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Representations of the Mapping Class Group of the Two Punctured Torus arising from Mathematical Physics

Ioannis Smyrnakis

Abstract

Two techniques for determining bases of horizontal sections of torus conformal block bundles are examined. The first technique consists of obtaining a Fuchsian differential equation whose solution space is the vector space of conformal blocks. This is done explicitly in a particular case. The second technique is based on free field representations. The particular case of the $\hat{sl}_2(C)$ free field representation obtained in [1] is used to derive a basis of horizontal sections of the conformal block bundle in the case of level k spin 1/2-spin 1/2 torus conformal blocks. Such a basis provides a monodromy representation of the mapping class group of the two punctured torus. An explicit presentation of this mapping class group is given in terms of generators and relations. Finally infinitely many representations of this mapping class group are derived using the above basis, for arbitrary level k.

1. Introduction

It has been known for some time that given a rational conformal field theory (RCFT), it is possible to associate finite dimensional linear representations of the braid group with n string on the sphere S (in the sense of Artin) [2] , [3]. This representations arise through the action of the braid group generators on the finite dimensional vector spaces of conformal blocks with n identical insertions. The particular case of the SU(2) Wess-Zumino-Witten (WZW) model on the sphere, has been studied in great detail in [4]. There it was also shown that apart from the braid group relations satisfied by the representatives of the braid group elements, there is an extra relation of algebraic type, the Hecke relation. This extra relation permitted the construction of a class of knot polynomial invariants among which are the Jones polynomials [5]. Furthermore the representatives of the interchange operators were linked to the quantum group $U_q(sl(2))$ R-matrix [6].

In a similar way to that of Artin it is possible to define braid groups of n strings on any Riemann surface [7]. The braid group $B_{g,n}$ corresponding to the Riemann surface M of genus g is defined to be $B_{g,n} = \pi_1(M^n - D/\Sigma_n)$. Here D is the subset of M^n where any two coordinates are identical, and Σ_n is the symmetry group of n elements. The difference from the sphere is that the braid group generators mix with the mapping class group generators of the unpunctured Riemann surface in a nontrivial way. In fact we have the following short exact sequence:

$$1 \to B_{g,n}/center \to \mathcal{M}_{g,n} \to \mathcal{M}_{g,0} \to 1 \tag{1}$$

Suppose now that we are dealing with the SU(2) level k WZW RCFT, whose symmetry algebra consists of two copies of the centrally extended loop group algebra $\hat{su}_2(C)$ with central element k. The complexification of this algebra is the algebra $\hat{sl}_2(C)$. We want to obtain a representation of the braid group of n strings on the sphere. We need to consider the configuration space $C_n^0 = (S^n - D)/\Sigma_n$. If we associate to every coordinate (every braid group string) an identical $\hat{sl}_2(C)$ highest weight irreducible module H_j then it is possible to define the vector bundle of conformal blocks over the configuration space. The vector space of conformal blocks over a point in C_n^0 is defined to be the vector space of $sl_2(C)(z_1, \cdots, z_n)$ invariant homomorphisms from $(H_j)^n$ to C. Of course the above algebra and its action on $(H_j)^n$ needs to be defined.

The algebra $sl_2(C)(z_1, \cdots, z_n)$ is defined to have elements

$$sl_2(C)(z_1, \cdots, z_n) = sl_2(C) \otimes \mathcal{M}_{z_1, \cdots, z_n} \tag{2}$$

where $\mathcal{M}_{z_1, \cdots, z_n}$ is the space of meromorphic functions with possible poles only at the points $z_1, \cdots, z_n$. The Lie algebra structure is given by

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg \tag{3}$$

Since a meromorphic function admits a Laurent expansion near each z_i, we have linear maps

$$\tau_i : sl_2(C)(z_1, \cdots, z_n) \to sl_2(C) \otimes C((w_i)) \tag{4}$$

where $w_i = z - z_i$. Here $C((w))$ stands for the vector space of formal Laurent series in the parameter w. Nevertheless τ_i are not Lie algebra homomorphisms because the Lie algebra structure we give to $sl_2(C) \otimes C((w_i))$ is the structure of the centrally extended loop group algebra $\hat{sl}_2(C)$. It is now possible to define the action of $X \otimes f \in sl_2(C)(z_1, \cdots, z_n)$ on $H_{j_1} \otimes \cdots \otimes H_{j_n}$ through the relation

$$X \otimes f(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_{i=1}^{n} \xi_1 \otimes \cdots \otimes \tau_i(X \otimes f)\xi_i \otimes \cdots \otimes \xi_n \tag{5}$$

