BULLETIN OF THE
GREEK MATHEMATICAL SOCIETY
Volume 51, 2005 (23–30)

Isolated umbilical points on surfaces in $\mathbb{R}^3$

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Abstract

Recent advances in the application of line congruence techniques to surfaces in $\mathbb{R}^3$ are used to generate surfaces with isolated umbilical points of all indices less than or equal to 1.

Keywords: line congruence, umbilical point

1. Introduction

Let S be a C^2 immersed surface in $\mathbb{R}^3$. The eigen-directions of the second fundamental form determine a pair of foliations on the surface, the principal foliations. These foliations have singularities at points on S where the eigenvalues of the second fundamental form are equal, that is, at umbilical points.

Given an isolated umbilical point on a surface, the index of the umbilic is defined to be the index of the principal foliation about the point. In general, since the foliations need not be oriented, the index of an isolated umbilic is an element of $\frac{1}{2}\mathbb{Z}$. Well-known examples of isolated umbilics arise on the triaxial and rotationally symmetric ellipsoid (of index $\frac{1}{2}$ and 1, respectively) .

The purpose of this note is to show how recent work [2] [3] on line congruences can be readily used to construct strictly convex surfaces with isolated umbilical points of any index less than or equal to 1. In particular we introduce the notion of a *regular* foliation about an isolated umbilic and completely classify such umbilics.

We find that, for each index less than 1 there is a finite-dimensional family of such surfaces, for index equal to 1 there is an infinite dimensional family, while for index greater than 1 there are none. This is in agreement with the Caratheodory conjecture [1], which states that the index of an umbilic cannot exceed 1. In addition, each of the families surfaces of index less than 1 are found to be strictly convex.

2. Line Congruence Approach to Surfaces

Definition 1. Let $\mathcal{L}$ be the set of oriented (affine) lines in euclidean $\mathbb{R}^3$.

Definition 2. Let $\Phi : TS^2 \to \mathcal{L}$ be the map that identifies $\mathcal{L}$ with the tangent bundle to the unit 2-sphere in euclidean $\mathbb{R}^3$, by parallel translation. This bijection gives $\mathcal{L}$ the structure of a differentiable 4-manifold [4].

Let (ξ, η) be holomorphic coordinates on TS^2, where ξ is obtained by stereographic projection from the south pole onto the plane through the equator, and we identify (ξ, η) with the vector

$$\eta \frac{\partial}{\partial \xi} + \overline{\eta} \frac{\partial}{\partial \overline{\xi}} \in T_\xi S^2.$$

Theorem 1. *[2] The map Φ takes $(\xi, \eta) \in TS^2$ to the oriented line given by*

$$z = \frac{2(\eta - \overline{\eta}\xi^2) + 2\xi(1 + \xi\overline{\xi})r}{(1 + \xi\overline{\xi})^2} \tag{2.1}$$

$$t = \frac{-2(\eta\overline{\xi} + \overline{\eta}\xi) + (1 - \xi^2\overline{\xi}^2)r}{(1 + \xi\overline{\xi})^2}, \tag{2.2}$$

where $z = x^1 + ix^2$, $t = x^3$, (x^1, x^2, x^3) are euclidean coordinates on $\mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R}$ and r is an affine parameter along the line.

Definition 3. A *line congruence* is a surface $\Sigma \subset \mathcal{L}$.

Consider line congruences that arise as graphs of local sections of the bundle map $\pi : \mathcal{L} \to S^2$, defined by projection onto the first factor. Such congruences are given by $\eta = F(\xi, \bar{\xi})$, where F is a complex valued function on an open neighbourhood of S^2. Substituting this function in (2.1) and (2.2) gives an explicit description for the 2-parameter family of lines in $\mathbb{R}^3$.

