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The Atiyah-Singer index theorem for the spin complex and the gravitational chiral anomaly in $d = 4k$ for a massless spin-1/2 field

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Abstract

We show that the index of the Dirac operator over a closed, orientable, curved, even-dimensional manifold is given by the integrated chiral anomaly for a one-dimensional linear sigma model with $N = 1$ rigid supersymmetry. We follow the path integral approach extended for curved manifolds with a Lagrangian including both bosonic and fermionic degrees of freedom.

Keywords: index theorem, path integral, chiral anomaly

1. Introduction

Let $\mathcal{M}$ be a Riemannian manifold of dimension $dim\mathcal{M} = d$. $\mathcal{M}$ is said to be *closed* if it is compact and has no boundary. A complex vector bundle V on $\mathcal{M}$ is said to be a *hermitian bundle* if it is endowed with a (*fibre*) *metric* and a compatible *connection*. Let now $V_\pm$ be two Hermitian vector bundles of fibre dimension $d_\pm$, $C^\infty(V_\pm)$ is the vector space of smooth (C^∞) sections of $V_\pm$ and

$$D : C^\infty(V_+) \to C^\infty(V_-) \tag{1}$$

be a differential operator which locally can be written as

$$D = \sum_{l=1}^{n} a^{\nu_1\cdots\nu_l}(x) D_{\nu_1}\cdots D_{\nu_l} + a(x). \tag{2}$$

The ν's take values from $1, \cdots, d$, D_ν's are defined by

$$D_\nu := -i\frac{\partial}{\partial x^\nu} \tag{3}$$

and $a^{\nu_1\cdots\nu_l}$'s, a are $d_+ \times d_-$ dimensional matrix-valued functions of x. The adjoint of D is a differential operator

$$D^\dagger : C^\infty(V_-) \to C^\infty(V_+) \tag{4}$$

of the same order. Let $s_\pm$ be arbitrary sections of $V_\pm$ then $D^\dagger$ is defined by the following expression

$$\int_{\mathcal{M}} (s_-, Ds_+)_- d\Omega = \int_{\mathcal{M}} (D^\dagger s_-, s_+)_+ d\Omega \tag{5}$$

where $(.,.)_\pm$ are the Hermitian inner products on $V_\pm$ [1] and $d\Omega$ denotes the volume form of $\mathcal{M}$. The adjoint operator $D^\dagger$ depends not only on the original operator D, but also on the geometric structures of the base manifold and the vector bundles.

Let us consider a linear operator

$$P : C^\infty(V_+) \to C^\infty(V_-) \tag{6}$$

the *kernel* and the *cokernel* are defined respectively by

$$\begin{aligned} ker(P) &:= \{s_+ \in C^\infty(V_+) : P(s_+) = 0\} \\ coker(P) &:= C^\infty(V_-)/im(P). \end{aligned} \tag{7}$$

Operators whose kernel and cokernel are finite dimensional are called *Fredholm operators*. For a Fredholm operator one defines the *analytic index* by

$$Index(P) := dim(ker(P)) - dim(coker(P)). \tag{8}$$

Elliptic operators on a *compact* manifold are Fredholm operators and one using the relation $coker(D) = ker(D^\dagger)$ can reexpress the analytic index as

$$Index(D) := dim(ker(D)) - dim(coker(D^\dagger)). \tag{9}$$

Define $\Delta_+ = D^\dagger D$ and $\Delta_- = DD^\dagger$ then one can show that

$$ker(D) = ker(\Delta_+) \qquad ker(D^\dagger) = ker(\Delta_-). \tag{10}$$

Let $\mathcal{C} : C^\infty(V_+) \to C^\infty(V_-)$ be an elliptic complex [2] and $\sigma \to \Sigma$ be the symbol bundle of D. Then the *topological index* of C (alternative of D) is defined by

$$Index_{top.}(D) := \int_\Sigma ch(\sigma) \wedge \pi^* \left[td(TM^* \otimes C)\right] \tag{11}$$

[1]If **h** denotes the hermitian metric on V then for any two sections s_1, s_2 of V their inner product is given by $(s_1, s_2) := \int_{\mathcal{M}} s^{i*}(x) h_{ij}(x) s^j(x) d\Omega$.

[2]Let $V_0, V_1, \cdots, V_k$ be a sequence of hermitian vector bundles over $\mathcal{M}$. Then the sequence $\mathcal{C}$: $C^\infty(V_0) \stackrel{D_0}{\to} C^\infty(V_1) \stackrel{D_1}{\to} \cdots \stackrel{D_{k-1}}{\to} C^\infty(V_k)$ is said to be a *complex* if $im(D_i) \subset ker(D_{i+1}), \forall i = 0, \cdots, k-2$. The D_i's are arbitrary differential operators such that $D_i : C^\infty(V_i) \to C^\infty(V_{i+1})$ $\forall i = 0, \cdots, k-1$.

where π^* denotes the pullback operation defined by the projection map $\pi : \Sigma \to M$ and ch and td are the Chern character and Todd class defined by [3]

$$\begin{aligned} ch(V) &:= \sum_{i=1}^{n} e^{x_i} = n + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \cdots \\ td(V) &:= \prod_{i=1}^{n} \frac{x_i}{1 - e^{-x_i}} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \cdots \end{aligned} \tag{12}$$

The Atiyah-Singer index theorem states that the analytic index (8) and the topological index (12) are indentical [1]. The motivation for defining the topological index is to give a closed expression for the analytic index in terms of some differential geometric quantities.