So, with respect to the above action, the vector space of conformal blocks over the point $(z_1, \cdots, z_n)$ in $S^n - D$ is

$$\mathcal{H}(z_1, \cdots, z_n; j_1, \cdots, j_n) = Hom_{sl_2(C)(z_1,\cdots,z_n)}(H_{j_1} \otimes \cdots \otimes H_{j_n}, C). \tag{6}$$

In the particular case the modules H_{j_i} are isomorphic it is possible to obtain in this way a vector bundle over C_n^0. In this case the vector space of conformal blocks is

$$\mathcal{H}(S, n, j) = Hom_{sl_2(C)(z_1,\cdots,z_n)}((H_j)^n, C). \tag{7}$$

Let us denote by V_j the $sl_2(C)$ irreducible spin j module. It is possible to show that there is an injective map (see e.g. [8])

$$i^* : \mathcal{H}(S^2, n, j) \to Hom_{sl_2(C)}((V_j)^n, C) \tag{8}$$

This shows that the vector space of conformal blocks is necessarily finite dimensional. The map i^* is given as the restriction of the conformal blocks $\Psi \in Hom_{sl_2(C)(z_1,\cdots,z_n)}$ $((H_j)^n, C)$ from $(H_j)^n$ to $(V_j)^n$. So to determine the conformal block Ψ we only need to know its restriction on $(V_j)^n$.

It is now possible to define the primary fields. First consider the conformal block

$$\Psi(z) : H_{j_0} \otimes H_j \otimes H^*_{j_\infty} \to C \tag{9}$$

where we have taken as insertion points the points $0, z, \infty$ and H^*_j is the dual module of H_j with respect to the invariant inner product. This conformal block exists if the representations satisfy the quantum Clebsch-Gordan condition at level k:

$$\begin{aligned} j_1 + j_2 + j_3 &\in Z \\ |j_1 - j_2| \leq j_3 &\leq j_1 + j_2 \\ j_1 + j_2 + j_3 &\leq k. \end{aligned} \tag{10}$$

The conformal block $\Psi(z)$ induces an operator

$$\tilde{\Psi}(z) : H_{j_0} \otimes H_j \to H_{j_\infty}. \tag{11}$$

If we restrict H_j to the level 0 part V_j then $\tilde{\Psi}(z)$ defines an operator

$$\phi(v, z) : H_{j_0} \to H_{j_\infty} \tag{12}$$

where v is an element in V_j. These operators $\phi(v, z)$ are the primary fields of the theory.

The finite dimensional vector bundle of the conformal blocks over the configuration space of the sphere admits a flat connection, the Knizhnik-Zamolodchikov (KZ) connection. This means in particular that the monodromy of the horizontal sections of the conformal block bundle with the KZ connection is a homotopy invariant. So we have a linear representation of $B(0,n) = \pi_1(C_n^0)$ on $GL(N_j^{(n)}, C)$. Here $N_j^{(n)}$ is the dimension of the space of the conformal blocks with n identical spin j insertions. In general this dimension is given by the Verlinde formula [9]. This representation has been computed in [4] in the particular case $j = 1/2$ for arbitrary n. This was used to obtain the jones polynomial [4] [10].

The next challenge is to extend this to the case of higher genus Riemann surfaces. The situation in this case is more complicated because the configuration space coordinates may be analytically continued around nontrivial cycles in the Riemann surface. Furthermore, there are only a few cases of conformal field theories on surfaces of genus $g \geq 2$, where we have an explicit expression for the horizontal sections of the conformal block bundle. For this reason we will restrict ourselves in this work, to the case of conformal blocks with two spin 1/2 insertions on the torus, for the case of the $sl_2(C)$ WZW RCFT.

