -\tfrac{(\Delta d)^2}{4}\omega g(\omega)R_\omega\\ f_0 &= 0, & f_1 &= -\tfrac{(\Delta d)^2}{4}\omega g(\omega)S_\omega\\ e_0 &= -\tfrac{1}{2}(\Delta h-\Delta dE), & e_1 &= \tfrac{(\Delta d)^2}{8}\omega^2 g(\omega)U_\omega\ . \end{aligned} \tag{21}$$

We call Q^{pol} and Q loop elements due to their close relationship to loop algebras (we leave the investigation of this direction for a future paper) and h_j, f_j, e_j for $j=0,1$, the potentials. We remark that the potentials depend on the solutions $E, R_\omega, S_\omega, U_\omega$ of the ib-rMB equations. Therefore *a precise description of the loop element is equivalent to having a set of solutions for our system.*

5. Bäcklund Transformation

One of the simplest examples of a Bäcklund transformation (BT) is the one that transforms a constant solution of a system to another time and space-depedent solution, called a one-soliton. We illustrate this by finding a constant solution to equations (4)-(7), and using it in the potentials (21). This defines the corresponding loop element, which we call Q_0, and depends on the potentials h_j, f_j, e_j, $j=0,1$. We then find a simultaneous, fundamental solution Ψ_1 to the Lax pair system $L\Psi=0, A\Psi=0$.
We define

$$\vec{\Phi_1}=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}:=\Psi_1(\lambda=\nu_1)\vec{c_1}\ ,$$

where $\vec{c_1} = \begin{pmatrix} c_1 \\ ic_2 \end{pmatrix}$ is a constant vector with $c_1, c_2 \in \mathbb{R}$, and the matrix N_1 as,

$$N_1 = \begin{pmatrix} \phi_1 & -\overline{\phi_2} \\ \phi_2 & \overline{\phi_1} \end{pmatrix} .$$

The BT matrix function is then constructed as: $G(\nu_1, \vec{c_1}; \lambda) = N_1 \begin{pmatrix} \lambda - \nu_1 & 0 \\ 0 & \lambda - \overline{\nu_1} \end{pmatrix} N_1^{-1}$.
Applying G to Ψ_1 yields a new fundamental solution Ψ_2:

$$\Psi_2(\nu_1, \vec{c_1}; \lambda) = G(\nu_1, \vec{c_1}; \lambda)\Psi_1(\nu_1, \vec{c_1}; \lambda) .$$

Define $\vec{\Phi}_2$ as:

$$\vec{\Phi}_2 = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} := \Psi_2(\nu_1, \vec{c_2}; \lambda = \nu_2)\vec{c_2}$$

where $\vec{c_2} = \begin{pmatrix} k_1 \\ ik_2 \end{pmatrix}$ is a constant vector with $k_1, k_2 \in \mathbb{R}$.

Construct N_2 in the same way N_1 was constructed but using the vector function $\vec{\Phi}_2$.
We can iterate the BT procedure to find Ψ_n:

$$\Psi_n(\nu_j, \vec{c_j}, 1 \le j \le n-1; \lambda) = G(\nu_j, \vec{c_j}, 1 \le j \le n-1; \lambda)\Psi_{n-1}(\nu_j, \vec{c_j}, 1 \le j \le n-1; \lambda) .$$

Define $\vec{\Phi}_n$ as:

$$\vec{\Phi}_n(\nu_j, \vec{c_j}, 1 \le j \le n) = \Psi_n(\nu_j, \vec{c_j}, 1 \le j \le n-1; \lambda = \nu_n)\vec{c_n} .$$

Construct N_n as described above using the vector function $\vec{\Phi}_n$.