Among the *classical complexes* whose indices give important topological information about the associated structures the *spin complex* is perhaps the most subtle and interesting. To begin with we construct (if it exists) the *spin bundle* of an oriented compact Riemannian manifold $\mathcal{M}$ of dimension d. For all $x \in \mathcal{M}$ one can use TM_x^* to construct the associated Clifford algebra $\mathcal{C}(TM_x^*)$ [4] . These may be viewed as the fibres of a global bundle structure over $\mathcal{M}$ which is called the *Clifford bundle* of $\mathcal{M}$. By lifting the transition functions of the Clifford bundle from $So(d)$ to $Spin(d)$ we construct the associated principal bundle which is called the spin bundle S. If the transition functions can be consistently defined then the manifold is a *spin manifold.*

Now we restrict ourselves to a four-dimensional spin manifold with Euclidean signature. Denote the set of sections of the spin bundle by $\Delta(M) = \Gamma(M, S(M))$. A Dirac spinor $\Psi \in \Delta(M)$ is an irreducible representation of the Clifford algebra but not that of $Spin(d)$. Irreducible representations of $Spin(d)$ are obtained by splitting $\Delta(M)$, according to the eigenvalues of $\gamma^{d+1} = (-i)^{\frac{d}{2}}\gamma^1\gamma^2\cdots\gamma^d$, [5] into two eigenspaces

$$\Gamma(\Delta) = \Delta^+(M) \oplus \Delta^-(M) \tag{13}$$

with local coordinates $(x^i, \Psi_\pm)$ and $\gamma^{d+1}\Psi^\pm = \pm\Psi^\pm$ for $\Psi^\pm \in \Delta^\pm(M)$. Define now the projection operators $P^\pm = \frac{1}{2}\left(1 \pm \gamma^{d+1}\right)$ having the property $P^\pm\Psi^\pm = \Psi^\pm$. In

[3] The *Chern classes* of a complex hermitian vector bundle over a closed Riemannian manifold $\mathcal{M}$ are defined by $c_1(V) := \sum_i x^i$, $c_2(V) := \sum_{i<j} x_i \wedge x_j$, $\cdots$, $c_n(V) := x_1 \wedge x_2 \wedge \cdots \wedge x_n$.

[4] Let E be an *inner product space* with an orthonormal basis $\{e_i\}$. The Clifford algebra $\mathcal{C}(E)$ of E is generated by e_i according to the Clifford multiplication rule $\{e_i, e_j\} = e_i{*}e_j + e_j{*}e_i = -2(e_i, e_j) = -2\delta_{ij}$ where * and (,) denote the Clifford multiplication and the inner product respectively.

[5] We have chosen a hermitian representation of the Clifford algebra $\mathcal{C}(E)$ spanned by $\{\epsilon_i = \gamma_\alpha\}$ satisfying $\{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha\beta}$. In Euclidean four-dimensional space-time such a representation is given by $\gamma_i = \begin{bmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{bmatrix}$, $\gamma_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\gamma_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ where $\sigma_i,\ i = 1, 2, 3$ are the usual $SO(3)$ generators.

curved space-time the Dirac operator (which is an elliptic operator) is given by [6]

$$\not{D}\Psi = \gamma^i(x)(\partial_i + \omega_i(x))\Psi = E^i_\alpha(x)\gamma^\alpha(\partial_i + \omega_i(x))\Psi \tag{14}$$

where ω_i is the one-form spin connection with components $\omega_i = \frac{1}{4}\omega_{i\alpha\beta}\gamma^\alpha\gamma^\beta$. [7] Applying Atiyah-Singer index theorem one finds that

$$Ind(D) = dim(ker(D)) - dim(ker(D^\dagger)) = n_+ - n_- = \int \hat{A}(\mathcal{M})|_{top} \tag{15}$$

where $D = \not{D}P^+, D^\dagger = \not{D}P^-$, $n_\pm$ is the number of zero energy modes of $\pm$ chirality respectively and the subscript "*top*" means that only the highest rank form in the power series expansion of (16) is integrated. The so-called *Dirac's $\hat{A}$-genus density* is given by

$$\hat{A} = \prod_{i=1}^{l} \frac{\Omega_i/2}{\sinh(\Omega_i/2)} \tag{16}$$

where $\Omega_i = \Omega_{i\mu\nu}dx^\mu \wedge dx^\nu$ are the 2-form skew eigenvalues of the block diagonalized curvature 2-form

$$\Omega := \frac{1}{2}R_{ij\mu\nu}dx^\mu \wedge dx^\nu := diag\begin{bmatrix} 0 & \Omega_i \\ -\Omega_i & 0 \end{bmatrix}, \quad i = 1, \cdots, l \tag{17}$$

considered as a matrix in the Lie algebra of $SO(2l)$. The *$\hat{A}$-genus density* is invariant under $\Omega_i \to -\Omega_i$ and all permutations of the Ω_i's so it can be expanded as