To define a basis of the horizontal sections of the $sl_2(C)$ WZW conformal blocks on the torus, we need the Sugawara construction of the Virasoro algebra. Suppose that we choose generators J^0, J^+, J^- of the $sl_2(C)$ algebra so that they satisfy the relations

$$[J^0, J^+] = J^+, \quad [J^0, J^-] = -J^-, \quad [J^+, J^-] = 2J^0. \tag{13}$$

The Casimir element in terms of the above generators is

$$C = 2(J^0)^2 + J^+J^- + J^-J^+ \tag{14}$$

and it is an element of the universal enveloping algebra of $sl_2(C)$. The algebra $\hat{sl}_2(C)$ now takes the form

$$\begin{aligned} \left[J_n^0, J_m^\pm\right] &= \pm J_{n+m}^\pm, \\ \left[J_n^0, J_m^0\right] &= \frac{nk}{2}\delta_{n,-m}, \\ \left[J_n^+, J_m^-\right] &= 2J_{n+m}^0 + nk\delta_{n,-m} \end{aligned} \tag{15}$$

The Sugawara construction gives the generators of the Virasoro algebra as elements of the universal enveloping algebra of $\hat{sl}_2(C)$:

$$L_n = \frac{1}{2(k+2)} \sum_{j \in Z} (: 2J_{-j}^0 J_{n+j}^0 : + : J_{-j}^+ J_{n+j}^- : + : J_{-j}^- J_{n+j}^+ :) \tag{16}$$

where :: stands for normal ordering, that is for placing generators with negative subscripts on the right of generators with positive subscripts. These generators satisfy the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12} c\delta_{m+n,0} \tag{17}$$

with central charge $c = \frac{3k}{k+2}$. It is possible to define generating functions for both the current algebra and the Virasoro algebra as follows

$$\begin{aligned} J^a(z) &= \sum_{n \in Z} J^a_{-n} z^{n-1}, \\ T(z) &= \sum_{n \in Z} L_{-n} z^{n-2}. \end{aligned} \tag{18}$$

For the complex torus with fundamental domain D, the configuration space is $C^1_n = T^n - D/\Sigma_n$. Given a point in C^1_n it is possible to represent it by a set of points $(z_1, \cdots, z_n)$ in the fundamental domain D. Consider now as special insertion points the points 0 and ∞ on the sphere containing the fundamental domain D, and assign to them the modules $H_{j'}$, $H^*_{j'}$. It is now possible to define the vector space

$$\mathcal{H}_{j'}(S, n, j) = Hom_{sl_2(C)(0, z_1, \cdots, z_n, \infty)}(H_{j'} \otimes (H_j)^n \otimes H^*_{j'}, C). \tag{19}$$

Since we do not want any dependence on the sphere points 0 and ∞, we need some kind of trace over the module $H_{j'}$. The right choice turns out to be the following twisted trace (see e.g. [11]):

$$Tr_{H_{j'}} q^{L_0 - \frac{c}{24}} : (H_j)^n \to C. \tag{20}$$

Here $q = e^{2\pi i \tau}$ where τ is the modular parameter of the complex torus. The vector space spanned by *all* such traces as linear maps from $(H_j)^n \to C$ associated to elements in $\mathcal{H}_{j'}(S, n, j)$ for arbitrary j', is the fiber of the vector bundle of conformal blocks over the configuration space C^1_n. More precisely, denote

$$\mathcal{H}_{j'}(D, n, j) = span\{Tr_{H_{j'}} q^{L_0 - \frac{c}{24}} / Traces\ assoc.\ to\ elements\ in\ \mathcal{H}_{j'}(S, n, j)\}. \tag{21}$$

The fiber $\mathcal{H}(T, n, j)$ over a point in C^1_n is

$$\mathcal{H}(T, n, j) = \bigoplus_{0 \le j' \le k/2} \mathcal{H}_{j'}(D, n, j). \tag{22}$$

Note that there is an explicit dependence of the elements in $\mathcal{H}_{j'}(D, n, j)$ associated to elements in $\mathcal{H}_{j'}(S, n, j)$ upon the moduli parameter of the torus.