The method described produces new fundamental solutions $\Psi_n, n \in \mathbb{N}$ for the corresponding Lax pair equations: $L_n\Psi_n = 0,\ A_n\Psi_n = 0$, for $L_n = \partial_x + Q_n,\ A_n = -\partial_t + Q_n^{pol}$, where

$$Q_n = \lambda(h_0^n\mathcal{H} + f_0^n\mathcal{F}) + e_0^n\mathcal{E} + \int_{-\infty}^{\infty} \frac{1}{(\omega^2 - \lambda^2)}[\lambda(h_1^n\mathcal{H} + f_1^n\mathcal{F}) + e_1^n\mathcal{E}]d\omega . \quad (22)$$

has the general structure that includes both a polynomial and a singular part. The upper indices of the potentials $h_0^n, f_0^n, e_0^n, h_1^n, f_1^n, e_1^n$, and the lower indices of the loop element Q_n are meant to indicate the level of the Bäcklund transformation. We note that the potentials h_0 and f_0 are invariant under the BT because they are constant quantities and thus $h_0^j = h_0, f_0^j = f_0 = 0$, for $j \in \mathbb{N}$. We also note that in the following calculations if there is no upper index for the potentials, then it is meant to be 0. The exact expressions for the potentials $h_0^n, f_0^n, e_0^n, h_1^n, f_1^n, e_1^n$, $n \in \mathbb{N}$ can be deduced from a result we present below (Theorem 1.).

We derived a formula for the new loop element Q_n, i.e. for $\{h_0^n, f_0^n, e_0^n, h_1^n, f_1^n, e_1^n\}$, in terms of the previous one Q_{n-1}, i.e. $\{h_0^{n-1}, f_0^{n-1}, e_0^{n-1}, h_1^{n-1}, f_1^{n-1}, e_1^{n-1}\}$, and the matrix N_n by explicit calculations that eliminate Ψ_n and Ψ_{n-1} from the Lax pair equations. Thus, the procedure of the BT can be summarized as follows:

$$\begin{array}{llll}
\Psi_1 & \to N_1 & \to Q_1 & \to \text{one-solitons} \\
G\downarrow & & & \\
\Psi_2 & \to N_2 & \to Q_2 & \to \text{two-solitons} \\
\vdots & & & \\
\Psi_n & \to N_n & \to Q_n & \to n\text{-solitons} \\
\vdots & & &
\end{array}$$

The relevant formulas for Q_n are given in the following two results. Theorem 1., that appears in [1] and was formulated for the rMB equations (sharp line case), gives the loop element Q_n in terms of the loop element Q_{n-1} and the matrix N_n (a matrix with entries that depend on BT data). This result is extendable to the inhomogeneously broadened case because the A operator in the Lax pair for the ib-rMB system has the same form as the A operator in the Lax pair for the sharp line case derived in [1], and the corresponding L operators have the same singularity structure with two finite poles.

Theorem 1.

$$\begin{aligned}
& N_n^{-1}Q_n(\lambda)N_n \\
= \; & \left[N_n^{-1}Q_{n-1}(\nu_n)N_n e_1, N_n^{-1}Q_{n-1}(\bar{\nu_n})N_n e_2\right] + ((\lambda - Re(\nu_n)) - Im(\nu_n)\mathcal{H})\cdot \\
& \left[N_n^{-1}\left(\frac{Q_{n-1}(\lambda)-Q_{n-1}(\nu_n)}{\lambda-\nu_n}\right)N_n\vec{e}_1, N_n^{-1}\left(\frac{Q_{n-1}(\lambda)-Q_{n-1}(\bar{\nu_n})}{\lambda-\bar{\nu_n}}\right)N_n\vec{e}_2\right] ,
\end{aligned} \tag{23}$$

where $\{\vec{e}_1, \vec{e}_2\}$ are the standard basis vectors in $\mathbb{R}^2$ and $[\vec{v}, \vec{u}]$ symbolizes the matrix with first column $\vec{v}$ and second column $\vec{u}$.

The following result, Corollary 2., is a special case of Theorem 1. when one uses the loop element Q_{n-1} as given in (22) and takes the specific value of the spectral parameter to be purely imaginary, $\nu_n = im_n \in i\mathbb{R}$, to ensure the reality of the potentials and consequently the solutions $E, R_\omega, S_\omega, U_\omega$ (As we will see in section 6 this condition can be relaxed when we iterate the BT twice).