Of particular interest are the integrable line congruences, that is, those that are orthogonal to a (possibly singular) foliation of $\mathbb{R}^3$:

Theorem 2. *[3] The graph of a local section of $\pi : \mathcal{L} \to S^2$ is integrable iff the defining function F satisfies*

$$\frac{\partial}{\partial \xi}\left(\frac{F}{(1 + \xi\bar{\xi})^2}\right) = \frac{\partial}{\partial \bar{\xi}}\left(\frac{\bar{F}}{(1 + \xi\bar{\xi})^2}\right). \tag{2.3}$$

Turning to the principal foliations on the integral surfaces of such a congruence, we have:

Theorem 3. *[3] Let Σ be an integrable congruence with defining function F and $S \subset \mathbb{R}^3$ a leaf of the orthogonal foliation. A point $p \in S$ is umbilic iff*

$$\bar{\partial} F(\xi_0) = \frac{\partial F}{\partial \bar{\xi}}(\xi_0) = 0,$$

where $\xi_0 \in S^2$ is the normal direction to S at p. Moreover, the principal foliations around p are given by the argument of $\bar{\partial}F$ and, in particular, the index of the umbilic is half the index of $\bar{\partial}F$.

Definition 4. An isolated umbilical point p on a surface S in $\mathbb{R}^3$ has a *regular* principal foliation if in a neighbourhood of p the argument of $\bar{\partial}F$ is a constant multiple of the argument of the canonical holomorphic coordinate $\xi = |\xi|e^{i\phi}$:

$$\bar{\partial}F = |\bar{\partial}F|e^{iN\phi}.$$

In this case, the index of the umbilical point is $N/2$.

Let p be an isolated umbilic of index $N/2$ on a real analytic surface S. We rotate the surface so that the direction of the normal to S at p is $\xi = 0$. In addition, we still have the freedom to translate the surface, which induces a quadratic holomorphic transformation on the defining function [3]:

$$F \to F - \alpha + 2a\xi + \bar{\alpha}\xi^2,$$

for $\alpha \in \mathbb{C}$ and $a \in \mathbb{R}$.

Theorem 4. *Let p be an isolated umbilical point of index $N/2$ on a real analytic surface, as above. The principal foliation cannot be regular for $N > 2$.*

For $N \leq 2$, after translation, the defining functions F for a real analytic surface with an isolated regular umbilic point of index $N/2$ are of the form:

N	$F(\xi, \bar{\xi})$
2	$\xi H(\lvert\xi\rvert^2)$
1	$\lambda\xi^2 + 2\bar{\lambda}\xi\bar{\xi} + \bar{\lambda}\xi^2\bar{\xi}^2$
0	$\lambda\xi^3 + 3\bar{\lambda}\xi\bar{\xi}^2 + 2\bar{\lambda}\xi^2\bar{\xi}^3$
< 0	$\lambda\xi^{3-N} + \mu\bar{\xi}^{1-N} + (2\mu + (3-N)\bar{\lambda})\xi\bar{\xi}^{2-N} + (\mu + (2-N)\bar{\lambda})\xi^2\bar{\xi}^{3-N}$

where H is an arbitrary real analytic function of $|\xi|^2$ and $\lambda, \mu \in \mathbb{C}$.

The $N = 2$ surfaces are rotationally symmetric, and for $N < 2$ the surfaces are open with a single isolated umbilical point of index $N/2$ at $\xi = 0$.

Proof. On a real analytic surface the Gauss map is real analytic and so the defining function F is real analytic. Thus it can be expanded in a power series about $\xi = 0$:

$$F = \sum_{n,m=0}^{\infty} A_{nm}\xi^n\bar{\xi}^m,$$

for $A_{nm} \in \mathbb{C}$. For the surface to have an umbilical point at $\xi = 0$ we must have $A_{01} = 0$.

The integrability condition (2.3) restricts the coefficients of the power series by

$$A_{10} = \overline{A}_{10} \tag{2.4}$$

$$2A_{20} = \overline{A}_{11} - 2\overline{A}_{00} \tag{2.5}$$

$$A_{21} = \overline{A}_{21} \tag{2.6}$$

$$(n+1)A_{n+1\,0} = \overline{A}_{1n} - 2\overline{A}_{0\,n-1} \tag{2.7}$$

$$(n+1)A_{n+1\,1} + (n-2)A_{n\,0} = 2\overline{A}_{2\,n} - \overline{A}_{1\,n-1} \tag{2.8}$$

$$(n+1)A_{n+1\,m} + (n-2)A_{n\,m-1} = (m+1)\overline{A}_{m+1\,n} + (m-2)\overline{A}_{m\,n-1}, \tag{2.9}$$

where $n, m = 2, 3, 4....$

Suppose the principal foliation is regular around p. Consider the Fourier decomposition of $\bar{\partial}F$ about $\xi = 0$:

$$\bar{\partial}F = \sum_{k=-\infty}^{\infty} \beta_k(|\xi|)e^{ik\theta},$$

for complex functions β_k of $|\xi|$.