2. Fuchsian Differential Equations for the Conformal Blocks

It was originally noticed in [12] that it is possible to write Fuchsian differential equations on the configuration space C^1_n whose solutions span the vector space of conformal blocks. These equations were originally derived from the operator product expansions of the currents with the primary fields for the SU(2) WZW model. However it was soon realized [12], [13] that such differential equations can be derived for all rational conformal field theories. The reason for this is the known singularity structure of the conformal blocks that arises from the operator product expansions of the primary fields.

Consider for example the case of the $k=1$ SU(2) WZW model. The $sl_2(C)$ algebra we described earlier is the complexification of the $su(2)$ algebra hence we can choose as generators for $su(2)$ the elements J^+, J^-, J^0 satisfying the same commutation relations as earlier. Suppose now that we want to study the space of spin 1/2-spin 1/2 torus conformal blocks. In the case $k=1$ there are only two integrable representations of the $\hat{su}(2)$ current algebra, H_0 and $H_{1/2}$. Since

$$F_j^{\alpha\beta}(z_1-z_2|\tau) = Tr_{H_j} < \phi(v^\alpha, z_1)\phi(v^\beta, z_2) q^{L_0-\frac{c}{24}} >= \epsilon^{\alpha\beta} N(z_1-z_2|\tau) \tag{23}$$

where $\epsilon^{+-} = -\epsilon^{-+} = -1$, there are only two linearly independent conformal blocks. Hence they should satisfy a second order differential equation of the form

$$\partial^2 F(z|\tau) + c_1(z|\tau)\partial F(z|\tau) + c_0(z|\tau)F(z|\tau) = 0. \tag{24}$$

The singularity structure of the above conformal blocks comes from the operator product expansion of the two primary fields. To get a nonzero trace the two primaries need to fuse to a field in the conformal class of the identity. Since the conformal dimension of the fields $\phi(v^\alpha, z_1)$, $\phi(v^\beta, z_2)$ is $j(j+1)/(k+2) = 1/4$, scaling invariance implies that

$$F_j^{\alpha\beta}(z) \equiv z^{-1/2}(a + bz + cz^2 + \cdots). \tag{25}$$

Furthermore, since the term bz in (25) appears when the two primaries fuse to a level 1 descendent of the identity, and the trace over such descendents is 0, we have that $b=0$. This means that by taking particular linear combinations of the conformal blocks we can construct two solutions $F^1(z)$, $F^2(z)$ of the differential equation with the following asymptotic behaviour near $z=0$

$$F^1(z) \sim z^{-1/2} \qquad F^2(z) \sim z^{3/2}. \tag{26}$$

This means that the asymptotic behaviour of the Wronskian is

$$W(z) = det \begin{pmatrix} F^1(z) & F^2(z) \\ \partial F^1(z) & \partial F^2(z) \end{pmatrix} \sim 1. \tag{27}$$

Furthermore the Wronskian has no more singularities, hence it is a constant. Since now $c_1(z|\tau) = -\frac{W'(z|\tau)}{W(z|\tau)}$ we have that $c_1(z|\tau) = 0$. Now for $c_0(z|\tau)$ we have that

$$c_0(z|\tau) = \frac{1}{W(z)} det \begin{pmatrix} \partial F^1(z) & \partial F^2(z) \\ \partial^2 F^1(z) & \partial^2 F^2(z) \end{pmatrix} \sim z^{-2}. \tag{28}$$

Hence $c_0(z|\tau)$ can have a double pole at the origin. But we know that for the torus there exist only one doubly periodic function with a double pole at 0 with no constant term, the Weierstrass $\mathcal{P}$ function. This tells us that the fuchsian differential equation for the spin 1/2-spin 1/2 conformal blocks takes the form

$$\partial^2 F(z|\tau) + c\mathcal{P}(z|\tau)F(z|\tau) = 0. \tag{29}$$

So to fully determine the differential equation we only need to find c. The indicial equation near zero for $F(z|\tau) \sim z^{\mu}$ takes the form

$$\mu(\mu - 1) + c = 0. \tag{30}$$

For these to have as solutions -1/2 and 3/2 we must have $c = -3/4$. Hence the fuchsian differential equation for the spin 1/2-spin 1/2 conformal blocks takes the final form

$$\partial^2 F(z|\tau) - \frac{3}{4}\mathcal{P}(z|\tau)F(z|\tau) = 0. \tag{31}$$

One aspect of this equation that we should remark is that this is not just an equation on the torus. It is rather an equation on the *complex* torus. This means that it is invariant under the transformations $(z, \tau) \to (z, \tau + 1)$ and $(z, \tau) \to (z/\tau, -1/\tau)$. This means that the vector space of solutions carries also a projective representation of the modular group of the complex torus. The projectivity arises from the choice of the path along which we analytically continue the solutions.

