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Geometric quantization and the integrability of Lie algebroids

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Abstract

Given a Poisson manifold P, if there exists a symplectic manifold Σ and a surjective submersion $\Sigma \to P$ then it is possible to quantize Σ and then "push" the results to P. This method of quantizing a Poisson manifold is known as symplectic realisation. In this paper we illustrate how this method is related with the integrability of Lie algebroids.

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0. Introduction

Roughly speaking, quantization is the process by which one passes from classical mechanics to quantum mechanics in physics. The main obstruction to this is the Heisenberg uncertainty principle, namely:

$$\vec{x} \cdot \vec{p} - \vec{p} \cdot \vec{x} = -\iota\hbar$$

where $\vec{x}$ and $\vec{p}$ are the position and the momentum of a moving particle respectively and $\hbar$ is Planck's constant. Due to this, the algebra of differentiable functions $C^\infty(M)$ of the phase space M for a physical system is no longer sufficient to describe the system in quantum mechanics. That is because the usual product is commutative, and the Heisenberg principle no longer allows commutativity in the quantum level. Therefore quantization in general can be thought of as the effort to introduce a non-abelian product in $C^\infty(M)$. There are indeed many ways to quantize a manifold, whenever this is possible of course. For example, if there is a way to map every differentiable function f to a differentiable operator $\hat{f}$, then one can define a $*$-product in $C^\infty(M)$ by

$$f * g = fg - \iota\hbar \cdot B_1(f,g) + (-\iota\hbar)^2 \cdot B_2(f,g) + \dots$$

where the B_i are certain bidifferential operators arising from the process of mapping f to $\hat{f}$ that we mentioned above. From this point of view, it is clear that classical mechanics is the limit of quantum mechanics when Planck's constant tends to zero. Moreover, the first term B_1 gives rise to a bracket in the algebra $C^\infty(M)$, namely:

$$\{f,g\} = \frac{1}{2}(B_1(f,g) - B_1(g,f)).$$

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This bracket is -bilinear, alternating, satisfies the Jacobi identity, as well as the following Leibniz identity:

$$\{f, gh\} = h\{f, g\} + g\{f, h\}.$$

Such brackets are called *Poisson* brackets, and a manifold with a Poisson bracket on the algebra of differentiable functions is called a Poisson manifold. This example illustrates the fact that Poisson brackets arise naturally in the process of quantization.

This paper focuses on geometric quantization, the origins of which lie in the work of Kostant [8], Kirillov [7], and Souriau. Kostant and Kirillov in particular, worked (independently) with symplectic manifolds. They produced a method which (whenever possible) maps every differentiable function to a skew-adjoint operator in a Hilbert space. Mainly, there are two reasons for working with symplectic manifolds: On one hand they have very rich dynamics and thus are extremely useful in physics, and on the other symplectic manifolds are very good examples of Poisson manifolds (see Section 1 below).

A geometric way to quantize a Poisson manifold, relating it to the quantization of symplectic manifolds, is symplectic realisation: Given a Poisson manifold P, if there exists a symplectic manifold Σ and a submersion $\Sigma \longrightarrow\!\!\!\!\gg P$ which preserves the Poisson structures, then it is possible to quantize Σ and "push" the results to P. The manifold Σ together with the submersion on P are called a *symplectic realisation* of P. This method was introduced by [14].

This paper aims to give an overview of how the existence of a symplectic realisation for a manifold P is related to the problem of the integrability of Lie algebroids. Section 1 gives the basic definitions of symplectic manifolds, as well as the example of the Poisson-Lie bracket on the dual of a Lie algebra. We restrict to the material needed in later sections. More on symplectic and Poisson geometry can be found in [1], [15], [17], [4]. In Section 2 we reformulate this example in order to introduce symplectic groupoids and their implication with the problem of symplectic realisation of a Poisson manifold. Finally, in Section 3 we discuss how the existence of symplectic groupoids over a Poisson manifold is equivalent to the problem of the integrability of Lie algebroids. We also present the latest results on this problem.

1. Symplectic and Poisson manifolds

Let us start with the definition of a symplectic manifold.