$$\begin{aligned} \hat{A} &= \prod_{i=1}^{l}\left[1 + \sum_{n\geq 1}(-1)^n \frac{2^{2n}-2}{(2n)!}B_n\left(\frac{\Omega_i}{2}\right)^{2n}\right] \\ &= 1 - \frac{1}{2^2}\frac{1}{6}\sum_{i=1}^{l}\Omega_i^2 + \frac{1}{2^4}\left[\frac{1}{36}\sum_{i<j}\Omega_i^2\wedge\Omega_j^2 + \frac{7}{360}\sum_{i=1}^{l}\Omega_i^2\wedge\Omega_i^2\right] + \cdots \\ &= 1 - \frac{1}{2^2}\frac{1}{6}p_1 + + \frac{1}{2^4}\left[\frac{7}{360}p_1^2 - \frac{1}{90}p_2\right] + \cdots \end{aligned} \tag{18}$$

where p_i's are polynomials of order $2i$ in Ω_i's, the so-called *Pontryagin classes*, given by

$$p_1 := \sum_{i=1}^{l}\Omega_i^2$$

[6]The vielbein $e^\alpha_i(x)$ and its inverse $E \equiv e^{-1}$, defined by $E^i_\alpha(x) = \delta_{\alpha\beta}g^{ij}(x)e^\beta_j(x)$ and satisfying $E^i_\alpha(x)e^\beta_i(x) = \delta^\beta_\alpha$, $E^i_\alpha(x)e^\alpha_j(x) = \delta^i_j$ are used to change tensor quantities referred to coordinate frames to tangent frames, and vice versa.

[7]We use Greek letters $(\alpha, \beta, \cdots)$ as orthonormal tangent frame indices and Latin letters $(i, j, \cdots)$ as coordinate frame indices.

$$
\begin{aligned}
p_2 &:= \sum_{i<j} \Omega_i^2 \wedge \Omega_j^2 \\
p_3 &:= \sum_{i<j<k} \Omega_i^2 \wedge \Omega_j^2 \wedge \Omega_k^2, \\
&\dots \\
p_l &:= \Omega_1 \wedge \Omega_2 \wedge \cdots \wedge \Omega_l. \qquad (19)
\end{aligned}
$$

Any such polynomial can be expanded in terms of the basis functions

$$
Tr\left(\frac{\Omega}{4\pi}\right)^{2j} = 2(-1)^j \sum_{a=1}^{l} \left(\frac{\Omega_a}{2}\right)^{2j} \qquad (20)
$$

which are $4j$-forms. The first symmetric invariant is the 4-form $p_1 = -\frac{1}{8\pi^2} Tr(\Omega^2)$. The next non-zero invariant is $p_2 = \frac{1}{2^7\pi^2}\left[(Tr(\Omega^2))^2 - 2Tr(\Omega^4)\right]$. In four dimensions only the 4-form p_1 contributes to the chiral anomaly giving the result

$$
n_+ - n_- = \frac{1}{24 \cdot 8\pi^2} \int_{\mathcal{M}} Tr(\Omega \wedge \Omega). \qquad (21)
$$

2. The Fujikawa Approach

In Quantum Field Theory the index of an operator is usually related to the nonconservation of currents signaling the breakdown of a symmetry at quantum level. [8]. In the case of the Feynman triangle diagram with a massless chiral fermion running in the triangle and coupled with two energy momentum tensors and one axial current, one finds, by imosing energy-momentum conservation in both energy-momentum channels, that chirality is not conserved. The index of the associated operator in this case can be shown to be given in $d = 4k$ dimensions by [2]

$$
\begin{aligned}
Index(D) &= \frac{1}{2}\int_{\mathcal{M}} d^{2l}x\sqrt{g}\partial_i J^i_{d+1} \\
&= \frac{1}{2}\int_{\mathcal{M}} d^{2l}x\sqrt{g}\lim_{\beta\to 0} Tr_{x\to y}\left(\gamma_{d+1} e^{-\beta \not{D}^2}\delta^{2l}(x-y)\right). \qquad (22)
\end{aligned}
$$

Consider the generating functional for the Green functions in Euclidean space

$$
Z[e] = \int \left[\mathcal{D}\bar{\Psi}\right]\left[\mathcal{D}\Psi\right] e^{-\frac{1}{\hbar}S_E(e,\Psi,\bar{\Psi})} \qquad (23)
$$

with Euclidean Lagrangian density $\mathcal{L}_E = |dete_i^a| \bar{\Psi}(x) E_a^i(x) \gamma^a D_i \Psi(x)$. The covariant derivative D_i is given by $D_i = \partial_i + \omega_i$ where ω_i is the spin connection.

[8] One of the most prominent decays in particle physics which created a puzzling situation for some time, whose solution led to the discovery of the anomaly, is the neutral pion decay into two photons $\pi^0 \to \gamma\gamma$.