This equation can be solved and doing so one obtains the following linearly independent solutions

$$\frac{\theta_2(z|2\tau)}{\sqrt{\theta_1(z|\tau)}} \qquad \frac{\theta_3(z|2\tau)}{\sqrt{\theta_1(z|\tau)}}. \tag{32}$$

These solutions still need to be normalized appropriately if they are to be identified with the conformal blocks $F_0^{\alpha\beta}(z|\tau)$ and $F_{1/2}^{\alpha\beta}(z|\tau)$. This can be done by demanding the appropriate monodromies under modular transformations. The results are the following expressions

$$\begin{aligned} F_0^{\alpha\beta}(z|\tau) &= \epsilon^{\alpha\beta}\frac{\theta_2(z|2\tau)}{\sqrt{\eta(\tau)}\sqrt{\theta_1(z|\tau)}} \\ F_{1/2}^{\alpha\beta}(z|\tau) &= \epsilon^{\alpha\beta}\frac{\theta_3(z|2\tau)}{\sqrt{\eta(\tau)}\sqrt{\theta_1(z|\tau)}}. \end{aligned} \tag{33}$$

These solutions span our space of conformal blocks.

3. Free Field Representation for the Conformal Blocks

There are a number of problems in using the above construction to find a basis of the vector space of conformal blocks and then read the monodromy representation of the mapping class group of the two punctured torus. First it gets increasingly difficult to construct Fuchsian differential equations as the level increases. In the case of level k spin 1/2-spin 1/2 $sl_2(C)$ blocks the dimension of the space of conformal blocks is $2k$, hence we need a Fuchsian differential equation of order $2k$. Also great difficulties arise in constructing the differential equation in the case we have more than two insertions. Another problem is solving these differential equations in a geometrically meaningfull way, so that it is possible to read the monodromies of the solutions.

Fortunately this is not the only way to treat the vector space of conformal blocks, at least in the case of the $sl_2(C)$ algebra. The reason is that in this case there exists a free field representation of both the currents $J^a(z)$ and the primary fields [14], [1]. Lets introduce a free bosonic field $\Phi(z)$ and a spin 0-spin 1 bosonic ghost system $\omega(z)$, $\omega^\dagger(z)$. These admit the following expansions into modes

$$i\partial\Phi(z) = \sum_{n\in Z} a_n z^{-n-1}, \quad \omega(z) = \sum_{n\in Z} \omega_n z^{-n}, \quad \omega^\dagger(z) = \sum_{n\in Z} \omega_n^\dagger z^{-n-1}. \tag{34}$$

The modes above satisfy the following nontrivial commutation relations

$$[a_n, a_m] = n\delta_{n+m,0}, \quad \left[\omega_n, \omega_m^\dagger\right] = \delta_{n+m,0}. \tag{35}$$

Now it is possible to represent the currents $J^a(z)$ as

$$\begin{aligned} J^+(z) &= \omega^\dagger(z), \\ J^0(z) &= :\omega(z)\omega^\dagger(z): + \gamma i\partial\Phi(z), \\ J^-(z) &= :\omega^2(z)\omega^\dagger(z): - 2\gamma i\partial\Phi(z)\omega(z) - k\partial\omega(z) \end{aligned} \tag{36}$$

where $\gamma = \sqrt{\frac{k+2}{2}}$. So it is possible to treat the $sl_2(C)$ current algebra as a subalgebra of the universal envaloping algebra of the algebra (35).