Corollary 2.

$$\begin{aligned}
& Q_n(\lambda) = \lambda h_0^{n-1}\mathcal{H} + m_n h_0^{n-1}[\mathcal{H}, N_n\mathcal{H}N_n^{-1}] + e_0^{n-1}\mathcal{E} \\
& + \int_0^\infty \frac{1}{(\omega^2-\lambda^2)}\frac{1}{(\omega^2+m_n^2)}\cdot \\
& \left\{\lambda\left[\omega^2(h_1^{n-1}\mathcal{H} + f_1^{n-1}\mathcal{F}) - (m_n)^2(N_n\mathcal{H}N_n^{-1})(h_1^{n-1}\mathcal{H} + f_1^{n-1}\mathcal{F})(N_n\mathcal{H}N_n^{-1})\right.\right. \\
& \left. + m_n e_1^{n-1}[\mathcal{E}, N_n\mathcal{H}N_n^{-1}]\right] + m_n\omega^2\left(h_1^{n-1}[\mathcal{H}, N_n\mathcal{H}N_n^{-1}] + f_1^{n-1}[\mathcal{F}, N_n\mathcal{H}N_n^{-1}]\right) \\
& \left. + \omega^2 e_1^{n-1}\mathcal{E} - (m_n)^2 e_1^{n-1}(N_n\mathcal{H}N_n^{-1})\mathcal{E}(N_n\mathcal{H}N_n^{-1})\right\} d\omega \,.
\end{aligned} \tag{24}$$

The expression above gives the loop element at the $n-th$ BT in terms of the potentials of the $(n-1)$ BT and the matrix N_n which can be constructed using data appearing at the $(n-1)$ BT. Therefore formula (24) iteratively produces the n-soliton potentials for $n \in \mathbb{N}$.

6. Soliton Solutions

To initialize the Bäcklund transformation we set the solutions to $E = \frac{\Delta h}{\Delta d}, S_\omega = 0, U_\omega = 0$ and $R_\omega = R_\omega^{init}$, a nonzero constant. The reader can easily verify that these constitute a set of solutions for the ib-rMB equations (4)-(7). The potentials then become:

$$\begin{array}{ll} h_0 = \frac{1}{2}, & h_1 = -\frac{(\Delta d)^2}{4}\omega g(\omega) R_\omega^{init} \\ f_0 = 0, & f_1 = 0 \\ e_0 = 0, & e_1 = 0\ . \end{array} \tag{25}$$

We specialize Corollary 2. for $n = 1$. We find Ψ_1, the fundamental solution of $L\Psi = 0, A\Psi = 0$, by multiplying the solutions of the systems: $\Psi_t = Q^{pol}\Psi$, $\Psi_x = -Q\Psi$, where the loops are evaluated at the potentials (25):

$$\begin{aligned} &\Psi_t = \lambda h_0 \mathcal{H}\Psi \\ &\Psi_x = -\int_{-\infty}^{\infty} \frac{\lambda}{\omega^2 - \lambda^2} h_1 \mathcal{H} d\omega\ . \end{aligned}$$

Thus Ψ_1 takes the form:

$$\Psi_1 = \begin{pmatrix} e^{-i \int_{-\infty}^{\infty} \frac{\lambda}{\omega^2-\lambda^2} h_1 d\omega x + i\lambda h_0 t} & 0 \\ 0 & e^{i \int_{-\infty}^{\infty} \frac{\lambda}{\omega^2-\lambda^2} h_1 d\omega x - i\lambda h_0 t} \end{pmatrix} .$$

We let $\alpha_1 = \int_{-\infty}^{\infty} \frac{m_1}{\omega^2+(m_1)^2} h_1 d\omega$ and then the fundamental solution evaluated at $\lambda = im_1$ becomes

$$\Psi_1 = \begin{pmatrix} e^{\alpha_1 x - m_1 h_0 t} & 0 \\ 0 & e^{-(\alpha_1 x - m_1 h_0 t)} \end{pmatrix} .$$