The contribution to the topological index stems from the fact that the path integral measure $\left[\mathcal{D}\bar{\Psi}\right][\mathcal{D}\Psi]$ is not invariant under infinitesimal local axial $U(1)$ transformations described by

$$\Psi(x) \rightarrow \Psi'(x) = \Psi(x) + ia(x)\gamma_{d+1}\Psi(x). \tag{24}$$

This can be proved by expanding the Ψ, $\bar{\Psi}$ in terms of the eigenfunctions [9]of the Hermitian operator $\not{D}$, namely

$$\Psi(x) = \sum_n a_n \Psi_n(x); \qquad \bar{\Psi}(x) = \sum_n \Psi_n^\dagger(x)\bar{b}_n \tag{25}$$

where $a_n, \bar{b}_m$ denote independent elements of the Grassmann algebra. Under the infinitesimal chiral rotation (24) the coefficients a transform according to

$$\begin{aligned} a'_n = \sum_m C_{nm} a_m; \qquad C_{nm} &= \int d^{2l}xe(x)\Psi_n^\dagger(x)e^{ia(x)\gamma_{d+1}}\Psi_m(x) \\ &\simeq \delta_{nm} + i\int d^{2l}xe(x)a(x)\Psi_n^\dagger(x)\gamma_{d+1}\Psi_m(x) + O(a^2) \end{aligned} \tag{26}$$

and the path integral measure transforms with the inverse determinant as follows [10]

$$\begin{aligned} [\mathcal{D}\Psi]\left[\mathcal{D}\bar{\Psi}\right] = \prod_n da_n d\bar{b}_n \rightarrow [\mathcal{D}\Psi']\left[\mathcal{D}\bar{\Psi}'\right] &= \prod_n da'_n d\bar{b}'_n \\ &= (\det C_{mn})^{-2}\prod_m da_m d\bar{b}_m \end{aligned} \tag{27}$$

where

$$(\det C_{mn})^{-1} = e^{-i\int d^{2l}xe(x)a(x)\sum_k \Psi_k^\dagger(x)\gamma_{d+1}\Psi_k(x)}. \tag{28}$$

The sum in the exponential of (28) is ill-defined

$$\sum_k \Psi_k^\dagger(x)\gamma_{d+1}\Psi_k(x) = Tr(\gamma_{d+1})\delta^{2l}(0) \tag{29}$$

where we used the completeness relation of the eigenfunctions $\{\Psi_n(x)\}$. Fujikawa regularized the previous sum by introducing a Gaussian cut-off M

$$\begin{aligned} &\int d^{2l}xe(x)\left(\sum_k \Psi_k^\dagger(x)\gamma_{d+1}\Psi_k(x)\right)_R \\ &= \lim_{M\rightarrow\infty}\sum_k\int d^{2l}xe(x)\Psi_k^\dagger(x)\gamma_{d+1}e^{-(\lambda_k/M)^2}\Psi_k(x) \\ &= \int d^{2l}xe(x)\lim_{\beta\rightarrow 0}Tr_{x\rightarrow y}\left(\gamma_{d+1}e^{-\beta\not{D}^2}\delta^{2l}(x-y)\right); \end{aligned} \tag{30}$$

[9]The set of eigenfunctions $\{\Psi_n(x)\}$ is orthonormal and complete, namely, they obey $(\Psi_n(x),\Psi_m(x)) = \int d^{2l}x\Psi_n^\dagger(x)\Psi_m(x) = \delta_{nm}$ and $\sum_k \Psi_n(x)\Psi_n^\dagger(y) = \delta^{2l}(x-y)$.

[10]In the derivation of the determinant of the matrix C_{nm} we use the identity $det(C_{nm}) = e^{Tr\ln(C_{nm})}$ and expand the logarithm $\ln(1+a) = a + O(a^2)$.

where only the $\beta = \frac{1}{M^2}$-independent terms contribute to the topological index. We should empasize that the anomaly is indepedent of the chosen regularization scheme for the large eigenvalue contributions in the sum (29). Instead of the exponential damping we could equally have chosen some other function which is smooth and decreasing sufficiently rapidly at infinity. Moreover one can show that the nonzero eigenvalues of the Dirac operator are paired, thus giving a vanishing contribution, while the $\pm$-chirality zero modes carry all the topological structure of the manifold.

3. The Path Integral Approach

In the Fujikawa approach one can show that the consistent regulator $\not{D}^2$ can be rewritten in the form

$$\hat{\mathcal{R}} = \not{D}^2 = \frac{1}{2} g^{-\frac{1}{2}} \tilde{D}_i \sqrt{g} g^{ij} \tilde{D}_j + \frac{1}{4} R; \qquad R = \frac{1}{2} R_{ijab} \gamma^i \gamma^j \gamma^a \gamma^b \tag{31}$$

where $\tilde{D}_i = \partial_i + \frac{1}{4}\omega_{iab}\gamma^a\gamma^b$. One can prove that $\hat{\mathcal{R}}$ commutes with the generator of infinitesimal general coordinate transformations

$$[\hat{\mathcal{R}}, \hat{G}_E] = 0; \qquad \hat{G}_E = \frac{1}{2i\hbar} \left[\partial_i \xi_i + \xi_i \partial_i \right]. \tag{32}$$

We would like to interpret $\hat{\mathcal{R}}$ as the Hamiltonian of some one-dimensional supersymmetric linear or nonlinear σ-model.