The next question is how to place irreducible representations of the current algebra inside representations of the algebra (35). To do this lets choose the following representation of the algebra (35)

$$\begin{aligned} a_0|j> &= \frac{j}{\gamma}|j>, \\ a_n|j> &= 0, \quad n>0 \\ \omega_n^\dagger|j> &= 0, \quad n\geq 0 \\ \omega_{n+1}|j> &= 0, \quad n\geq 0. \end{aligned} \tag{37}$$

The highest weight state has the property that $J_0^0|j>= j|j>$. The module is built by acting on the highest weight state $|j>$ with the negative modes of the generators (37) while respecting the commutation relations. This is a Fock module. Let now $n_j = 2j+1$. Denote by F_{n_j} the above Fock module. It is possible to find an operator

Q_n, the BRST charge, so that $Q_nQ_{(k+2)-n} = Q_{(k+2)-n}Q_n = 0$ [1]. Since this operator behaves like a differential operator, it can be used to construct the following complex

$$\cdots \stackrel{Q_{n_j}}{\rightarrow} F_{-n_j+2(k+2)} \stackrel{Q_{(k+2)-n_j}}{\rightarrow} F_{n_j} \stackrel{Q_{n_j}}{\rightarrow} F_{-n_j} \stackrel{Q_{(k+2)-n_j}}{\rightarrow} F_{n_j-2(k+2)} \stackrel{Q_{n_j}}{\rightarrow} \cdots . \tag{38}$$

If we assign a degree to the above Fock modules by setting $deg(F_{n_j-2i(k+2)}) = 2i$, $deg(F_{-n_j-2i(k+2)}) = 2i+1$, we have the following theorem due to [1] for the corresponding cohomology groups:

Theorem 1.

$$H^i = KerQ^{(i)}/ImQ^{(i-1)} = 0, \quad i \neq 0$$

$$H^0 = KerQ^{(0)}/ImQ^{(-1)} = H_j \quad i = 0.$$

Here H_j is the irreducible $\hat{sl}_2(C)$ module based on a spin j representation. Since the BRST operator Q_n commutes with the currents the above identification permits us to compute traces over H_j as an alternating sum of traces over F_n. Take for example the character

$$\chi_j(\tau,\nu) = Tr_{H_j}\left(q^{(L_0-\frac{c}{24})}e^{2\pi i\nu J_0^0}\right). \tag{39}$$

This takes the form (for $q = e^{2\pi i\tau}$)

$$\begin{aligned}\chi_j(\tau,\nu) &= \sum_{i\in z} Tr_{F_{n_j-2i(k+2)}}\left(q^{(L_0-\frac{c}{24})}e^{2\pi i\nu J_0^0}\right) \\ &- \sum_{i\in z} Tr_{F_{-n_j-2i(k+2)}}\left(q^{(L_0-\frac{c}{24})}e^{2\pi i\nu J_0^0}\right)\end{aligned} \tag{40}$$

Since it is possible in general to compute traces over Fock modules, this can be calculated to give [14]

$$\chi_j(\tau,\nu) = \frac{\Theta^{k+2}_{n_j}(\tau,\nu) - \Theta^{k+2}_{-n_j}(\tau,\nu)}{2isin(\pi\nu)q^{3/24}\prod_{l=1}^{\infty}(1-q^le^{2\pi i\nu})(1-q^le^{-2\pi i\nu})(1-q^l)} \tag{41}$$

where

$$\Theta^a_b(\tau,\nu) = \sum_{j\in Z} e^{2\pi i a\tau(j+b/2a)^2} e^{2\pi i a\nu(j+b/2a)}. \tag{42}$$

To compute a basis for the space of conformal blocks we need a free field representation for the primary fields. This has been derived in [15], [1]. To obtain it we need to restrict our primary fields $\phi(v,z)$ to operarors from $H_{j_1} \to H_{j_3}$. Lets also suppose that v is in the restriction V_{j_2} of the module H_{j_2}. Denote the restricted operator by

$\phi^{n_3}_{n_2,n_1}(v,z)$, where $n_i = 2j_i + 1$. If by v^μ we denote the J^0 normalized eigenvector with eigenvalue μ then we have that

$$\phi^{n_3}_{n_2,n_1}(v^\mu, z) = \int_{\mathcal{C}} \prod_{i=2}^{r} dz_i V^{\mu}_{j_2}(z) V(z_2) \cdots V(z_r) \tag{43}$$

where

$$V^{\mu}_{j}(z) =: \omega^{j-\mu}(z) :: e^{i\frac{j}{\gamma}\Phi(z)} : \tag{44}$$

are vertex operators necessary to establish the right conformal dimension and

$$V(z) = \omega^{\dagger}(z) : e^{-\frac{i}{\gamma}\Phi(z)} : \tag{45}$$

are screening operators necessary for the charge balance. The integration contour consists of nested circles around the origin starting and ending at z, as in [16]. The parameter $r = j_1 + j_2 - j_3$ and it is an integer.

