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On the bi-Hamiltonian structure of Bogoyavlensky-Volterra systems

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Abstract

Results on the Volterra model which is associated to the simple Lie algebra of type A_n are extended to the Bogoyavlensky-Volterra systems of type B_n,C_n and D_n. In particular we find Lax pairs, Hamiltonian and Casimir functions and multi-Hamiltonian structures. Moreover, we investigate recursion operators, higher Poisson brackets and master symmetries. In additions we give, for the first time a bi-Hamiltonian formulation of the Volterra-B_n system using a negative recursion operator.

1. Introduction

The Volterra lattice is the system of o.d.e.'s

$$\frac{dv_i}{dt} = v_i\left(v_{i+1} - v_{i-1}\right) \ , \quad i = 1, 2, \ldots, n \ , \tag{1.1}$$

where $v_0 = v_{n+1} = 0$. These equations were studied originally by Volterra in ref.[17] to describe population evolution in a hierarchical system of competing species. The importance of this system derives from the fact that it can be considered as a discrete analogue of the Korteweg-de Vries equation. This system was solved by Kac and Van Moerbeke [10] using a discrete version of inverse scattering. There is also an explicit solution by Moser in [12]. Finally, Damianou [4] constructed Multi Hamiltonian structures and master symmetries for the system.

Bogoyavlensky in 1988 constructed dynamical systems connected with simple Lie algebras that generalize the Volterra system. In particular, the Volterra lattice (also known as KM system) is related to the root system of a simple Lie algebra of type A_n. For more details see ref.[1],[2].

In this paper we investigate the Bogoyavlensky-Volterra systems associated with the classical Lie algebras.

In Section 2 we describe the construction of the systems. We obtain the Bogoyavlensky-Volterra (BV) system for each classical Lie algebra $\mathcal{G}$.

In Section 3 we investigate the BV system of type B_{n+1}. We find a Lax-pair (L, B) for every $n \geq 2$. When n is even, we define two compatible brackets π_1, π_3 which define a recursion operator $\mathcal{R} = \pi_3 \pi_1^{-1}$. This recursion operator produces compatible Poisson brackets $\pi_{2j+1} = \mathcal{R}^j \pi_1$ and the constants of motion are in involution for every $j = 1, 2, 3, \ldots$. Finally, we give a bi-Hamiltonian formulation of the BV B_{n+1} system. In Section 4 we find master symmetries of the BV B_{n+1} system as well as the relations which they satisfy. We do not present the analogous results for the C_{n+1} system since it is equivalent to the B_{n+1} system.
In Section 5 we investigate the BV system of type D_{n+1}. We find again a Lax pair (L, B) for every $n \geq 4$ and, when n is odd, we define two compatible Poisson brackets π_1, π_3. We also describe the Hamiltonian formulation and compute the Casimirs. A bi-Hamiltonian formulation of the BV D_{n+1} system is still an open problem.

2. Definition of the systems

We now describe the construction of the generalized Volterra systems of Bogoyavlensky (see [1],[2]).
Let $\mathcal{G}$ be a simple Lie algebra (rank $\mathcal{G} = n$) and $\Pi = \{\omega_1, \omega_2, \ldots, \omega_n\}$ the Cartan-Weyl basis of simple roots in $\mathcal{G}$ (ref.[3]). There are unique positive integers k_i such that

$$k_0\omega_0 + k_1\omega_1 + \cdots + k_n\omega_n = 0 \ , \tag{2.1}$$

where $k_0 = 1$ and ω_0 is the minimal negative root.
We consider the following Lax pairs:

$$\begin{aligned} \dot{L} &= [B, L] \ , \\ L(t) &= \sum_{i=1}^{n} b_i(t)\, e_{\omega_i} + e_{\omega_0} + \sum_{1 \leq i < j \leq n} \left[e_{\omega_i}, e_{\omega_j}\right] \ , \\ B(t) &= \sum_{i=1}^{n} \frac{k_i}{b_i(t)} e_{-\omega_i} + e_{-\omega_0} \ . \end{aligned} \tag{2.2}$$

Let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{G}$. For every root $\omega_a \in \mathcal{H}^*$ there is a unique $H_{\omega_a} \in \mathcal{H}$ such that $\omega(h) = k(H_{\omega_a}, h)\ \forall\ h \in \mathcal{H}$, where k is the Killing form and $\mathcal{H}^*$ is the dual space of $\mathcal{H}$. We also have an inner product on $\mathcal{H}^*$such that $\langle \omega_a, \omega_b \rangle = k(H_{\omega_a}, H_{\omega_b})$.
We set

$$c_{ij} = \begin{cases} 1 & \text{if } \langle \omega_i, \omega_j \rangle \neq 0 \text{ and } i < j \\ 0 & \text{if } \langle \omega_i, \omega_j \rangle = 0 \text{ or } i = j \\ -1 & \text{if } \langle \omega_i, \omega_j \rangle \neq 0 \text{ and } i > j \end{cases} \tag{2.3}$$