Definition 1.1 A *symplectic manifold* is a manifold M together with a real-valued differential 2-form $\omega : TM \times TM \to M \times$ which is:

(i) closed, i.e. $d\omega = 0$

(ii) non-degenerate, i.e. the map $\omega^\flat : T^*M \to TM$ is a vector bundle isomorphism

Example 1.2 *For every manifold M, its cotangent bundle T^*M has a natural symplectic structure. For every $\theta \in T^*M$ define the* Liouville *form $\alpha_\theta : T_\theta T^*M \to M \times$ in the following way:*

$$\alpha_\theta(u) = \theta((pr)_*(u))$$

for all $u \in T_\theta T^*M$. Here $pr : T^*M \to M$ is the canonical vector bundle projection. Now $\omega = d\alpha$ is a symplectic form.

Symplectic manifolds are particularly useful in classical mechanics because they have very rich dynamics. Every $f \in C^\infty(M)$ defines a vector field $X_f \in \Gamma TM$ by the following formula:

$$\omega(X_f, Y) = Y(f)$$

for all $Y \in \Gamma TM$. This is the *Hamiltonian* vector field corresponding to f.

There is, however, another feature that makes symplectic manifolds interesting. The symplectic form ω defines a Poisson bracket on the algebra $C^\infty(M)$ by

$$\{f, g\} = \omega(X_f, X_g) = X_g(f) = -X_f(g).$$

A Poisson manifold P also has Hamiltonian vector fields, defined by

$$X_f(g) = \{g, f\}$$

for all $f,\ g \in C^\infty(P)$.

The question that arises is whether every Poisson manifold is symplectic. The answer in general is no (see [4]). In fact, every Poisson manifold has a symplectic foliation, i.e. a foliation such that every leaf is a symplectic submanifold. The following example illustrates exactly this argument.

Example 1.3 *Consider a Lie algebra $\mathfrak{g}$ with finite dimension. The derivative of a function $f \in C^\infty(\mathfrak{g}^*)$ is a 1-form $df\colon\ \Gamma T\mathfrak{g}^* \to C^\infty(\mathfrak{g}^*)$. For every $\theta \in \mathfrak{g}^*$ it is possible to identify $d_\theta f$ with an element of $\mathfrak{g}^{**}$. Now since the dimension of $\mathfrak{g}$ is finite, we have $\mathfrak{g}^{**} = \mathfrak{g}$. Therefore, it makes sense to define the following bracket on $C^\infty(\mathfrak{g}^*)$:*

$$\{f, g\}(\theta) = \theta([d_\theta(f), d_\theta g]).$$

From the properties of the Lie bracket in $\mathfrak{g}$ it follows that this is a Poisson bracket. This is the Poisson-Lie bracket *on $\mathfrak{g}^*$. The symplectic leaf of some $\theta \in \mathfrak{g}^*$ is the coadjoint orbit* $\mathrm{Ad}^*(\theta)$ *([7]).*

2. Poisson manifolds and Symplectic groupoids

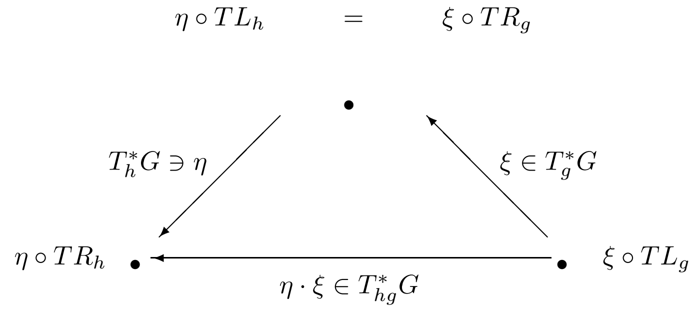
There is, however, another way to realise the symplectic leaves in the previous example. Consider the connected and simply connected Lie group G that $\mathfrak{g}$ integrates to and focus on its cotangent bundle T^*G. Every element $\xi \in T^*_g G$ can be thought of as an arrow, as in the following diagram:

$$\mathfrak{g}^* \ni \xi \circ TR_g \quad \bullet \xleftarrow{\ \ \xi\ \ } \bullet \quad \xi \circ TL_g \in \mathfrak{g}^*$$