Consider in Euclidean space the Langrangian density [4]

$$\begin{aligned} L &= \frac{1}{2} g_{ij}(x) \dot{x}^i \dot{x}^j + \frac{1}{2} g_{ij}(x) \Psi_1^i \left(\dot{\Psi}_1^j + \Gamma^j_{kl} \dot{x}^k \Psi_1^l \right); \\ \Gamma^j_{kl} &= E^j_a \left(\omega_l{}^a{}_b e^b_k + \partial_l e^a_k \right) \end{aligned} \tag{33}$$

where dots over the bosonic and fermionic fields indicate differentiation w.r.t. time. Also $\Psi_1^i(t)$ are one-component Majorana fermions. The configuration space of (33) is a (d, d) dimensional supermanifold with x and Ψ denoting its bosonic (commuting) and fermionic (anticommuting) coordinates, respectively. The symmetries of the above action are

— global $N = 1$ supersymmetry

$$\delta x^i = \epsilon \Psi_1^a E_a^i; \qquad \delta \Psi_1^a = -e_i^a \dot{x}^i \epsilon - \delta x^j \omega_j{}^a{}_b \Psi_1^b \tag{34}$$

— local $SO(d)$ rotations

$$\Psi_1^a \to L_b^a(x) \Psi_1^b; \qquad \omega_j{}^a{}_b \to L_c^a \omega_i{}^c{}_d d L_b^d + L_c^a \partial_i L_d^c. \tag{35}$$

where ϵ is a constant spinor and $L_b^a(x)$ are local Lorentz transformations.

The canonical momenta conjugate to x_i and Ψ_1^a are

$$\pi_i(t) = \frac{\partial L}{\partial \dot{x}^i} = \dot{x}_i + \frac{1}{2}\omega_{iab}\Psi_1^a\Psi_1^b; \qquad \Pi^a(t) = \frac{\partial L}{\partial \dot{\Psi}_a} = \frac{1}{2}\Psi_1^a. \tag{36}$$

Note that the fermionic momenta Π^a are not independent of the coordinates Ψ_1^a thus one faces the problem of canonically quantizing this superclassical system. One then needs to consider the appropriate (fermionic) first class constraints [3]. In the Peierls quantization scheme this is completely avoided. This is because, in the Peierls program there is no need to define momentum variables to carry out the quantization. Promoting the canonical bosonic variables to operators satisfying the commutation relation

$$\left[\hat{\pi}_i, \hat{x}^j\right] = -i\hbar\delta_i^j \tag{37}$$

where $\hat{\pi}_i = \hat{p}_i - \frac{i\hbar}{2}\omega_{iab}\Psi_1^a\Psi_1^b$. One can construct the quantum Noether supersymmetry charge operator

$$\hat{Q} = g^{\frac{1}{4}}E_a^i\Psi_1^a\hat{\pi}_i g^{-\frac{1}{4}} = g^{-\frac{1}{4}}E_a^i\Psi_1^a\hat{\pi}_i g^{\frac{1}{4}}. \tag{38}$$

The second equality in (38) is a direct consequence of Hermiticity. The Hamiltonian operator is then given by

$$\hat{H} = \frac{1}{2}\{\hat{Q},\hat{Q}\} = \frac{1}{2}g^{-\frac{1}{4}}\hat{\pi}_i\sqrt{g}g^{ij}\hat{\pi}_j g^{-\frac{1}{4}} - \frac{1}{8}\hbar^2 R. \tag{39}$$

and shows that it is supersymmetric hence supersymmetry is preserved at quantum level.

The integrated contribution to the chiral anomaly (topological index) comes from the following path integral [5]

$$An(chiral) = \frac{(-i)^{\frac{d}{2}}}{2^{\frac{d}{2}}}\lim_{\beta\to 0} Tr\left(\prod_{a=1}^{d}\left(\hat{\Psi}^a + \hat{\Psi}^{a\dagger}\right)e^{-\frac{\beta}{\hbar}\hat{H}}\right) \tag{40}$$

in which we have introduced *new free* operators $\hat{\Psi}_2^a$ (satisfying $\{\Psi_2^a, \Psi_2^b\} = \delta^{ab}$ and $\{\Psi_1^a, \Psi_2^b\} = 0$) and define $\hat{\Psi}^a \equiv (\hat{\Psi}_1^a + i\hat{\Psi}_2^a)/\sqrt{2}$, $\hat{\Psi}_a^\dagger \equiv (\hat{\Psi}_1^a - i\hat{\Psi}_2^a)/\sqrt{2}$. The Hamiltonian operator $\hat{H}$ is given by (39).

In order to calculate the contribution of the bosonic Hamiltonian to the path integral, in the configuration space, we need the completeness relations

$$\int d^dx\sqrt{g(x)}\,|x\rangle\langle x| = I = \int d^dp\,|p\rangle\langle p| \tag{41}$$

where $|x\rangle, |p\rangle$ are the eigenstates of the Hermitian position and momentum operators. Their inner products are

$$\langle x|y\rangle = \frac{1}{\sqrt{g}}\delta^{(d)}(x-y); \qquad \langle p|p'\rangle = \delta^{(d)}(p-p');$$

$$\langle x|p\rangle = \frac{1}{(2\pi\hbar)^{\frac{n}{2}}}e^{\frac{i}{\hbar}p_i x^i}g^{-\frac{1}{4}}. \tag{42}$$

