BULLETIN OF THE
GREEK MATHEMATICAL SOCIETY
Volume 51, 2005 (67–76)

The Kowalevski Top

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Abstract

We present an alternative derivation of some well–known results of Sophia Kowalevski concerning the top that bares her name. To this end we use the same ideas of the bi–Hamiltonian approach to the theory of separation of variables for the solution of the Hamilton–Jacobi equation.

1. Introduction

In 1899, in a renowned paper [1], Sophia Kowalevski identified a third integrable case in the dynamics of tops, and she was able to solve the equations of motion by using the technique of separation of variables. If one denotes by (L_1, L_2, L_3) the components of the angular momentum of the top along the principal axes of inertia, and by $(\gamma_1, \gamma_2, \gamma_3)$ the components of the gravity along the same axes, the equations of motion of the top read:

$$\begin{aligned}
\dot{L}_1 &= \frac{1}{2}L_2L_3 \\
\dot{L}_2 &= -\frac{1}{2}L_1L_3 - c_0\gamma_3 \\
\dot{L}_3 &= c_0\gamma_2 \\
\dot{\gamma}_1 &= L_3\gamma_2 - \frac{1}{2}L_2\gamma_3 \\
\dot{\gamma}_2 &= \frac{1}{2}L_1\gamma_3 - L_3\gamma_1 \\
\dot{\gamma}_3 &= \frac{1}{2}L_2\gamma_1 - \frac{1}{2}L_1\gamma_2
\end{aligned}$$

where c_0 is a constant related to the position of the center of mass of the top in its equatorial plane. These equations are Hamiltonian with respect to the Poisson bracket defined by

$$\begin{aligned}\{L_1, L_2\} &= -L_3\\ \{L_1, \gamma_2\} &= -\gamma_3\\ \{\gamma_1, \gamma_2\} &= 0\end{aligned}$$

and their cyclic permutations. This bracket is nowadays known and the Lie–Poisson bracket on the dual of the Lie algebra of the group of isometries of the three–dimensional Euclidean space. It has two Casimir functions given by

$$2l = L_1\gamma_1 + L_2\gamma_2 + L_3\gamma_3$$

$$a = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 \ .$$

The Hamiltonian of the Kowalevski top is

$$6\,l_1 = \frac{L_1^2}{2} + \frac{L_2^2}{2} + L_3^2 - 2c_0\gamma_1 \ .$$

The first remarkable result of Kovalevski was that the function

$$16k^2 = (L_1^2 - L_2^2 + 4c_0\gamma_1)^2 + (2L_1L_2 + 4c_0\gamma_2)^2$$

is an additional constant of motion of the top. Since the flow takes place on the four–dimensional symplectic leaves defined by the two Casimir functions, this result allows to conclude that the Kowalevski top is an integrable system. The effective solution of the equations of motion remains, notwithstanding a formidable problem. The second truly remarkable result of Kowalevski was to show that the zeros of the following polynomial

$$(x_1 - x_2)^2(s - \frac{1}{2}l_1)^2 - R(x_1, x_2)(s - \frac{1}{2}l_1) - \frac{1}{4}R_1(x_1, x_2) = 0$$

where

$$x_1 = \frac{1}{2}(L_1 + iL_2) \qquad x_2 = \frac{1}{2}(L_1 - iL_2)$$

and

$$\begin{aligned}R(x_1x_2) &= -x_1^2x_2^2 + 6l_1x_1x_2 + 2lc_0(x_1 + x_2) + c_0^2a - k^2\\ R_1(x_1x_2) &= -6l_1x_1^2x_2^2 - (c_0^2a - k^2)(x_1 + x_0)^2 - alc_0(x_1 + x_2)x_1x_2 + cl_1(c_0^2a - k^2)\\ &\quad -4l^2c_0^2\\ R(x_1) &= R(x_1, x_1)\\ R(x_2) &= R(x_2, x_2)\end{aligned}$$

are separation coordinates for the given equations. This means that, on each symplectic leaf, the Hamilton–Jacobi equation associated with the restriction of the top on the leaf admits a complete integral which is additively separated in those coordinates. The general solution of the equations of motion can then be found explicitly according to a well–known theorem of Jacobi. In the paper of Kowalevski the discovery of these coordinates is the fruit of divine intuition. The polynomial is brought out from nothing and nobody so far has been able to provide convincing arguments in support of this discovery. In this paper we try to attack this question from the point of view of the bi–Hamiltonian approach to the separation of the Hamilton–Jacobi equation. The paper is a preliminary report of work in progress. Accordingly we allow ourselves to omit all the proofs, concentrating our attention only on the main leading ideas.