The vector equation (2.2) is equivalent to the dynamical system

$$\dot{b}_i = -\sum_{j=1}^{n} \frac{k_j c_{ij}}{b_j} \ . \tag{2.4}$$

We determine the skew-symmetric variables

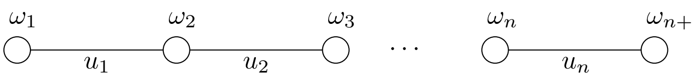
$$x_{ij} = c_{ij} b_i^{-1} b_j^{-1}, \quad x_{ji} = -x_{ij}, \quad x_{jj} = 0 \ , \tag{2.5}$$

which correspond to the edges of the Dynkin diagram for the Lie algebra $\mathcal{G}$, connecting the vertices ω_i and ω_j.
The dynamical system (2.4) in the variables x_{ij} takes the form

$$\dot{x}_{ij} = x_{ij} \sum_{s=1}^{n} k_s \left(x_{is} + x_{js} \right) \ . \tag{2.6}$$

We recall that the vertices ω_i, ω_j of the Dynkin diagram are joined by edges only if $\langle \omega_i, \omega_j \rangle \neq 0$. Hence $x_{ij} = 0$ if there are no edges connecting the vertices ω_i and ω_j of the diagram. We call the equations (2.6) Bogoyavlensky-Volterra system associated with $\mathcal{G}$ (BV system for short).
We shall now describe the BV system for each simple Lie algebra $\mathcal{G}$. The number of independent variables $x_{ij}(t)$ is equal to $n-1$ and is one less than the number of variables $b_j(t)$. We use the standard numeration of vertices of the Dynkin diagram and define the variables $u_k(t) = x_{ij}(t)$ corresponding to the edges of the Dynkin diagram with increasing order of the vertices ($i<j$).
The phase space consists of variables u_i , $1 \le i \le n$, with $u_i > 0$.

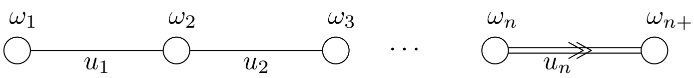
$\mathbf{A}_{n+1}$



$\omega_0 = -\left(\omega_1 + \omega_2 + \cdots \omega_{n+1}\right) \qquad k_i = 1 \qquad i = 1, \ldots, n+1$

$$\dot{u}_1 = u_1 u_2, \quad \dot{u}_i = u_i(u_{i+1} - u_{i-1}) \ 2 \le i \le n-1, \quad \dot{u}_n = -u_{n-1} u_n \qquad (\mathbf{BV\ A_{n+1}})$$

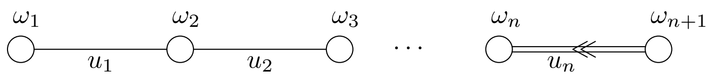
$\mathbf{B}_{n+1}$



$\omega_0 = -\left(\omega_1 + 2\omega_2 + \cdots + 2\omega_{n+1}\right) \qquad k_1 = 1, \ k_i = 2 \quad i = 2, \ldots, n+1$

$$\dot{u}_1 = u_1 \left(u_1 + 2u_2\right), \quad \dot{u}_2 = u_2 \left(2u_3 - u_1\right), \ \dot{u}_n = -2u_{n-1} u_n \qquad (\mathbf{BV\ B_{n+1}})$$
$$\dot{u}_i = 2u_i(u_{i+1} - u_{i-1}) \quad 3 \le i \le n-1$$

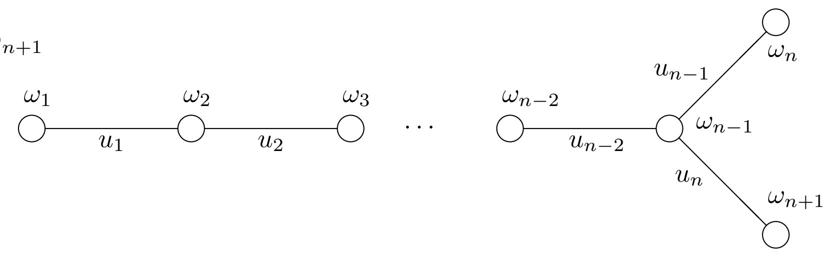
$\mathbf{C}_{n+1}$



$\omega_0 = -(2\omega_1 + \cdots + 2\omega_n + \omega_{n+1}) \qquad k_i = 2 \quad i = 1, \ldots, n \ , \ k_{n+1} = 1$

$$\begin{gathered} \dot{u}_1 = 2u_1u_2, \quad \dot{u}_i = 2u_i(u_{i+1} - u_{i-1}) \ \ 2 \leq i \leq n-2 \\ \dot{u}_{n-1} = u_{n-1}(u_n - 2u_{n-2}), \quad \dot{u}_n = -u_n(u_n + 2u_{n-1}) \end{gathered} \tag{BV $\mathbf{C_{n+1}}$}$$