The fermionic Fock space is constructed using a complete basis of coherent states defined by

$$|\eta\rangle = e^{\hat{\psi}^\dagger \eta}\,|0\rangle \qquad \langle\bar{\eta}| = \langle 0|\, e^{\bar{\eta}\hat{\psi}} \tag{43}$$

where the operators $\hat{\psi}^a, \hat{\psi}^\dagger_a$, $a = 1, \cdots, d$ play the role of annihilation ($\hat{\psi}^a\,|0\rangle = 0$) and creation opertors ($|a, b\rangle = \hat{\psi}^\dagger_a \hat{\psi}^\dagger_b\,|0\rangle$) for the system, and satisfy $\hat{\psi}\,|\eta\rangle = \eta\,|\eta\rangle$ and $\langle\bar{\eta}|\,\hat{\psi}^\dagger = \langle\bar{\eta}|\,\bar{\eta}$. These eigenvectors have the following inner product and decomposition of unity

$$\langle\bar{\eta}|\xi\rangle = e^{\bar{\eta}\xi} \qquad I = \int d\bar{\eta}d\xi\,|\xi\rangle\, e^{-\bar{\eta}\xi}\,\langle\bar{\eta}| \tag{44}$$

where our convention for the ordering of the anticommuting variables is that $d\bar{\eta} = d\bar{\eta}_d \cdots d\bar{\eta}_1$ while $d\xi = d\xi_1 \cdots d\xi_d$. The trace of an operator over the fermionic Fock space is then given by

$$Tr(\hat{O}) = \int d\xi d\bar{\eta} e^{\bar{\eta}\xi} < \bar{\eta}|\hat{O}|\xi > . \tag{45}$$

In the discretised version of the path integral after the insertion of N complete sets of momentum eigenstates and $N-1$ complete sets of position eigenstates into the propagator, we face the problem of ordering ambiguities. These can be treated consistently by rewriting the operators $\exp\left(-\frac{\epsilon}{\hbar}\hat{H}(\hat{x},\hat{p})\right)$ in Weyl ordered form. This means that we symmetrize in all operators it contains. The Weyl ordering of a polynomial operator $\hat{B}_W(\hat{x},\hat{p})$ [11] leads to the midpoint rule

$$\begin{aligned}\langle z|\,\hat{B}_W(\hat{x},\hat{p})\,|y\rangle &= \int d^d p\,\langle z|\,\hat{B}_W(\hat{x},\hat{p})\,|p\rangle\,\langle p|y\rangle \\ &= \int d^d p\,\langle z|p\rangle\,\langle p|y\rangle\, B_W(\frac{1}{2}(z+y),p)\end{aligned} \tag{46}$$

where in the last line the Weyl ordered operator $\hat{B}_W$ has been replaced by a function simply by substituting $\hat{p} \to p$, $\hat{x} \to \frac{1}{2}(z+y)$. This is an exact result. For a fermionic Weyl ordered polynomial operator [12] $\hat{F}_W(\hat{\psi},\hat{\psi}^\dagger)$ one can also derive the midpoint rule for coherent states

$$\begin{aligned}\langle\bar{\eta}|\,\hat{F}_W(\hat{\psi},\hat{\psi}^\dagger)\,|\eta\rangle &= \int d\bar{\xi}d\xi\,\langle\bar{\eta}|\,\hat{F}_W(\hat{\psi},\hat{\psi}^\dagger)\,|\xi\rangle\, e^{-\bar{\xi}\xi}\,\langle\bar{\xi}|\eta\rangle \\ &= \int d\bar{\xi}d\xi\,\langle\bar{\eta}|\xi\rangle\,\langle\bar{\xi}|\eta\rangle\, e^{-\bar{\xi}\xi} F_W(\xi,\frac{1}{2}(\bar{\eta}+\bar{\xi})).\end{aligned} \tag{47}$$

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[11] For a polynomial one may prove that $(\hat{x}^m\hat{p}^r)_W = \frac{1}{2^m}\sum_{l=0}^{m}\binom{m}{l}\hat{x}^{m-l}\hat{p}^r\hat{x}^l$ and then it follows that $\langle z|\,(\hat{x}^m\hat{p}^r)_W\,|y\rangle = \int \langle z|p\rangle\,\langle p|y\rangle\left(\frac{z+y}{2}\right)^m p^r d^n p$.

[12] For polynomials of anticommuting variables we define the Weyl ordering by $(\psi^n\psi^{\dagger m})_W = \frac{1}{(m+n)!}\partial^n_{\bar{\xi}}\partial^m_{\xi}(\bar{\xi}\hat{\psi} + \xi\hat{\psi}^\dagger)^{m+n}$.

In general Weyl ordering and exponentiation do not commute, $\left[\exp\left(-\frac{\epsilon}{\hbar}\hat{H}\right)\right]_W \neq \exp\left(-\frac{\epsilon}{\hbar}\hat{H}\right)_W$. A closed expression for $\left[\exp\left(-\frac{\epsilon}{\hbar}\hat{H}\right)\right]_W$ cannot be written down but for path integrals their difference cancels.