2. The classical theory of separation of variables

In this section we present the geometric view of the classical theory of separation of variables due to Jacobi and Levi Civita. For the sake of presentation we consider the classical case of a Hamiltonian vector field X_H on a cotangent bundle T^*Q, which is separable by a change of coordinates in the configuration space Q. We notice, however, that all the results reached in this case can be extended to general Hamiltonian systems on Poisson manifolds, like the Kowalevski top. The assumption of separability means the following. According to classical mechanics a basic task in the study of a given Hamiltonian system is the computation of a complete integral of the Hamilton–Jacobi equation

$$H\left(x_1,\dots,x_n,\frac{\partial W}{\partial x_1},\dots,\frac{\partial W}{\partial x_n}\right)=E\ .$$

This computation is usually very hard and depends on the choice of the coordinates x_j on the configuration space Q.. The system is said to be separable if there exists a privileged system of coordinates λ_j on Q such that the above PDE splits into a system of ODE's

$$\Phi_j(\lambda_j,\frac{dW_i}{d\lambda_j};E_1,\dots,E_n)=0$$

called the separation equations, depending on n arbitrary constants $(E_1,\dots,E_n)$, called the separation constants. In this case the complete integral of the Hamilton–Jacobi equation is easily obtained by adding together the solutions of the separated ODE's.

To understand the geometry of a separable system we have found useful to introduce in the configuration space Q, the tensor field L of type $(1,1)$ having as eigenvalues the separation coordinates $\lambda_j(x_1,\dots,x_n)$ and as eigenvectors their differentials. This tensor field is defined by

$$Ld\lambda_i=\lambda_i d\lambda_i\ ,$$

and one can easily show that its torsion vanishes. In a second time, this tensor field is prolonged to T^*Q by the process of complete lifting. This is accomplished by introducing the modified Liouville 1–form

$$\theta' = \sum L_k^i(x_1, \dots, x_n) y_i dx^k$$

and the associated 2–form $\omega' = d\theta'$. By comparison of ω' with the canonical symplectic 2–form ω of T^*Q one ends up with the tensor field N defined by

$$\omega' = \omega \circ N \ .$$

It can be shown that it is a tensor field of type $(1,1)$ without torsion, and that its eigenvalues are still the functions $\lambda_i(x_1, \dots, x_n)$ (each counted two times). This tensor field is called the Nijenhuis tensor (or recursion operator) associated with the separation coordinates. The cotangent bundle T^*Q endowed with the 2–form ω and the Nijenhuis tensor N is a particular instance of a class of manifolds introduced in [2] under the name of ωN–manifolds.

To see the effect of the existence of N, let us introduce also the second character of our play, namely the Hamiltonian vector field X_H. By the iterated action of N on X_H one readily obtains the distribution

$$D_H = < X_H, NX_H, \dots, N^{n-1}X_H >$$

called the Levi–Civita distribution associated with X_H. Then the main result of the theory of separable systems is that X_H is separable in the coordinate system defined by the eigenvalues of N iff the distribution D_H is integrable in the Frobenius sense. This statement gives a completely intrinsic characterization of the separable systems. It is obtained at the price of trading the separation coordinates $\lambda_i(x_1, \dots, x_n)$ by the more complicated tensor field N. This trade, however, may be rewarding in so far it suggests new perspectives which would be hard to conceive inside the usual conceptual scheme. One of the aims of the present paper is to support some evidence in favor of this claim.