The elements $\xi \circ TL_g$ and $\xi \circ TR_g$ of $\mathfrak{g}^*$ are called the *source* and *target* of ξ and denoted $\alpha(\xi)$ and $\beta(\xi)$ respectively. An appropriate pair of such arrows, i.e. a pair which belongs to

$$T^*G * T^*G = \{(\eta, \xi) \in T^*G \times T^*G\colon\ \alpha(\eta) = \beta(\xi)\}$$

can be multiplied in the following way:



where $\eta \cdot \xi$ is defined as

$$\eta \cdot \xi = \eta \circ TR_{g^{-1}} = \xi \circ TL_{h^{-1}}.$$

Moreover, every arrow ξ has an inverse ξ^{-1}, such that $\alpha(\xi^{-1}) = \beta\xi$ and $\beta\xi^{-1} = \alpha(\xi)$. In terms of category theory, T^*G is a groupoid over $\mathfrak{g}^*$. The fact that $T^*G, \mathfrak{g}$ are differentiable manifolds, $\alpha, \beta : T^*G \to \mathfrak{g}^*$ and the multiplication $\cdot : T^*G * T^*G \to T^*G$ are differentiable maps make $T^*G \rightrightarrows \mathfrak{g}^*$ a *Lie groupoid*. The full definition of a Lie groupoid is:

Definition 2.1 A *Lie groupoid* consists of two manifolds Ω and M, called respectively the *groupoid* and the *base*, together with two surjective submersions $\alpha, \beta\colon\ \Omega \to M$, called respectively the *source* and *target* projections, a smooth map $1 : M \to \Omega$ and a smooth multiplication $(\eta, \xi) \mapsto \eta\xi$ in Ω defined on the set $\Omega * \Omega = \{(\eta, \xi) \in \Omega \times \Omega \mid \alpha(\eta) = \beta(\xi)\}$, all subject to the following conditions:

(i) $\alpha(\eta\xi) = \alpha(\xi)$ and $\beta(\eta\xi) = \beta(\eta)$ for all $(\eta, \xi) \in \Omega * \Omega$;

(ii) $\zeta(\eta\xi) = (\zeta\eta)\xi$ for all $\zeta, \eta, \xi \in \Omega$ such that $\alpha(\zeta) = \beta(\eta)$ and $\alpha(\eta) = \beta(\xi)$;

(iii) $\alpha(1_x) = \beta(1_x) = x$ for all $x \in M$;

(iv) $\xi 1_{\alpha(\xi)} = \xi$ and $1_{\beta(\xi)}\xi = \xi$ for all $\xi \in \Omega$;

(v) each $\xi \in \Omega$ has a (two-sided) inverse ξ^{-1} such that $\alpha(\xi^{-1}) = \beta(\xi)$, $\beta(\xi^{-1}) = \alpha(\xi)$ and $\xi^{-1}\xi = 1_{\alpha(\xi)}, \xi\xi^{-1} = 1_{\beta(\xi)}$. The inversion map $\xi \mapsto \xi^{-1}$ is smooth.

Now we have

$$\mathrm{Ad}^*(\theta) = \{\theta \circ TR_{g^{-1}} \circ TL_g \mid g \in G\} = \beta(\alpha^{-1}(\theta))$$

Note that this particular Lie groupoid, T^*G, is also a symplectic manifold as was shown in 1.2. Relevant to this, we have the following definition:

Definition 2.2 A *symplectic groupoid* over a Poisson manifold P is a Lie groupoid $\Sigma \rightrightarrows P$ with a symplectic structure on Σ such that the graph of the multiplication

$$\mathcal{C} = \{(\eta\xi, \eta, \xi) \mid \alpha(\eta) = \beta(\xi)\} \subseteq \overline{\Sigma} \times \Sigma \times \Sigma$$

is a Lagrangian submanifold.