4. Chiral Anomaly in the Center-of-Mass Approach

To calculate (40) we employ the background field method in the *center-of-mass-approach* [6] according to which the bosonic trajectories $x^i(t)$ decompose into a constant zero-mode part x_0^i and a periodic quantum fluctuating part $q^i(t)$ orthogonal to the constant. The fluctuations obey

$$\int_{-1}^{0} q^i(t)dt = 0, \qquad q^i(t) = q^i(t+1). \tag{48}$$

The Green function of the fluctuations on the interval $[-1, 0]$ which satisfies

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\Delta_{cm}(t,t') &= \delta(t-t') - 1, \\ \Delta_{cm}(t,t') &= \Delta_{cm}(t+1,t') = \Delta_{cm}(t,t') = \Delta(t,t'+1) = 0 \end{aligned} \tag{49}$$

is given by

$$\Delta_{cm}(t-t') = -2\sum_{n=1}^{\infty}\frac{\cos(2n\pi(t-t'))}{(2n\pi)^2} = \frac{1}{2}(t-t')\epsilon(t-t') - \frac{1}{2}(t-t')^2 - \frac{1}{12}. \tag{50}$$

The complex fermionic trajectories $\psi^a(t), \bar{\psi}^a(t)$ split into constant Grassmann variables and quantum fluctuations as follows

$$\psi^a(t) = \chi^a + \psi^a_{qu}(t) \qquad \bar{\psi}^a(t) = \bar{\eta}^a + \bar{\psi}_{qu,a}(t). \tag{51}$$

The Green function for the Dirac fermions $\psi^a(t), \bar{\psi}^a(t)$ depends on the (anti)-periodic boundary conditions and one finds that

$$\begin{aligned} < \psi^a(t)\bar{\psi}^b(t') >_{AP} &= \delta^{ab}\sum_{-\infty}^{+\infty}\frac{e^{(2n+1)i\pi(t-t')}}{(2n+1)i\pi} = \frac{1}{2}\delta^{ab}\epsilon(t-t') \\ < \psi^a(t)\bar{\psi}^b(t') >_{P} &= \delta^{ab}\sum_{-\infty}^{+\infty}{}'\frac{e^{2ni\pi(t-t')}}{2ni\pi} = \delta^{ab}\left(\frac{1}{2}\epsilon(t-t') - (t-t')\right). \end{aligned} \tag{52}$$

Using the trace formula (45) and the completeness relations (41), (44) we obtain

$$\begin{aligned} An(chiral) &= \frac{(-i)^{d/2}}{2^{d/2}}\int\left(\prod_{i=1}^{d} dx_0^i\right)\sqrt{g(x_0)}\prod_{a=1}^{d}(d\bar{\eta}_a d\eta^a d\chi^a d\bar{\chi}_a) \\ & e^{\bar{\chi}\chi}\langle\bar{\chi}|\prod_{a=1}^{d}(\hat{\psi}^a + \hat{\psi}^\dagger_a)|\eta\rangle e^{-\bar{\eta}\eta}\langle\bar{\eta}, x_0| e^{-\frac{\beta}{\hbar}\hat{H}}|\chi, x_0\rangle. \end{aligned} \tag{53}$$

The first matrix element between fermionic coherent states in the second line can be evaluated and gives

$$\langle\bar{\chi}|\prod_{a=1}^{d}(\hat{\psi}^a+\hat{\psi}_a^\dagger)\,|\eta\rangle = \langle\bar{\chi}|\eta\rangle\prod_{a=1}^{d}(\bar{\chi}^a+\eta^a) = \prod_{a=1}^{d}(\bar{\chi}^a+\eta^a). \tag{54}$$

Using the identity

$$e^{\bar{\chi}\chi}e^{-\bar{\eta}\eta}\prod_{a=1}^{d}(\bar{\chi}^a+\eta^a) = e^{-\frac{1}{2}(\eta-\bar{\chi})(\chi-\bar{\eta})}\prod_{a=1}^{d}(\bar{\chi}^a+\eta^a) \tag{55}$$

and performing the integrals over η and $\bar{\chi}$ (rewriting the measure $d\bar{\chi}^a d\eta^a$ in terms of the variables $\eta-\bar{\chi}$ and $\eta+\bar{\chi}$ as $2^n d(\bar{\chi}^a+\eta^a)d(\eta^a-\bar{\chi}^a)$) we find

$$An(chiral) = \frac{(-i)^{d/2}}{(2\pi\beta\hbar)^{d/2}}\int\left(\prod_{i=1}^{d}dx_0^i\right)\int\left(\prod_{a=1}^{d}d\psi_{1,bg}^a\right)e^{-\frac{1}{\hbar}S_{loops}(x_0,\psi_{1,bg}^a)}. \tag{56}$$

In the previous expression $\psi_{1,bg}^a = (\chi^a+\bar{\eta}^a)/\sqrt{2}$ is the background value of Ψ_1^a. The only contribution to the chiral anomaly is given by the vertex

$$\begin{aligned}
-\frac{1}{\hbar}S_{loops}(x_0,\psi_{1,bg}^a) &= -\frac{1}{2\beta\hbar}\int_{-1}^{0}dt\dot{q}^i\omega_{iab}(x_0+q)\Psi_1^a(t)\Psi_1^b(t)\\
&= -\frac{1}{4\beta\hbar}R_{ijab}(\omega(x_0))\Psi_{1,bg}^a\Psi_{1,bg}^b\int_{-1}^{0}dtq^i\dot{q}^j.
\end{aligned} \tag{57}$$