3. The search for separable coordinates

The main problem we are concerned with is to find the separation coordinates, if any, of a given integrable system on a symplectic manifold (M, ω). We therefore assume that a family of involutive functions $(I_1, \dots, I_n)$ is given on M, and we set the question of finding the coordinates which separate simultaneously all the HJ equations associated with these functions. According to the point of view presented in the previous section, if these coordinates exist, it should be possible to introduce on M a tensor field N of type $(1,1)$ obeying the following three conditions

1 N is torsionless

2 $\omega' = \omega \circ N$ is a closed 2–form

3 The Lagrangian foliation defined by $(I_1, \ldots, I_n)$ is invariant under the action of N.

The construction of this tensor field is a challenging problem which can be reduced in difficulty if we add an extra assumption:
4. The Lagrangian foliation is a cyclic Levi–Civita foliation for N.

To explain this last assumption it is suitable to consider the minimal polynomial of N:

$$\Delta(\lambda) = \lambda^n - (p_1\lambda^{n-1} + \cdots + p_n) \ .$$

If N verifies the first two conditions listed above, one can show that the coefficients $(p_1, \ldots, p_n)$ are in involution with respect to the symplectic 2–form ω

$$\{p_i, p_k\} = 0 \ ,$$

and that they verify the cyclic relations

$$\begin{aligned} N^* dp_1 &= dp_2 + p_1 dp_1 \\ N^* dp_2 &= dp_3 + p_2 dp_1 \\ \vdots &= \vdots \\ N^* dp_n &= \quad +p_n dp_1 \end{aligned}$$

with respect to N. Assume that they are functionally independent. The associated Lagrangian foliation is the prototype of cyclic Levi–Civita foliation. More generally, a cyclic Levi–Civita foliation is a Lagrangian foliation generated by n functions $(J_1, \ldots, J_n)$ obeying the recursive relations

$$\begin{aligned} N^* dJ_1 &= dJ_2 + J_1 dJ_1 \\ N^* dJ_2 &= dJ_3 + J_2 dJ_1 \\ \vdots &= \vdots \\ N^* dJ_n &= \quad +J_n dJ_1 \ . \end{aligned}$$

The existence of these foliations is a basic fact in the geometry of ωN–manifolds. We can now explain the meaning of the fourth assumption. We notice that what is important in a problem of separation of variables is the Lagrangian foliation and not the specific basis of functions $(I_1, \ldots, I_n)$ which is used to specify it. Thus by assumption we do not mean that $(I_1, \ldots, I_n)$ is a cyclic basis, in the previous sense, but only there exist a cyclic basis $(J_1, \ldots, J_n)$, which is still unknown, adapted to the Lagrangian foliation. Since we do not know either the recursion operator N or the cyclic basis $(J_1, \ldots, J_n)$ one may rightly doubt, at this point, that the fourth assumption is of any use. This impression however is incorrect. The existence of a cyclic basis $(J_1, \dots, J_n)$ leaves an imprinting on any other basis $(I_1, \dots, I_n)$. It is this imprinting on $(I_1, \dots, I_n)$ of the existence of a cyclic basis $(J_1, \dots, J_n)$ which is the real content of the fourth assumption. We shall now spell out this property in the particular case $n = 2$. So, henceforth, we shall restrict our study to integrable systems with two degrees of freedom. This study is already highly nontrivial and interesting, and it is sufficient to cover examples like the Kowalevski top.

We proceed in an inductive way. We assume the existence of N and of the cyclic basis (J_1, J_2), and we slowly come back to the initial basis (I_1, I_2). We first notice that the first and the fourth assumptions listed above allow to write the Nijenhuis tensor in the form

$$\begin{aligned} N^* dp_1 &= dp_2 + p_1 dp_1 \\ N^* dp_2 &= dp_3 + p_2 dp_1 \\ N^* dJ_1 &= dJ_2 + p_1 dJ_1 \\ N^* dJ_2 &= dJ_3 + p_2 dJ_1 \ . \end{aligned}$$

The first two equations follow from the assumption that N is torsionless. The remaining two equations are just the definition of a cyclic basis. In writing these equations we have in mind that N is known, and we are ready to interpret these equation as the expression of constraints which must be satisfied by the coefficients (p_1, p_2) of the minimal polynomial of N and by the elements (J_1, J_2) of any cyclic basis of N. But we can equally well change our point of view. We can choose arbitrarily four independent functions (p_1, p_2, J_1, J_2) on M and we can define N according to the previous equations. It will automatically verify assumptions 1 and 4.