Thus the vector function $\vec{\Phi}_1 := (\phi_1,\ \phi_2)^t$ is composed of $\phi_1 = c_1 e^{\alpha_1 x - m_1 h_0 t}$, $\phi_2 = ic_2 e^{\alpha_1 x - m_1 h_0 t}$ and this gives,

$$N_1 = c_1 e^{\alpha_1 x - m_1 h_0 t}\mathcal{I} + c_2 e^{-(\alpha_1 x - m_1 h_0 t)}\mathcal{E}\ .$$

Using the potentials (25) in Corollary 2. reduces equation (24) to:

$$\begin{aligned} &Q_1(\lambda) = \lambda h_0 \mathcal{H} + m_1 h_0 [\mathcal{H}, N_1 \mathcal{H} N_1^{-1}] \\ &+ \int_{-\infty}^{\infty} \frac{1}{(\omega^2-\lambda^2)} \frac{1}{(\omega^2+(m_1)^2)} \cdot \\ &\left\{\lambda \left[\omega^2 h_1 \mathcal{H} - (m_1)^2 h_1 (N_1 \mathcal{H} N_1^{-1})\mathcal{H}(N_1 \mathcal{H} N_1^{-1})\right] + m_1 \omega^2 h_1 [\mathcal{H}, N_1 \mathcal{H} N_1^{-1}]\right\} d\omega\ . \end{aligned} \tag{26}$$

By definition the new loop element Q_1 is:

$$Q_1(\lambda) = \lambda h_0^1 \mathcal{H} + e_0^1 \mathcal{E} + \int_{-\infty}^{\infty} \frac{1}{(\omega^2-\lambda^2)} \left[\lambda(h_1^1 \mathcal{H} + f_1^1 \mathcal{F}) + e_1^1 \mathcal{E}\right] d\omega\ . \tag{27}$$

We calculate the following expressions that appear in equation (26):

$$[\mathcal{H}, N_1\mathcal{H}N_1^{-1}] = 2\mathrm{sech}(x_1)\mathcal{E} \tag{28}$$
$$(N_1\mathcal{H}N_1^{-1})\mathcal{H}(N_1\mathcal{H}N_1^{-1}) = (2\mathrm{sech}^2(x_1)-1)\mathcal{H} - 2\mathrm{sech}(x_1)\tanh(x_1)\mathcal{F}\ , \tag{29}$$

where $x_1 = 2(\alpha_1 x - m_1 h_0 t) + \ln(c_1/c_2)$. Using expressions (28)-(29) into (26) and equating the coefficients of the different powers of λ for (26) and (27) we obtain the following:

$$\begin{aligned}
\lambda^0 :\ e_0^{1-sol} &= 2m_1 h_0 \mathrm{sech}(x_1)\\
\frac{1}{\omega^2-\lambda^2} :\ e_1^{1-sol} &= \left(\frac{2m_1\omega^2 h_1}{\omega^2+m_1^2}\right)\mathrm{sech}(x_1)\\
\frac{\lambda}{\omega^2-\lambda^2} :\ h_1^{1-sol} &= \left(\frac{1}{\omega^2+m_1^2}\right)\left[\omega^2 h_1 - m_1^2 h_1(2\mathrm{sech}^2(x_1)-1)\right]\\
f_1^{1-sol} &= \left(\frac{2m_1^2 h_1}{\omega^2+m_1^2}\right)\mathrm{sech}(x_1)\tanh(x_1)
\end{aligned}$$

If we use the potentials given in (21), we obtain the one-soliton solutions of the ib-rMB equations (4)-(7):

$$\begin{aligned}
E^{1-sol}(t,x) &= \frac{2m_1}{\Delta d}\mathrm{sech}(x_1) + \frac{\Delta h}{\Delta d}\\
S_\omega^{1-sol}(t,x) &= \frac{2m_1^2}{\omega^2+m_1^2}R_\omega^{init}\mathrm{sech}(x_1)\tanh(x_1)\\
R_\omega^{1-sol}(t,x) &= \frac{1}{\omega^2+m_1^2}R_\omega^{init}\left[\omega^2+m_1^2-2m_1^2\mathrm{sech}^2(x_1)\right]\\
U_\omega^{1-sol}(t,x) &= \frac{-4m_1\omega}{\omega^2+m_1^2}R_\omega^{init}\mathrm{sech}(x_1)\ ,
\end{aligned}$$