Integrating both sides of this against $e^{il\theta}$, using regularity, we find that

$$\beta_l = 0 \qquad \text{for } l \neq N.$$

On the other hand

$$\bar{\partial}F = \sum_{n,m=0}^{\infty} mA_{nm}\xi^n\bar{\xi}^{m-1},$$

and so, for a regular foliation with index $N/2$,

$$A_{rs} = 0 \qquad \Longleftarrow \qquad r + 1 - s \neq N \quad \text{and} \quad s \neq 0. \tag{2.10}$$

The theorem follows from finding which terms of the power series are non-vanishing when (2.10) is repeatedly combined with (2.4) to (2.9). We must treat the cases $N > 2$, $N = 2$, $N = 1$, $N = 0$ and $N < 0$ seperately.

For $N > 2$, equation (2.10) implies $A_{11} = 0$ and so, from equations (2.4) and (2.5), an appropriate translation sets A_{00}, A_{10} and A_{20} to zero. Equation (2.6) is identically satisfied as, by (2.10), $A_{21} = 0$. Equation (2.10) also ensures that each term of the right hand side of (2.7) vanishes, and so $A_{n+1\,0} = 0$ for $n \geq 2$. Similarily, the right hand side of (2.8) also vanishes, and so we conclude that $A_{n+1\,1} = 0$ for $n \geq 2$. Finally, by (2.10) each term of equation (2.9) vanishes unless either $n + 2 - m = N$ or $m + 2 - n = N$. In either case, each term on one side of the equation vanishes and the remaining equation reads:

$$(m + N - 1)A_{m+N-1\,m} + (m + N - 4)A_{m+N-2\,m-1} = 0,$$

for $m \geq 2$. By (2.7) the second of these terms vanishes for $m = 2$, and proceeding inductively we find that all of the coefficients must vanish. Thus there are no regularly foliated umbilical points for $N > 2$, as claimed.

For $N = 2$, as before, $A_{11} = 0$ and a translation sets A_{00}, A_{10} and A_{20} to zero. Equation (2.7) reduces to $A_{n+1\,0} = 0$ for $n \geq 2$ and we conclude from equation (2.8) that $A_{n+1\,1} = 0$ for $n \geq 2$, both by virtue of (2.10). An inductive argument reduces the last equation to $A_{n+1\,n} = \overline{A}_{n+1\,n}$ for $n \geq 2$. Thus the power series is ξ times a real function of $\xi\bar{\xi}$, as claimed.

For $N = 1$, by (2.10) $A_{21} = 0$ and, from equation (2.7), $A_{n+1\,0} = 0$ for $n \geq 2$. The only non-trivial equation in (2.8) is for $n = 2$, from which we conclude that $A_{11} = 2A_{22}$. The final equation reduces to $mA_{mm} + (m - 3)A_{m-1\,m-1} = 0$ for $m \geq 3$, which inductively forces all the higher coefficients to zero. Letting $A_{00} = \alpha$, $A_{10} = -2a$, $A_{22} = \bar{\lambda}$ we find the defining function is

$$F = \alpha - 2a\xi + (\lambda - \bar{\alpha})\xi^2 + 2\bar{\lambda}\xi\bar{\xi} + \bar{\lambda}\xi^2\bar{\xi}^2.$$

Finally, a translation sets a and α to zero and yields the stated function.

As before, when $N = 0$ we can set A_{00}, A_{10} and A_{20} to zero and $A_{21} = 0$. Applying (2.10), equation (2.7) says only that $3A_{30} = \overline{A}_{12}$, while from (2.8) we find that $A_{30} = 2\overline{A}_{23} - \overline{A}_{12}$. Again, an inductive argument shows that equation (2.9) forces the vanishing of all higher coefficients. Letting $A_{30} = \lambda$ gives the result stated.