We are then left with the other two assumptions. By assumption 3 the functions (J_1, J_2) must depend only on the given functions (I_1, I_2):

$$\begin{aligned} J_1 &= F_1(I_1, I_2) \\ J_2 &= F_2(I_1, I_2) \ . \end{aligned}$$

No restriction is so far imposed on this dependence. Moreover, the functions (p_1, p_2) are still arbitrary. The clue for fixing these functions is the second assumption. The question is: when $\omega' = \omega \circ N$ is a closed form? The answer is provided by the following set of three commutation relations:

$$\begin{aligned} \{p_1, p_2\} &= 0 \\ \{p_1, J_2\} &= \{p_2, J_1\} \\ \{p_2 + p_1^2, J_2 &= \{p_1 p_2, J_1\} \ . \end{aligned}$$

(Tacitly we also use the fact that the brackets$\{I_1, I_2\}$ and therefore $\{J_1, J_2\}$ vanish). One first result is that any set of four independent functions (p_1, p_2, J_1, J_2) fulfilling these commutation relations, with (J_1, J_2) depending only on the given functions (I_1, I_2), allows to construct a recursion operator N fulfilling all the four assumptions required by the ωN theory. The eigenvalues of operator are the separation coordinates we looking for. So, at this point, we have a definite idea of what conditions we must solve in order to construct the separation coordinates.

We can still simplify the problem by a suitable change of variables. Instead of (p_1, p_2) we introduce two new functions (s, t) according to:

$$-p_1 = \frac{2J_{12}J_{22}s + (J_{11}J_{22} + J_{12}J_{21})t + 2J_{11}J_{21}}{J_{12}^2 s + J_{11}J_{12}t + J_{11}^2}$$

$$-p_2 = \frac{J_{22}^2 s + J_{11}J_{22}t + J_{21}^2}{J_{12}^2 s + J_{11}J_{12}t + J_{11}^2}$$

where $J_{ik} = \frac{\partial J_i}{\partial J_k}$.

It is not easy to explain briefly the origin of this transformation, but it is easy to see its effect. The second and third commutation relations simply become

$$\begin{aligned} \{t, I_2\} &= \{s, I_1\} \\ \{-s + t^2, I_2\} &= \{st, I_1\} \ . \end{aligned}$$

These relations do no longer contain (J_1, J_2). So we have actually split our problem in two parts. The primary problem is to find a pair of functions (s, t) fulfilling these commutation relations with the given functions (I_1, I_2). The secondary problem is to find a pair of functions $J_1(I_1, I_2)$ and $J_2(I_1, I_2)$ such that the functions (p_1, p_2) defined by the above equations (where (s, t) are provided by the solution of the primary problem) commute. The secondary problem can be further simplified but we shall not discuss anymore this problem here. We concentrate our attention to the first problem.

4. The method of Kowalevski

We presently associate with any solution of the primary problem a matrix K which will be called the Kowalevski matrix. If allows to give the problem of finding the separation coordinates a more algebraic aspect. The initial step is to use a coordinate system (x_1, x_2, I_1, I_2) adapted to the Lagrangian foliation. For brevity, these coordinates will be referred to as Kowalevski coordinates. Coherently the functions (s, t) are thought off as functions of these coordinates. With any solution of the primary problem let us associate the new functions

$$U = \frac{s_1 s_2 - t_2 s_1 t + t_1 t_2 s}{t_1 s_2 - t_2 s_1}$$

$$V = \frac{s_2 s_2 - t_2 s_2 t + t_2 t_2 s}{t_1 s_2 - t_2 s_1}$$

$$W = \frac{-s_1 s_1 + t_1 s_1 t - t_1 t_1 s}{t_1 s_2 - t_2 s_1}$$

$$Z = \frac{-s_1 s_2 + t_1 s_2 t - t_1 t_2 s}{t_1 s_2 - t_2 s_1}$$

constructed by means of the functions (s,t) and their derivatives $s_i = \frac{\partial s}{\partial s_i}$, $t_i = \frac{\partial t}{\partial t_i}$. The Kowalevski matrix is simply the 2×2 matrix