where,
$x_1 = 2(\alpha_1 x - m_1 h_0 t) + \ln(c_1/c_2)$, $\alpha_1 = \int_{-\infty}^{\infty}\frac{m_1}{\omega^2+(m_1)^2}h_1 d\omega$, $h_1 = -\frac{(\Delta d)^2}{4}\omega g(\omega)R_\omega^{init}$.

Next we apply Corollary 2. for $n=2$. We repeat the procedure to find the two-soliton potentials in terms of the one-soliton potentials.

$$e_0^2 = e_0^1 - 4m_2\frac{i\phi_1\phi_2}{(\phi_1^2-\phi_2^2)}h_0^1 \tag{30}$$
$$e_1^2 = e_1^1 - \frac{2m_2}{(\omega^2+m_2^2)}\left\{m_2 e_1^1 + 2\omega^2\frac{i\phi_1\phi_2}{(\phi_1^2-\phi_2^2)}h_1^1 + \omega^2\frac{(\phi_1^2+\phi_2^2)}{(\phi_1^2-\phi_2^2)}f_1^1\right\} \tag{31}$$
$$h_1^2 = h_1^1 + \frac{4m_2}{(\omega^2+m_2^2)}\frac{i\phi_1\phi_2}{(\phi_1^2-\phi_2^2)}\left\{e_1^1 - 2m_2\frac{i\phi_1\phi_2}{(\phi_1^2-\phi_2^2)}h_1^1 - m_2\frac{(\phi_1^2+\phi_2^2)}{(\phi_1^2-\phi_2^2)}f_1^1\right\} \tag{32}$$
$$f_1^2 = f_1^1 + \frac{2m_2}{(\omega^2+m_2^2)}\frac{(\phi_1^2+\phi_2^2)}{(\phi_1^2-\phi_2^2)}\left\{e_1^1 - 2m_2\frac{i\phi_1\phi_2}{(\phi_1^2-\phi_2^2)}h_1^1 - m_2\frac{(\phi_1^2+\phi_2^2)}{(\phi_1^2-\phi_2^2)}f_1^1\right\}, \tag{33}$$

where ϕ_1 and ϕ_2 are

$$\phi_1 = i\left[k_1(m_2 - m_1\tanh(x_1))e^{y_2} - m_1k_2\mathrm{sech}(x_1)e^{-y_2}\right] \tag{34}$$

$$\phi_2 = m_1k_1\mathrm{sech}(x_1)e^{y_2} - k_2(m_2 + m_1\tanh(x_1))e^{-y_2}\ , \tag{35}$$

$x_1 = 2(\alpha_1 x - m_1 h_0 t) + \ln(c_1/c_2)$ and $y_2 = \alpha_2 x - m_2 h_0 t$.

The two-soliton potential e_0^2, for instance, can be expressed in terms of elementary functions if we substitute (34) and (35) for ϕ_1 and ϕ_2 in (30):

$$e_0^2 = 2h_0\left(\frac{m_1^2 - m_2^2}{m_1^2 + m_2^2}\right)\frac{m_1\mathrm{sech}(x_1) - m_2\mathrm{sech}(x_2)}{1 - \frac{2m_1m_2}{m_1^2+m_2^2}\left(\tanh(x_1)\tanh(x_2) - \mathrm{sech}(x_1)\mathrm{sech}(x_2)\right)}\ , \tag{36}$$

where $x_1 = 2(\alpha_1 x - 2m_1h_0t) + \ln(c_1/c_2)$, $x_2 = 2(\alpha_2 x - 2m_2h_0t) + \ln(k_1/k_2)$, $\alpha_j = \int_{-\infty}^{\infty}\frac{m_j}{(\omega^2+m_j^2)}h_1 d\omega$, for $j = 1, 2$, and $h_1 = -\frac{(\Delta d)^2}{4}\omega g(\omega)R_\omega^{init}$. The two-soliton electric field, E^{2-sol}, can then be obtained via $e_0^{2-sol} = -\frac{1}{2}(\Delta h - \Delta d E^{2-sol})$.