In the above expression we have Taylor expand the spin connection around the classical constant trajectory x_0, use Lorentz invariance to vanish $\omega(x_0)$ and integrate over the background values of $\Psi_{1,bg}^a$. Hence the *center-of-mass* approach with (50) yields

$$\begin{aligned}
An(chiral) &= \frac{(-i)^{d/2}}{(2\pi)^{d/2}}\int\left(\prod_{i=1}^{d}dx_0^i\sqrt{g(x_0)}\right)\int\left(\prod_{a=1}^{d}d\psi_{1,bg}^a\right)\\
&\quad\exp\left[\sum_{k=1}^{\infty}\left(-\frac{1}{\beta\hbar}\right)^k\frac{(k-1)!}{k!}2^{k-1}Tr\left(\frac{R_{ij}}{4}\right)^k(-\beta\hbar)^k I_k\right]
\end{aligned} \tag{58}$$

where $R_{ij} = R_{ijab}\psi_{1,bg}^a\psi_{1,bg}^b$ and

$$I_k = \int_{-1}^{0}dt_1\int_{-1}^{0}dt_2\cdots\int_{-1}^{0}dt_k\Delta_{cm}^{\bullet}(t_1-t_2)\Delta_{cm}^{\bullet}(t_2-t_3)\cdots\Delta_{cm}^{\bullet}(t_k-t_1) \tag{59}$$

with $\Delta_{cm}^{\bullet}(t_1-t_2) = \frac{\partial}{\partial t_2}\Delta_{cm}(t_1-t_2)$. The multiple integral in (59) can be evaluated by using

$$\begin{aligned}
\int_{-1}^{0}\Delta_{cm}^{\bullet}(t_1,t_2)\Delta_{cm}^{\bullet}(t_2,t_3)dt_2 &= \Delta_{cm}(t_1,t_3)\\
\int_{-1}^{0}\Delta_{cm}(t_1,t_2)\Delta_{cm}^{\bullet}(t_2,t_3)dt_2 &= \left(\frac{\partial}{\partial t_3}\right)^{-1}\Delta_{cm}(t_1,t_3), \qquad etc.
\end{aligned} \tag{60}$$

For odd values of k both $Tr(R^k)$ and I_k vanish while for even k one obtains the above integral which is given by

$$I_{2k} = 2(-1)^k \sum_{l=1}^{\infty} \frac{1}{(2\pi l)^{2k}}. \tag{61}$$

Substituting this result into (58), performing first the summation over k then over l, and scaling the constant fermionic backgrounds $\psi^a_{1,bg} = \sqrt{i}\tilde{\psi}^a_{1,bg}$, one gets

$$\begin{aligned} An(chiral) &= \frac{1}{(2\pi)^{d/2}} \int \left(\prod_{i=1}^{d} dx_0^i \sqrt{g(x_0)} \right) \int \left(\prod_{a=1}^{d} d\tilde{\psi}^a_{1,bg} \right) \\ & \det{}^{\frac{1}{2}} \left(\frac{iR_{ij}(x_0)/4}{i \sinh(R_{ij}(x_0)/4)} \right) \end{aligned} \tag{62}$$

where we used the identity $\det A = e^{Tr \ln A}$. To calculate the determinant we diagonalize the skew symmetric matrix $iR_{ij}(x_0) = iR_{ijab}(x_0)\tilde{\psi}^a_{1,bg}\tilde{\psi}^b_{1,bg}$ by an element of $GL(d, C)$. The eigenvalues denoted by $\lambda_i = \lambda_{iab}\tilde{\psi}^a_{1,bg}\tilde{\psi}^b_{1,bg}$ come into pairs of opposite sign. The determinant can be expanded, following steps similar to those presented is Section 1, in a sum as

$$\begin{aligned} \det{}^{\frac{1}{2}} \left(\frac{iR_{ij}/4}{i \sinh(R_{ij}/4)} \right) &= \prod_{i=1}^{d/2} \left(\frac{\lambda_i/4}{\sinh(\lambda_i/4)} \right) \\ &= 1 + \frac{1}{24}\frac{1}{8} R_{ijab}R_{jicd}\tilde{\psi}^a_{1,bg}\tilde{\psi}^b_{1,bg}\tilde{\psi}^c_{1,bg}\tilde{\psi}^d_{1,bg} + O(\tilde{\psi}^8_{1,bg}). \end{aligned} \tag{63}$$

In four dimensions only the second term of the sum survives upon integration over the Grassmann variables [13] and gives

$$An(chiral) = \frac{1}{8\pi^2}\frac{1}{24} \int \left(\prod_{i=1}^{d} dx_0^i \sqrt{g(x_0)} \right) Tr(R^2(x_0)) \tag{64}$$

where $Tr(R^2(x_0)) = \frac{1}{4} R_{ijab}(x_0)R_{jicd}(x_0)\epsilon^{abcd}$. This result is in perfect agreement with (21). Clearly there is a gravitational γ_{d+1} anomaly only in $d = 4k$ dimensions.

5. Conclusions

In this article we have shown an alternative way to compute the index of the Dirac operator for spin-$\frac{1}{2}$ fields in curved space-time. Using the path integral formalism for a one-dimensional supersymmetric model with a Weyl ordered Hamiltonian as a regulator, we derived the correct result predicted by Atiyah-Singer index theorem. The same scheme can also be applied to compute the index of other operators in Quantum Field Theory or String Theory [6].

[13] Recall that $\int d\theta_i = 0$ and $\int d\theta_i \theta_j = \frac{\partial \theta_j}{\partial \theta_i} = \delta_{ij}$.

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