$$K = \begin{pmatrix} U & V \\ W & Z \end{pmatrix}.$$

It enjoys several remarkable properties. We mention only the following two. First, it is clear from the definition that

$$t = \operatorname{Tr} K \qquad s = \det K \ .$$

So the solution of the primary problem can be considered as the spectral invariants of the Kowalevski matrix. Furthermore, it realizes the recursion relation

$$K\dot{x} = x'$$

where $\dot{x}$ is the 2–component vector whose entries are $\dot{x}_i = \{x_1, I_1\}$ and $\dot{x}_2 = \{x_2, I_2\}$ while x' is the 2–component vector of entries $x'_1 = \{x_1, I_2\}$, $x'_2 = \{x_2, I_2\}$. This relation reminds us of the recursion relations defined by N, Indeed, the matrix K can be considered as the last vestige of the recursion operator on the leaves of the Lagrangian foliation generated by (I_1, I_2). The discovery of the above recursion relation opens a new perspective on our problem. So far we have considered (s,t) as given (by solving the commutation relations of the primary problem) and K as a derived object. However, nothing prevent us from inverting the point of view, and to consider the matrix K as the primary object. From this perspective the procedure will run as follows. Given the integrable system through the functions (I_1, I_2) we first choose a pair of (suitable) Kowalevski coordinates (x_1, x_2), and then one computes the vectors x and x'. Next one studies the matrices which realize the recursion relation $K\dot{x} = x'$. Of course this relation does not define the matrix K uniquely. However, it may happen that the particular form of the equations should suggest a preferred matrix. For instance, in the case of the Kowalevski top, one easily realizes that there exists a unique matrix K whose entries are rational functions of the Kowalevski coordinates. This matrix is

$$K = -2 \begin{pmatrix} \frac{R(x_1\, x_2)}{(x_1 - x_2)^2} & \frac{R(x_1)}{(x_1 - x_2)^2} \\ \frac{R(x_2)}{(x_1 - x_2)^2} & \frac{R(x_1\, x_2)}{(x_1 - x_2)^2} \end{pmatrix}.$$

So, the form of the equations of the Kowalevki top suggests the proper choice of the matrix K. One then computes the trace and the determinant of this matrix and plugs these functions into the commutation relations of the primary problem. If they are not verified one must go back and look for a different matrix. In the opposite case (as is the case of the Kowalevksi top) it means that one is on a good track. We can then pass to the secondary problem. In the case of the Kowalevski top there is a natural gradation of variables and of the integrals of motion. This gradation suggests that the final unknown functions (J_1, J_2) must be homogeneous polynomials of (I_1, I_2). A careful examination of the degree of homogeneity suggests the following form of these functions

$$\begin{aligned} J_1 &= aI_1 \\ J_2 &= bI_2 + cI_1^2 \ , \end{aligned}$$

where (a, b, c) are arbitrary constants. Indeed in the Kowalevski top the integrals of motion have degrees 2 and 4 and the above ansatz is the simplest guess to have new integrals with the same degree. By means of the functions s, t, J_1, J_2 one then constructs the coefficients (p_1, p_2) of the minimal polynomial of N. The last condition is that these coefficients commute. Using this condition one readily finds the correct coefficients (a, b, c)

$$a = c, \qquad b = 1, \qquad c = \frac{1}{12} \ .$$

At this point the problem is completely solved. It remains only to use the coefficients (p_1, p_2) to construct the polynomial

$$\lambda^2 - p_1\lambda - p_2 \ .$$

The theory guarantees that the zeroes of this polynomial are the separation variables.

In conclusion we can say that the search for separation variables amounts to solving a well–defined set of commutation relations. Under certain circumstances, which may vary from problem to problem (and which must be the object of further investigation), the previous problem may be replaced by the study of the algebraic recursion relation $K\dot{x} = x'$, supplemented by a set of additional conditions required to fix uniquely the matrix K. If the additional conditions are properly chosen, the trace and the determinant of the matrix K solve the required commutation relations. In this way, the search for separation variables, that in principle requires the solution of a set of partial differential equations, can be performed in purely algebraic way. We call this procedure the Kowalevski algorithm.

References

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