$\mathbf{D}_{n+1}$



$\omega_0 = -(\omega_1 + 2\omega_2 + \cdots + 2\omega_{n-1} + \omega_n + \omega_{n+1})$
$k_1 = 1 \ , \ k_n = 1 \ , \ k_{n+1} = 1 \ \ , \ k_i = 2 \quad i = 2, \ldots, n-1$

$$\begin{gathered} \dot{u}_1 = u_1(2u_2 + u_1) \ , \ \dot{u}_2 = u_2(2u_3 - u_1) \ , \\ \dot{u}_i = 2u_i(u_{i+1} - u_{i-1}) \quad 3 \leq i \leq n-3 \ , \ \dot{u}_{n-2} = u_{n-2}(u_n + u_{n-1} - 2u_{n-3}) \ , \\ \dot{u}_{n-1} = u_{n-1}(u_n - u_{n-1} - 2u_{n-2}) \ , \ \dot{u}_n = -u_n(u_n - u_{n-1} + 2u_{n-2}) \ . \end{gathered} \tag{BV $\mathbf{D_{n+1}}$}$$

3. The BV $\mathbf{B_{n+1}}$ system and its Poisson bracket

Recall the *BV* B_{n+1} system $(u_i > 0)$.

$$\begin{aligned} \dot{u}_1 &= u_1(u_1 + 2u_2), \ \dot{u}_2 = u_2(2u_3 - u_1), \ \dot{u}_n = -2u_{n-1}u_n \ . \\ \dot{u}_i &= 2u_i(u_{i+1} - u_{i-1}) \qquad i = 3, \ldots, n-1 \ . \end{aligned} \tag{3.1}$$

We rescale the coordinates

$$v_1 = u_1 \ \ , \ \ v_i = 2u_i \qquad i = 2, \ldots, n \ , \tag{3.2}$$

to obtain the equivalent system

$$\dot{v}_1 = v_1(v_1 + v_2), \ \dot{v}_i = v_i(v_{i+1} - v_{i-1}), \ \dot{v}_n = -v_{n-1}v_n \quad i = 2, \ldots, n-1. \tag{3.3}$$

Before giving the Lax pair for the system (3.3) we introduce some matrix notations:

$$
\begin{aligned}
X_i &= \begin{pmatrix} \sqrt{v_i} & 0 \\ 0 & i\sqrt{v_i} \end{pmatrix}, \; O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\
Y_i &= \frac{1}{2}\begin{pmatrix} \sqrt{v_i v_{i+1}} & 0 \\ 0 & \sqrt{v_i v_{i+1}} \end{pmatrix}, \; Y_0 = \frac{i}{2}\begin{pmatrix} 0 & v_1 \\ -v_1 & 0 \end{pmatrix}.
\end{aligned} \tag{3.4}
$$

It turns out the equations (3.3) are equivalent to the Lax pair $\dot{L} = [L, B]$, where L, B are $(n+1)\times(n+1)$ matrices

$$
L = \begin{bmatrix}
0 & 0 & \cdots & 0 & \sqrt{v_1} & i\sqrt{v_1} \\
0 & O & X_n & O & \cdots & O \\
\vdots & X_n & O & \ddots & \ddots & \vdots \\
0 & O & \ddots & \ddots & X_3 & O \\
\sqrt{v_1} & \vdots & \ddots & X_3 & O & X_2 \\
i\sqrt{v_1} & O & \cdots & O & X_2 & O
\end{bmatrix}, \tag{3.5}
$$

$$
B = \begin{bmatrix}
0 & \cdots & \cdots & 0 & -\frac{1}{2}\sqrt{v_1 v_2} & -\frac{1}{2}\sqrt{v_1 v_2} & 0 & 0 \\
\vdots & O & O & Y_{n-1} & O & \cdots & \cdots & O \\
\vdots & O & O & O & \ddots & \ddots & & \vdots \\
0 & -Y_{n-1} & O & O & \ddots & Y_4 & O & \vdots \\
\frac{1}{2}\sqrt{v_1 v_2} & O & \ddots & \ddots & \ddots & O & Y_3 & O \\
\frac{1}{2}\sqrt{v_1 v_2} & \vdots & \ddots & -Y_4 & O & O & O & Y_2 \\
0 & \vdots & & O & -Y_3 & O & O & O \\
0 & O & \cdots & \cdots & O & -Y_2 & O & Y_0
\end{bmatrix}.
$$