7. Graphs

In this section we observe the qualitative behavior of the four fields, E, S, R, U for varying values of the dipole strength $\Delta\mu$ and the inhomogeneous broadening parameter σ. We use a gaussian distribution for the probability function: $g(\omega) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(\omega-\omega_0)^2}{2\sigma^2}}$. In the following graphs we set: $\ln(c_1/c_2) = \ln(k_1/k_2) = 0,\ \omega_0 = 1,\ \mu = 1,\ \Delta\omega = 1,\ R^{init} = -1$.

In figure 1 we display snapshots ($t = 0$) of the four fields when the two spectral parameters take the values $\nu_1 = -2i,\ \nu_2 = -i$, the dipole parameter $\Delta\mu = 0$ and we vary the value that controls inhomogeneous broadening: $\sigma = 0.1$ (dash), 0.5 (dash-dot), 1 (solid).

In figure 2 we display snapshots ($t = 0$) of the four fields when the values of the two spectral parameters are $\nu_1 = -2i,\ \nu_2 = -i,\ \sigma = 0.5$ and the permanent dipole varies: $\Delta\mu = 0$ (dash), 2 (dash-dot), 4 (solid).

From figures 1-2 we observe that the effects of the permanent dipole and inhomogeneous broadening work against each other. Increase of the permanent dipole strength tends to produce more localized pulses, whereas increase of the inhomogeneous broadening parameter spreads the pulses out.