It is possible to consider the above restricted primary fields as operators from $F_{n_1} \to F_{n_3}$ for n_2 an *arbitrary* integer (not necessarily corresponding to an integrable $\hat{sl}_2(C)$ representation). As such they almost commute with the BRST charge Q_n. More precisely they satisfy the following relation

$$Q_{n_3}\phi^{n_3}_{n_2,n_1}(v^\mu, z) = e^{-\pi i n_3 j_2 \gamma^{-2}} \phi^{-n_3}_{n_2,-n_1}(v^\mu, z) Q_{n_1} \tag{46}$$

This deformed commutation relation is sufficient to permit computation of the conformal block traces as traces over Fock modules. Suppose that we need to compute the conformal block

$$F^{[k_i]}_{n_j[n_i]}(z_1 \cdots z_N|\tau,\nu) = Tr_{H_j}\left(q^{L_0 - c/24} e^{2\pi i \nu J^0_0} \prod_{i=1}^{N} \phi^{k_i}_{n_i k_{i+1}}(v^{\mu_i}, z_i)\right) \tag{47}$$

Lets define the operators $\Xi_{2l} : F_{n_j - 2l(k+2)} \to F_{n_j - 2l(k+2)}$ and $\Xi_{2l+1} : F_{-n_j - 2l(k+2)} \to F_{-n_j - 2l(k+2)}$ as

$$\Xi_{2l} = q^{L_0 - c/24} e^{2\pi i \nu J^0_0} \prod_{i=1}^{N} \phi^{k_i - 2l(k+2)}_{n_i \quad k_{i+1} - 2l(k+2)}(v^{\mu_i}, z_i), \tag{48}$$

$$\Xi_{2l+1} = q^{L_0 - c/24} e^{2\pi i \nu J^0_0} e^{-2\pi i \quad {}_{i=1}^{N} j_i k_i/(k+2)} \prod_{i=1}^{N} \phi^{-k_i - 2l(k+2)}_{n_i \quad -k_{i+1} - 2l(k+2)}(v^{\mu_i}, z_i). \tag{49}$$

In terms of these operators our conformal block takes the form

$$F^{[k_i]}_{n_j[n_i]}(z_1 \cdots z_N|\tau,\nu) = \sum_{l\in Z} Tr_{F_{n_j - 2l(k+2)}}(\Xi_{2l}) - \sum_{l \in Z} Tr_{F_{-n_j - 2l(k+2)}}(\Xi_{2l+1}) \tag{50}$$

where the last expression can be calculated since it involves traces over Fock modules.

Lets restrict now to the expressions for the spin 1/2-spin 1/2 two point conformal blocks at arbitrary level k. Then by computing the Fock traces and moving to the parallelogram representation of the torus we get that

$$\begin{aligned} F^{\pm}_{n_j}(z-w|\tau) &= \lim_{\nu\to\infty} F^{[n_j,n_j\pm 1]}_{n_j[2,2]}(z,w|\tau,\nu) \\ &= \lim_{\nu\to\infty} Tr_{H_j}\left(q^{L_0-c/24}e^{2\pi i\nu J^0_0}\phi^{n_j}_{n_1 n_{j\pm 1/2}}(v^-,z)\phi^{n_j\pm 1/2}_{n_1 n_j}(v^+,w)\right) \\ &= \frac{1}{\eta^3(\tau)}\int_0^1 \Theta\begin{pmatrix}\pm\frac{n_j}{k+2}\\ 0\end{pmatrix}(-(z-w)-2t,2(k+2)\tau)\cdot \\ &\quad E(-t|\tau)^{-\frac{1}{k+2}}E(z-w+t|\tau)^{-\frac{1}{k+2}} \\ &\quad E(-(z-w)|\tau)^{\frac{1}{2(k+2)}}\frac{d^2}{dt^2}ln\frac{E(z-w+t|\tau)}{E(t|\tau)}. \end{aligned} \tag{51}$$