The functions $H_{2k} = \frac{1}{2k} Tr(L^{4k})$, $k = 1, 2, \ldots$ are constants of motion for the system. We use the old variables b_j appearing in the equations (2.4) in order to find a cubic bracket π_3 for the system. The equations (2.4) in the case of the Lie algebra B_{n+1} become

$$
\begin{aligned}
\dot{b}_1 &= -2b_2^{-1}, \quad \dot{b}_2 = -2b_3^{-1} + b_1^{-1}, \quad \dot{b}_{n+1} = 2b_n^{-1} \\
\dot{b}_j &= -2(b_{j+1}^{-1} - b_{j-1}^{-1}) \qquad j = 3, \ldots, n
\end{aligned} \tag{3.6}
$$

The dynamical system (3.6) can be written in Hamiltonian form $\dot{b}_j = \{b_j, H\}$, with Hamiltonian $H = \log b_1 + 2\sum_{j=2}^{n+1} \log b_j$ and a constant Poisson bracket

$$
\{b_j, b_{j+1}\} = -\{b_{j+1}, b_j\} = 1 \quad , \text{for } j = 1, 2, \ldots, n\ . \tag{3.7}
$$

All other brackets are zero. In terms of the variables v_j ($v_1 = b_1^{-1}b_2^{-1}$, $v_k = 2b_k^{-1}b_{k+1}^{-1}$, $k = 2, \ldots, n$) the above skew-symmetric bracket, which we denote by π_3, is given by

$$\begin{aligned}
\{v_1, v_2\} &= v_1 v_2 (2v_1 + v_2) \\
\{v_i, v_{i+1}\} &= v_i v_{i+1} (v_i + v_{i+1}), \ i = 2, \ldots, n-1 \\
\{v_i, v_{i+2}\} &= v_i v_{i+1} v_{i+2}, \ i = 1, \ldots, n-2 ,
\end{aligned} \tag{3.8}$$

and all other brackets are zero.
Suppose that n is even ($n = 2l$) and we look for a bracket π_1 which satisfies

$$\pi_3 \nabla H_2 = \pi_1 \nabla H_4.$$

We define the skew-symmetric matrix

$$\omega = \begin{pmatrix}
0 & -\frac{1}{v_1} & \cdots & -\frac{1}{v_1} & -\frac{1}{v_1} & -\frac{1}{v_1} \\
\frac{1}{v_1} & 0 & -\frac{1}{v_2} & \cdots & -\frac{1}{v_2} & -\frac{1}{v_2} \\
\vdots & \frac{1}{v_2} & \ddots & \ddots & \vdots & \vdots \\
\frac{1}{v_1} & \vdots & \ddots & 0 & -\frac{1}{v_{n-2}} & -\frac{1}{v_{n-2}} \\
\frac{1}{v_1} & \frac{1}{v_2} & \cdots & \frac{1}{v_{n-2}} & 0 & -\frac{1}{v_{n-1}} \\
\frac{1}{v_1} & \frac{1}{v_2} & \cdots & \frac{1}{v_{n-2}} & \frac{1}{v_{n-1}} & 0
\end{pmatrix}, \tag{3.9}$$

and we define $\pi_1 = \omega^{-1}$ (i.e. $\{v_i, v_j\}_{\pi_1} = \left(\omega^{-1}\right)_{ij}$).

Theorem 3.1 *The brackets π_1, π_3 satisfy:*
(i) π_1 , π_3 *are Poisson.*
(ii) The function $\frac{1}{4}H_2 = \frac{1}{8}Tr(L^4) = \sum_{i=2}^{n}\left(\frac{1}{2}v_i^2 + v_{i-1}v_i\right)$ *is the Hamiltonian of the BV* B_{n+1} *system with respect to the bracket* π_1.
(iii) π_1 , π_3 *are compatible.*

Proof. (i) Changing variables in the Poisson tensor (3.7) preserves the Jacobi identity and therefore π_3 is a Poisson bracket.

In order to prove that π_1 is a Poisson bracket we consider the 2-form

$$\omega = \frac{1}{2}\sum_{i,j=1}^{n} \omega_{ij} dv_i \wedge dv_j = \sum_{1 \le i < j \le n} -\frac{1}{v_i} dv_i \wedge dv_j \ . \tag{3.10}$$

Since the 2-form ω is closed, (i.e. $d\omega = 0$), $\pi_1 = \omega^{-1}$ satisfies the Jacobi identity (see [15], page 11) and therefore π_1 is Poisson.
(ii) follows from simple calculations.

(*iii*) It is well-known, see [5], that if a Poisson tensor is a Lie derivative of another, then the two tensors are compatible. We will see later, in the next section, that π_3 is the Lie derivative of π_1 in the direction of a master symmetry and this fact makes π_1 ,π_3 compatible. □

Finally, we define a sequence of Poisson brackets π_{2j-1} , $j = 1, 2, \dots$ which are compatible and the constants of motion are in involution with respect to each π_{2j-1}. Since the 2-tensor π_1 is invertible we can define the recursion operator $R = \pi_3 \pi_1^{-1}$. We define the higher order Poisson tensors

$$\pi_{2j+1} = R^j \pi_1 \ , \ j = 1, 2, \dots \tag{3.11}$$

Using standard theory of recursion operators [5], [11], [14] we obtain the following theorem.