A few clarifications are necessary: First, the notation $\overline{\Sigma}$ stands for the manifold Σ equipped with the opposite symplectic structure. That is to say, if ω is the symplectic structure of Σ, then the symplectic form of $\overline{\Sigma}$ is $-\omega$. Second, a Lagrangian submanifold $\mathcal{C}$ of a symplectic manifold (M, ω) is a submanifold such that $T\mathcal{C}^{\perp} \subseteq T\mathcal{C}$. Here,

$$T\mathcal{C}^{\perp} = \{X \in TM \mid \omega(X, Y) = 0 \ for \ all \ Y \in T\mathcal{C}\}.$$

The above definition appears in [6]. The fact that the graph of the multiplication is a Lagrangian submanifold practically means that the multiplication preserves the Poisson structures (see [4]).

3. Lie algebroids and symplectic realisation

Given a Poisson manifold P, the existence of a symplectic groupoid $\Sigma \rightrightarrows P$ is important mainly for two reasons: First, as we pointed out in the previous section, the symplectic leaves of P can then be realised as the orbits of Σ. Second, the existence of a symplectic groupoid are a major step towards the quantization of P. It is an immediate consequence of definition 2.2 that the target map β is a Poisson map and the source map α is anti-Poisson. If one of them happens to be a surjective submersion, then we have a symplectic realisation $\Sigma \to P$ of P.

The question that arises now is whether symplectic groupoids exist over a given Poisson manifold P. To answer this question, one needs to reformulate the notion of a Poisson structure on P. The Leibniz identity of the Poisson bracket actually means that it depends only on the first derivatives of the functions. In other words, there exists a bivector field $\Pi : T^*P \wedge T^*P \to$ such that

$$\{f, g\} = \Pi(df, dg).$$

This gives rise to a map

$$\sharp : T^*P \to (T^*P)^* = TP.$$

Locally, this map is given by

$$\sharp(f \cdot dg) = f \cdot X_g.$$

Moreover, we have an alternating, -bilinear bracket $[\ ,\ ]$: $\Gamma T^*P \times \Gamma T^*P \to \Gamma T^*P$ defined by

$$[\alpha, \beta] = -\mathcal{L}_{\sharp\alpha}\beta + -\mathcal{L}_{\sharp\beta}\alpha - d\Pi(\alpha, \beta)$$

In local coordinates this is

$$[udv, u'dv'] = uu'd\{vv'\} + uX_v(u')dv' - u'X_{v'}(u)dv$$

for all $u,\ u'\ , v\ , v'\ \in C^\infty(U)$, where U is an open subset of P. This bracket, together with $\sharp$ satisfy:

(i) $\sharp[\alpha, \beta] = [\sharp\alpha, \sharp\beta]$

(ii) $[f\alpha, \beta] = f[\alpha, \beta] + \sharp\alpha(f)\beta$

(iii) $[[\alpha, \beta], \gamma] + [[\gamma, \alpha], \beta] + [[\beta, \gamma], \alpha] = 0$

for all $\alpha, \beta, \gamma \in \Gamma T^*P$ and $f \in C^\infty(P)$. A vector bundle $A \to M$ with a vector bundle morphism $\sharp : A \to TM$ and a Lie bracket on its sections which satisfy (i)-(iii) is a *Lie algebroid*. The map $\sharp$ is called the *anchor*. Lie algebroids are the infinitesimal objects that Lie groupoids differentiate to (see [10]). Although these objects seem analogous to Lie algebras and Lie groups, they have a fundamental difference: Given a Lie algebroid A, there doesn't always exist a Lie groupoid differentiating to A. The first example of a non-integrable Lie algebroid can be found in [2].

Now, as far as the particular Lie algebroid $T^*P \to P$ which arises from the Poisson structure on P is concerned, Mackenzie and Xu [13] gave the following result:

Theorem 3.1 Given a connected Poisson manifold P, if there exists an α-connected Lie groupoid $\Sigma \rightrightarrows P$ which differentiates to T^*P, then there exists a canonical symplectic structure on Σ such that Σ is a symplectic groupoid over P.

This shows that the integrability of Lie algebroids is inextricably linked with the symplectic realisation of Poisson manifolds. Mackenzie [10] gave the (cohomological) obstruction to the integrability of transitive Lie algebroids. These are Lie algebroids whose anchor map is surjective. If, however, the anchor map $\sharp : T^*P \to TP$ is surjective, then it has maximal rank everywhere, and the symplectic foliation of P has one leaf. This reduces the Poisson manifold P to a symplectic one.