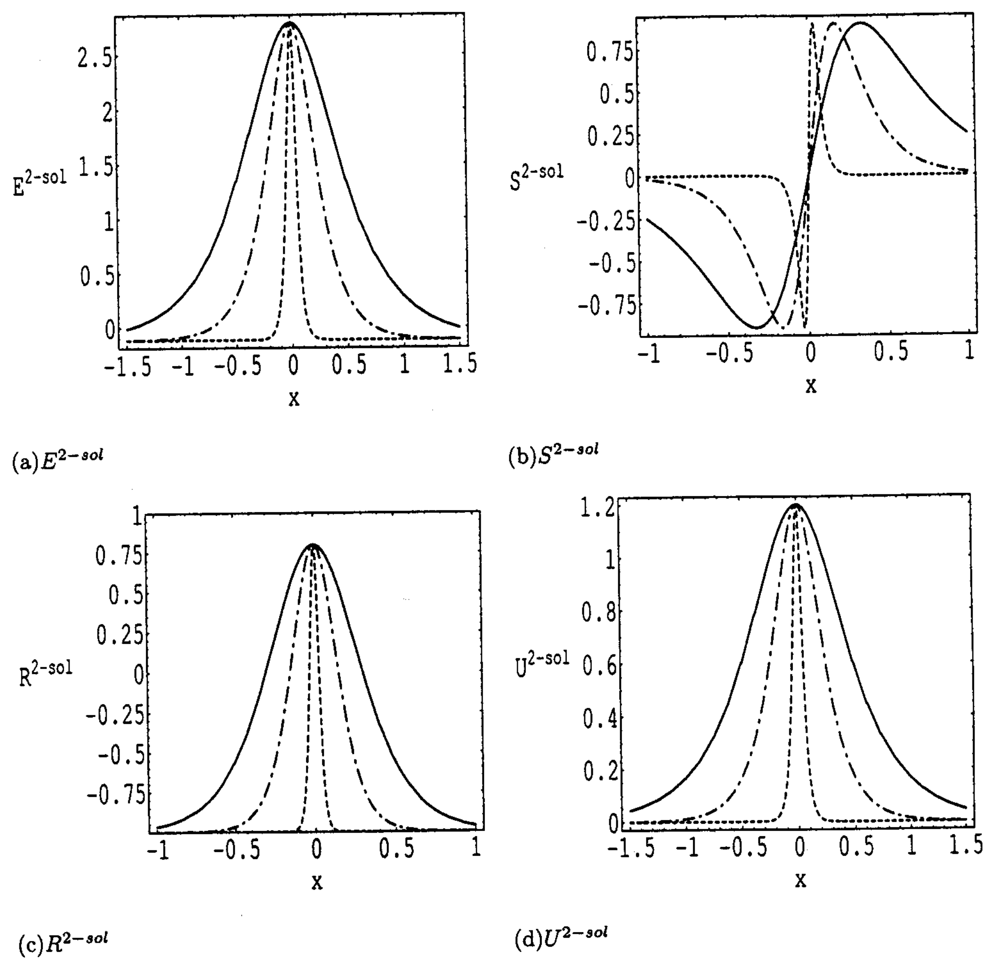

Figure 1: Simultaneous snapshots of the two-soliton solutions for varying values of the broadening parameter.

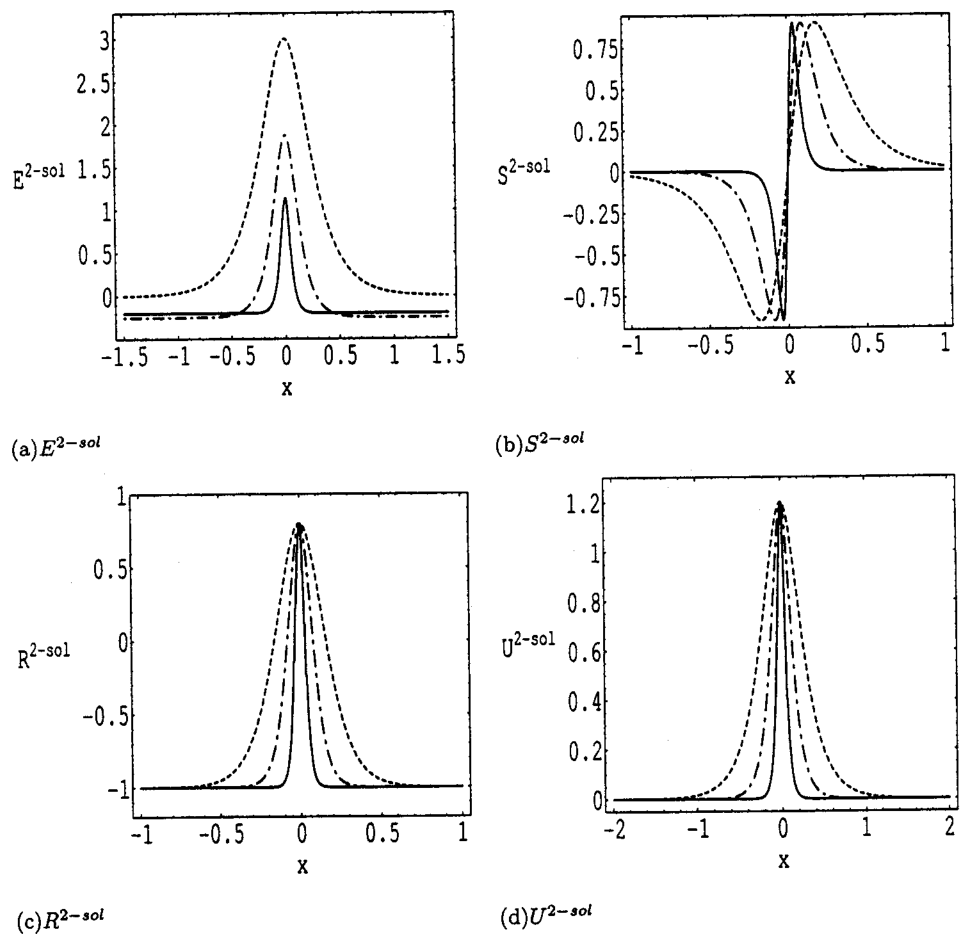

Figure 2: Snapshots of the two-soliton solutions for varying dipole strengths.

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