Theorem 3.2 *The sequence of higher Poisson tensors and invariants satisfy:*
(*i*) $\pi_{2j+1} \nabla H_{2i} = \pi_{2j-1} \nabla H_{2i+2}$, $\quad \forall \ i \ , j$.
(*ii*) H_{2i} *are in involution with respect to all Poisson brackets.*
(*iii*) π_{2j+1} *are all compatible Poisson brackets.*

To define a bi-Hamiltonian formulation of the $BV \ B_{n+1}$ system ($n = 2l$) we use an idea due to Damianou [6].
We define the inverse of the recursion operator R

$$\begin{aligned} N &= R^{-1} = \pi_1 \pi_3^{-1} \\ \pi_{-1} &= N \pi_1 = \pi_1 \pi_3^{-1} \pi_1. \end{aligned}$$

Then the poisson bracket π_{-1} satisfies:

$$\pi_{-1} \nabla H_4 = \pi_1 \nabla H_2.$$

Therefore the $BV \ B_{n+1}$ system has a bi-Hamiltonian formulation.
We give an example of the bracket π_{-1} for $n = 4$. First we define the skew-symmetric matrix A by

$$\begin{aligned} a_{12} &= v_1^2 v_3 \left(v_3^2 + v_4^2 + 2v_2 v_3 + 2v_3 v_4 \right) \\ a_{23} &= v_1 v_3^2 \left(v_3^2 + v_4^2 + 2v_1 v_2 + v_2 v_3 + 2v_3 v_4 \right) \\ a_{34} &= v_1 v_3^2 \left(v_3^2 + v_2^2 + 2v_1 v_2 + 2v_2 v_3 + v_3 v_4 \right) \\ a_{24} &= -v_1 v_3^2 \left(v_2^2 + v_3^2 + v_4^2 + 2v_1 v_2 + 2v_2 v_3 + 2v_3 v_4 + v_2 v_4 \right) \\ a_{13} &= -v_1^2 v_3^2 v_2^{-1} \left(2v_2^2 + v_3^2 + v_4^2 + 2v_1 v_2 + 2v_2 v_3 + 2v_3 v_4 \right) \\ a_{14} &= v_1^2 v_3^2 v_2^{-1} \left(v_2^2 + v_3^2 + v_4^2 + 2v_1 v_2 + 2v_2 v_3 + 2v_3 v_4 \right). \end{aligned}$$

The matrix of the tensor π_{-1} is defined by $\pi_{-1} = \frac{1}{d} A$ where

$$d = \sqrt{\det \pi_3} = v_1 v_2 v_3 v_4 \left(2v_1 v_3 + 2v_1 v_4 + v_2 v_4 \right).$$

More generaly, we define

$$\pi_{-(2j+1)} = N^j \pi_{-1} \qquad j = 1, 2, 3, \dots$$

and we obtain a multi-Hamiltonian formulation

$$\pi_1 \nabla H_2 = \pi_{-1} \nabla H_4 = \pi_{-3} \nabla H_6 = \cdots .$$

Remark 1: The Poisson bracket π_3 is invertible since $det\pi_3$ is equal to the product of $\sqrt{\det \pi_1}$ with the non–zero eigenvalues of L.
Remark 2: Since the functions $H_2, H_4, \dots, H_{2l}$ are independent and in involution the $BV\ B_{2l+1}$ system is integrable.

4. Master symmetries of the BV $\mathbf{B_{n+1}}$ system

The master symmetries were used to generate nonlinear Poisson brackets and higher order invariants. For the definition and examples of master symmetries see [7], [8], [9], [13], [14], [16]. In this section we find master symmetries for the system (3.3) and derive the relations which they satisfy.
We consider

$$\begin{aligned} \pi_3 \quad &= \quad v_1 v_2 \left(2v_1 + v_2\right) \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2} + \sum_{i=2}^{n-1} v_i v_{i+1} \left(v_i + v_{i+1}\right) \frac{\partial}{\partial v_i} \wedge \frac{\partial}{\partial v_{i+1}} \\ &+ \sum_{i=2}^{n-2} v_i v_{i+1} v_{i+2} \frac{\partial}{\partial v_i} \wedge \frac{\partial}{\partial v_{i+2}} \ , \quad \pi_1^{-1} = \sum_{1 \le i < j \le n} -\frac{1}{v_i} dv_i \wedge dv_j . \end{aligned}$$

The recursion operator is then

$$R = \pi_3 \pi_1^{-1} = \sum_{i,j=1}^{n} \alpha_{ij} dv_j \otimes \frac{\partial}{\partial v_i} \ , \tag{4.1}$$