The latest result on the general non-transitive case (even when the anchor map has non-constant rank) was given lately by Crainic and Fernandes [5]. They gave two obstructions to the integrability of general Lie algebroids which are computable in many examples. In general, given a Lie algebroid $A \xrightarrow{\pi} M$ with anchor map $\sharp : A \to TM$, there always exists a path space Ω consisting of these paths $\alpha : [0,1] \to A$ such that $\sharp\alpha(t) = \frac{d}{dt}(\pi \circ \alpha)$ which integrates A. This is a groupoid over M in the categorical sense (with source and target the beginning and ending of paths respectively and concatenation of paths as multiplication), however it is not always a manifold. The obstructions given by Crainic and Fernandes are exactly the obstructions to this particular path space being a manifold. A thorough account of the nature of the integrability obstructions in the case of the cotangent bundle of a Poisson manifold can be found in [9].

References

[1] Ralph Abraham and Jerrold E. Marsden. *Foundations of mechanics*. Benjamin/Cummings Publishing Co. Inc. Advanced Book Program, Reading, Mass., 1978, Second edition, revised and enlarged, With the assistance of Tudor Raţiu and Richard Cushman.

[2] Rui Almeida and Pierre Molino. Suites d'Atiyah et feuilletages transversalement complets. *C. R. Acad. Sci. Paris Sér. I Math.*, 300(1):13–15, 1985.

[3] Iakovos Androulidakis. Connections on Lie algebroids and the Weil-Kostant theorem. In *Proceedings of the 4th Panhellenic Conference on Geometry (Patras, 1999)*, volume 44, pages 51–57, 2000.

[4] Ana Cannas da Silva and Alan Weinstein. *Geometric models for noncommutative algebras*, volume 10 of *Berkeley Mathematics Lecture Notes*. American Mathematical Society, Providence, RI, 1999.

[5] Marius Crainic and Rui Loja Fernandes. Integrability of Lie brackets. *Ann. of Math. (2)*, 157(2):575–620, 2003.

[6] M.V. Karasëv and V.P. Maslov. *Nonlinear Poisson brackets*, volume 119 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1993. Geometry and quantization, Translated from the Russian by A. Sossinsky [A.B. Sosinskiĭ] and M. Shishkova.

[7] A.A. Kirillov. *Elements of the theory of representations.* Springer-Verlag, Berlin, 1976. Translated from the Russian by Edwin Hewitt, Grundlehren der Mathematischen Wissenschaften, Band 220.

[8] Bertram Konstant. Quantization and unitary representations. I. Prequantization. In *Lectures in modern analysis and applications, III*, pages 87–208. Lecture Notes in Math., Vol. 170. Springer, Berlin, 1970.

[9] Crainic M. and Fernandes R. Integrability of Poisson manifolds. *math-DG/0210152*, October 2002.

[10] K. Mackenzie. *Lie groupoids and Lie algebroids in differential geometry*, volume 124 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1987.

[11] Kirill Mackenzie. Integrability obstructions for extensions of Lie algebroids. *Cahiers Topologie Géom. Différentielle Catég.*, 28(1):29–52, 1987.

[12] Kirill Mackenzie. A unified approach to Poisson reduction. *Lett. Math. Phys.*, 53(3):215–232, 2000.

[13] Kirill C.H. Mackenzie and Ping Xu. Integration of Lie bialgebroids. *Topology*, 39(3):445–467, 2000.

[14] Jerrold Marsden and Alan Weinstein. Reduction of symplectic manifolds with symmetry. *Rep. Mathematical Phys.*, (1):121–130, 1974.

[15] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998.

[16] Norman Steenrod. *The topology of fibre bundles.* Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1999. Reprint of the 1957 edition, Princeton Paperbacks.

[17] Alan Weinstein. Symplectic groupoids and Poisson manifolds. *Bull. Amer. Math. Soc. (N.S.)*, 16(1):101–104, 1987.

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