The case $N < 0$ is similar to the $N = 0$ case, the only non-vanishing coefficients satisfy $(3 - N)A_{3-N\,0} = \overline{A}_{1\,2-N} - 2\overline{A}_{0\,1-N}$ and $(1 - N)A_{3-N\,0} = 2\overline{A}_{2\,3-N} - \overline{A}_{1\,2-N}$. Letting $A_{3-N\,0} = \lambda$ and $A_{0\,1-N} = \mu$ completes the proof. □

3. Discussion of Examples

In a sufficiently small neighbourhood of the umbilic the surfaces in the above theorem with $N < 2$ are real analytic. For further insight, we can use the line congruence description to draw the surfaces in $\mathbb{R}^3$.

Given an integrable congruence with defining function $F(\xi, \bar{\xi})$ we construct the integral surfaces in $\mathbb{R}^3$ by solving

$$\bar{\partial} r = \frac{2F}{(1 + \xi\bar{\xi})^2},$$

for the real function r. This is just the potential that must exist as a consequence of equation (2.3). By inserting $F(\xi,\bar{\xi})$ and $r(\xi,\bar{\xi})+C$ in (2.1) and (2.2), each choice of real integration constant C yields a parametric surface in $\mathbb{R}^3$, the leaves of the orthogonal foliation. The surface in $\mathbb{R}^3$ thus obtained is parameterised locally by the inverse of the Gauss map.

For the examples given above the function r is:

N	$r(\xi,\bar{\xi})$
1	$\frac{2}{1+\xi\bar{\xi}}(\lambda\xi^2\bar{\xi}+\bar{\lambda}\xi\bar{\xi}^2)$
0	$\frac{2}{1+\xi\bar{\xi}}(\lambda\xi^3\bar{\xi}+\bar{\lambda}\xi\bar{\xi}^3)$
<0	$\frac{2}{1+\xi\bar{\xi}}(\lambda\xi^{3-N}\bar{\xi}+\bar{\lambda}\xi\bar{\xi}^{3-N})+\frac{2}{2-N}(\mu\bar{\xi}^{2-N}+\bar{\mu}\xi^{2-N})$

Figure 1 shows the $N=2$ surface with $F=\xi(2+\xi\bar{\xi})$ and $C=1$. The lines on the surface are the image of the lines of longitude and lattitude on S^2 under the inverse of the Gauss map which takes the north pole to the umbilical point.

Note that, while the line congruence is smooth everywhere, the surface has a conical singularity away from the umbilic.

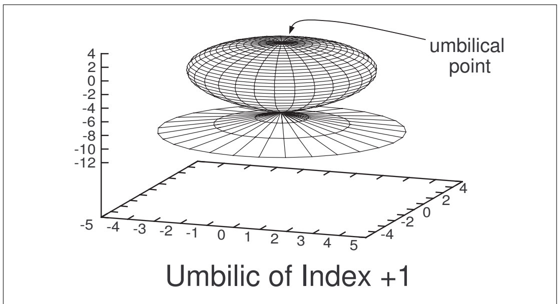
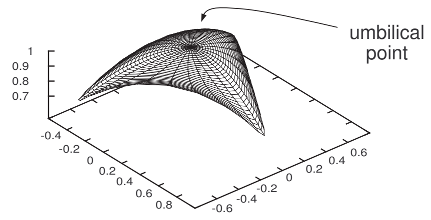


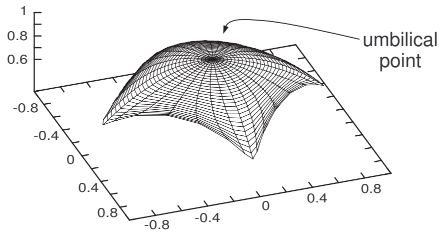
FIGURE 1

Figures 2 and 3 show surfaces containing umbilics of index $-1/2$ and -1, respectively, obtained by setting $C=1$, $\mu=0$ and $\lambda=1$.



Umbilic of Index -1/2

FIGURE 2



Umbilic of Index -1

FIGURE 3

The shape of the lower index families can be surmised from these two: an index $N/2$ surface is shaped like a $2-N$-spoked umbrella.

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