We now prove that π_1 and π_3 are compatible. It is enough to show that $\pi_3 = L_{Z_1} \pi_1$ for some vector field Z_1. We define

$$Z_1 = R\left(Z_0\right) = \left(\sum_{i,j} \alpha_{ij} dv_j \otimes \frac{\partial}{\partial v_i} \right) \ (Z_0) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} v_j a_{ij} \right) \frac{\partial}{\partial v_i} \ ,$$

where Z_0 is the Euler vector field

$$Z_0 = \sum_{i=1}^{n} v_i \frac{\partial}{\partial v_i} \ . \tag{4.2}$$

Using the formula

$$\{f,g\}_{L_X\pi} = X\{f,g\}_\pi - \{f,X(g)\}_\pi - \{X(f),g\}_\pi \tag{4.3}$$

it is easy to check that

$$L_{Z_1}(\pi_1) = -3\pi_3 \ , \tag{4.4}$$

and therefore π_3 is the Lie-derivative of π_1 in the direction of the vector field Z_1. This makes π_1 compatible with π_3 and completes the proof of Theorem 1.
Using the recursion operator we generate the master symmetries

$$Z_i = R^i Z_0 \ . \tag{4.5}$$

One calculates that

$$L_{Z_0}(\pi_1) = -\pi_1 \ , \ L_{Z_0}(\pi_3) = \pi_3 \ , \ L_{Z_0}(H_2) = 2H_2 \ . \tag{4.6}$$

Therefore Z_0 is a conformal symmetry for π_1 , π_3, and H_2. According to a theorem of Oevel [14] we end up with the following deformation relations:

$$\begin{aligned}[][Z_i, X_j] = (1+2j)\,X_{i+j}, \ \ [Z_i, Z_j] = 2\,(j-i)\,Z_{i+j}, \ L_{Z_i}(\pi_{2j+1}) \\ = (2j-2i-1)\,\pi_{2(i+j)+1}\end{aligned} \tag{4.7}$$

where $X_1 = \pi_3 dH2 = \pi_1 dH4$ and $X_i = R^{i-1}X_1$. We also have

$$Z_i(H_{2j}) = 2\,(i+j)\,H_{2(i+j)} \ . \tag{4.8}$$

We will not present the results for the $BV\ C_{n+1}$ system. In fact the $BV\ C_{n+1}$ system is equivalent to the $BV\ B_{n+1}$ system through the transformation

$$u_1 \longmapsto -u_n \ , \ u_2 \longmapsto -u_{n-1} \ , \ \ldots \ , \ u_{n-1} \longmapsto -u_2 \ , \ u_n \longmapsto -u_1 \ . \tag{4.9}$$

5. The BV $\mathbf{D_{n+1}}$ system and its Poisson bracket.

We recall the $BV\ D_{n+1}$ system $(u_i > 0)$.

$$\begin{aligned}
\dot{u}_1 &= u_1(u_1+2u_2), \quad \dot{u}_2 = u_2(2u_3-u_1), \\
\dot{u}_i &= 2u_i(u_{i+1}-u_{i-1}) \qquad i=3,\ldots,n-3 \\
\dot{u}_{n-2} &= u_{n-2}(u_n+u_{n-1}-2u_{n-3}), \ \dot{u}_{n-1} = u_{n-1}(u_n-u_{n-1}-2u_{n-2}), \\
\dot{u}_n &= -u_n(u_n-u_{n-1}+2u_{n-2}) \ .
\end{aligned} \tag{5.1}$$

We make a linear transformation

$$v_1 = u_1 \ \ , \ \ v_i = 2u_i \quad i=2,\ldots,n-2 \ \ , \ \ v_{n-1} = u_{n-1} \ \ , \ \ v_n = u_n \ , \tag{5.2}$$

to obtain the equivalent system

$$\begin{aligned} \dot v_1 &= v_1\,(v_1+v_2)\,, \quad \dot v_i \;= v_i(v_{i+1}-v_{i-1}) \qquad i=2,\dots n-3 \\ \dot v_{n-2} &= v_{n-2}(v_n+v_{n-1}-v_{n-3}),\ \dot v_{n-1}=v_{n-1}(v_n-v_{n-1}-v_{n-2}), \\ \dot v_n &= -v_n(v_n-v_{n-1}+v_{n-2})\ . \end{aligned} \tag{5.3}$$

We consider again the 2×2 matrices which were defined in (3.4) and we also set

$$X=\begin{pmatrix} \sqrt{v_n} & i\sqrt{v_n} \\ -\sqrt{v_{n-1}} & i\sqrt{v_{n-1}} \end{pmatrix},\ Y=\frac{1}{2}\begin{pmatrix} \sqrt{v_{n-2}v_n} & \sqrt{v_{n-2}v_n} \\ -\sqrt{v_{n-2}v_{n-1}} & \sqrt{v_{n-2}v_{n-1}} \end{pmatrix}, \tag{5.4}$$
$$W=\frac{i}{2}\begin{pmatrix} 0 & v_{n-1}-v_n \\ v_n-v_{n-1} & 0 \end{pmatrix}.$$

Equations (5.3) can be written in a Lax Pair form $\dot L=[L,B]$, where

$$L=\begin{bmatrix} 0 & 0 & \cdots & 0 & \sqrt{v_1} & i\sqrt{v_1} \\ 0 & O & X & O & \cdots & O \\ \vdots & X^t & O & X_{n-2} & \ddots & \vdots \\ 0 & O & X_{n-2} & \ddots & \ddots & O \\ \sqrt{v_1} & \vdots & \ddots & \ddots & O & X_2 \\ i\sqrt{v_1} & O & \cdots & O & X_2 & O \end{bmatrix}, \tag{5.5}$$

$$B=\begin{bmatrix} 0 & \cdots & \cdots & 0 & -\frac12\sqrt{v_1v_2} & -\frac12\sqrt{v_1v_2} & 0 & 0 \\ \vdots & O & O & Y & O & \cdots & \cdots & O \\ \vdots & O & W & O & Y_{n-3} & \ddots & & \vdots \\ 0 & -Y^t & O & O & \ddots & \ddots & O & \vdots \\ \frac12\sqrt{v_1v_2} & O & -Y_{n-3} & \ddots & \ddots & O & Y_3 & O \\ \frac12\sqrt{v_1v_2} & \vdots & \ddots & \ddots & O & O & O & Y_2 \\ 0 & \vdots & & O & -Y_3 & O & O & O \\ 0 & O & \cdots & \cdots & O & -Y_2 & O & Y_0 \end{bmatrix}.$$

The invariant polynomials of this system are given by the functions

$$H_2, H_4, \dots, H_{n-1} \quad \text{when } n \text{ is odd}\ ,$$
$$H_2, H_4, \dots, H_{n-2}, H_{n-1} \quad \text{when } n \text{ is even}\ ,$$

where $H_k = \frac{1}{k} Tr(L^{2k})$.
As in the case of the BV B_{n+1} system we use the variables b_j , $1 \le j \le n+1$ of the equations (2.4) in order to find a cubic bracket π_3 of the BV D_{n+1} system. The dynamical system (2.4) in the case of the Lie algebra of type D_{n+1} can be written in Hamiltonian form $\dot{b}_j = \{b_j, H\}$, with Hamiltonian

$$H = \log b_1 + 2\sum_{j=2}^{n-1} \log b_j + \log b_n + \log b_{n+1} \ , \tag{5.6}$$

and Poisson bracket

$$\begin{aligned} \{b_j, b_{j+1}\} &= -\{b_{j+1}, b_j\} = 1 \ \ , \text{for } j = 1, 2, \ldots, n-1 \\ \{b_{n-1}, b_{n+1}\} &= -\{b_{n+1}, b_{n-1}\} = 1; \end{aligned} \tag{5.7}$$

all other brackets are zero. In the new variables v_j $\;(v_1 = b_1^{-1} b_2^{-1},\ v_k = 2b_k^{-1} b_{k+1}^{-1},$ $k = 2, \ldots, n-2,\ \ v_{n-1} = b_{n-1}^{-1} b_n^{-1}\ ,\ v_n = b_{n-1}^{-1} b_{n+1}^{-1}\ )$ the above skew-symmetric bracket, which we denote by π_3, is given by

$$\begin{aligned} \{v_1, v_2\} &= v_1 v_2 \left(2v_1 + v_2\right) \\ \{v_i, v_{i+1}\} &= v_i v_{i+1} \left(v_i + v_{i+1}\right)\ ,\ i = 2, \ldots, n-3 \\ \{v_{n-2}, v_{n-1}\} &= v_{n-2} v_{n-1} \left(2v_{n-1} + v_{n-2}\right) \\ \{v_{n-1}, v_n\} &= 2v_{n-1} v_n \left(v_n - v_{n-1}\right) \\ \{v_i, v_{i+2}\} &= v_i v_{i+1} v_{i+2}\ ,\ i = 1, \ldots, n-3 \\ \{v_{n-2}, v_n\} &= v_{n-2} v_n \left(v_{n-2} + 2v_n\right) \\ \{v_{n-3}, v_n\} &= v_{n-3} v_{n-2} v_n\ . \end{aligned} \tag{5.8}$$

All other brackets are zero. As in the case of KM system we suppose that n is odd ($n = 2l+1$) and we look again for a bracket π_1 which satisfies $\pi_3 \nabla H_2 = \pi_1 \nabla H_4$.
We define

$$\tau_{ij} = -\tau_{ji} = v_{2i-1} \prod_{k=i}^{j-1} \frac{v_{2k+1}}{v_{2k}} \ \text{ for } i < j \ \ ,\ \ \tau_{ii} = v_{2i-1}\ , \tag{5.9}$$

and we let π_1 be the bracket which is defined as follows:

$$\begin{aligned} \{v_i, v_j\} &= (-1)^{i+j-1} \tau_{[\frac{i}{2}]+1, [\frac{j+1}{2}]} \ \text{ for } \ 1 \le i < j \le n-2\ , \\ \{v_i, v_{n-1}\} &= \{v_i, v_n\} = \frac{(-1)^{i+n}}{2} \tau_{[\frac{i}{2}]+1, [\frac{n}{2}]} \ \text{ for } \ \ i = 1, \ldots, n-2\ , \\ \{v_{n-1}, v_n\} &= -\{v_n, v_{n-1}\} = \frac{1}{2}\left(v_n - v_{n-1}\right)\ . \end{aligned} \tag{5.10}$$

We obtain the following Theorem:

Theorem 5.3 *(i)* π_1 , π_3 *are Poisson.*
(ii) The function

$$\frac{1}{4}H_2 = \frac{1}{8}Tr(L^4) = v_{n-2}v_n + 2v_{n-1}v_n + \sum_{i=1}^{n-2} v_i v_{i+1} + \frac{1}{2}\sum_{i=2}^{n-2} v_i^2 \ ,$$

is the Hamiltonian of the BV D_{n+1} *system with respect to the bracket* π_1.
(iii) The function

$$h_n = (v_n - v_{n-1}) \prod_{i=1}^{n-2} v_i \ ,$$

is the Casimir of the BV D_{n+1} *system in the bracket* π_1.
(iv) π_1 , π_3 *are compatible.*

Proof. *(i)* We denote $\{\}_d$ the bracket π_1 of $BV\ D_{n+1}$ system and $\{\}_b$ the Poisson bracket π_1 of $BV\ B_n$ system $(n = 2l+1)$. Then $\{\}_d$ can be defined as follows:

$$\begin{aligned}
\{v_i, v_j\}_d &= \{v_i, v_j\}_b \ , \quad 1 \le i, j \le n-2 \\
\{v_i, v_{n-1}\}_d &= \{v_i, v_n\}_d = \frac{1}{2}\{v_i, v_{n-1}\}_b \, , \quad 1 \le i \le n-2 \\
\{v_{n-1}, v_n\}_d &= \frac{1}{2}(v_n - v_{n-1}) \, .
\end{aligned}$$

We set

$$[v_i, v_j, v_k] = \{v_i, \{v_j, v_k\}\} + \{v_j, \{v_k, v_i\}\} + \{v_k, \{v_i, v_j\}\}$$

For $i, j, k = 1, 2, \ldots, n-2$

$$[v_i, v_j, v_k]_d = [v_i, v_j, v_k]_b = 0 \ .$$

For $i, j = 1, 2, \ldots, n-2$

$$[v_i, v_j, v_{n-1}]_d = [v_i, v_j, v_n]_d = \frac{1}{2}[v_i, v_j, v_{n-1}]_b = 0 \ .$$

For $i = 1, 2, \ldots, n-2$

$$\begin{aligned}
[v_i, v_{n-1}, v_n]_d &= \{v_i, \{v_{n-1}, v_n\}_d\}_d + \{v_{n-1}, \{v_n, v_i\}_d\}_d + \{v_n, \{v_i, v_{n-1}\}_d\}_d \\
&= \frac{1}{2}\{v_i, v_n - v_{n-1}\}_d + \frac{1}{2}\{v_{n-1}, \{v_{n-1}, v_i\}_b\}_d + \frac{1}{2}\{v_n, \{v_i, v_{n-1}\}_b\}_d \\
&= \frac{1}{2}\{v_i, v_{n-1} - v_{n-1}\}_b - \frac{1}{4}\{v_{n-1}, \{v_i, v_{n-1}\}_b\}_b + \frac{1}{4}\{v_{n-1}, \{v_i, v_{n-1}\}_b\}_b \\
&= 0 \ .
\end{aligned}$$

Therefore, $\{\}_d$ is Poisson. Relation (5.7) implies that π_3 is Poisson as well.

(ii) , (iii) follow from simple calculations.

(iv) The proof that the bracket $\pi_1 + \pi_3$ is Poisson is similar to the above proof that the $\{\}_d$ is Poisson. □

Remark: Since the functions $H_2, H_4, \ldots, H_{n-2}, h_n$ are independent and in involution (when n is even, $n = 2l$) the $BV\ D_{2l+1}$ system is integrable using the bracket π_3 and since the functions $H_2, H_4, \ldots, H_{n-1}, h_n$ are independent and in involution (when n is odd, $n = 2l - 1$) the $BV\ D_{2l}$ system is integrable using the brackets π_1 and π_3 .

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