Here $n_j = 1\cdots k+1$, and $F^-_1(z-w|\tau) = F^+_{k+1}(z-w|\tau) = 0$, hence the dimension of the space of conformal blocks is $2k$. $E(z|\tau) = \frac{\Theta_1(z,\tau)}{\Theta_1'(0,\tau)}$ is the prime form on the torus, $\eta(\tau)$ is the Jacobi eta function and

$$\Theta\begin{pmatrix} a \\ b\end{pmatrix}(z,\tau) = \sum_{n\in Z} e^{\pi i(n-a)^2\tau}e^{2\pi i(n-a)(z-b)}. \tag{52}$$

In computing the above conformal blocks we have regularized the expressions by analytic continuation in k from a region where the integrals converge.

4. Monodromy of the Conformal Blocks

Having now computed a basis for the space of conformal blocks we can read monodromies. Lets define the one-forms

$$\begin{aligned} \omega^{\pm}_{n_j}(Z|\tau) &= \frac{1}{\eta^3(\tau)}\Theta\begin{pmatrix}\pm\frac{n_j}{k+2}\\ 0\end{pmatrix}(-Z-2t,2(k+2)\tau)\cdot \\ &\quad E(-t|\tau)^{-\frac{1}{k+2}}E(Z+t|\tau)^{-\frac{1}{k+2}}E(-Z|\tau)^{\frac{1}{2(k+2)}}\frac{d^2}{dt^2}ln\frac{E(Z+t|\tau)}{E(t|\tau)}dt. \end{aligned} \tag{53}$$

Then the conformal blocks take the form

$$F^{\pm}_{n_j}(Z|\tau) = \int_{\gamma_0}\omega^{\pm}_{n_j}(Z|\tau) \tag{54}$$

where γ_0 is the element of $\pi_1(T-\{0,Z\})$ corresponding to a path from 0 to 1 above 0 and below Z.

It can be shown [17] that any integral of the forms $\omega^{\pm}_{n_j}(Z|\tau)$ along a nontrivial element γ of $\pi_1(T-\{0,Z\})$ can be expressed as a linear combination of the above basis of the conformal blocks. This permits us to find representations of both the braid group $B_{1,2}$ and the mapping class group $\mathcal{M}_{1,2}$ through their action on $\pi_1(T-\{0,Z\})$.

In particular the center of $B_{1,2}$ acts trivially on the conformal blocks because of translation invariance. Since $B_{1,2}/center$ can be thought of as a subgroup of $\mathcal{M}_{1,2}$ it is enaugh to read the monodromy representation of $\mathcal{M}_{1,2}$.

However to read the monodromy representation of $\mathcal{M}_{1,2}$ we need first to derive a presentation of this group. This can again be done through its action on $\pi_1(T-\{p,q\})$. To start with choose as generators the usual generators S, T of $\mathcal{M}_{1,0} = SL(2,Z)$. They satisfy the relations $S^4 = I$, $TSTSTS^{-1} = I$. We need to lift these generators to two generators $\tilde{S}, \tilde{T}$ in $\mathcal{M}_{1,2}$ so that they project to S, T under the natural projection of $\mathcal{M}_{1,2} \xrightarrow{\pi} SL(2,Z)$. This projection consists essentially of forgetting the two special points of the two punctured torus. The kernel of this projection is the braid group $B_{1,2}/center$. Since the center of $B_{1,2}$ consists of global translations of the two insertion points along cycles, we can choose as generators of $B_{1,2}/center$ translations of Z along A and B cycles , the generators a and b, as well as one more generator σ that exchanges the points 0 and Z in a clockwise direction. These generators inject in the mapping class group $\mathcal{M}_{1,2}$ since we can choose tubular neighbourhoods of the translation paths and define mapping class group elements that are nonzero only in these tubular neighbourhoods, and they carry the translations dictated by the braid group elements. So we can consider a, b, σ as elements in $\mathcal{M}_{1,2}$. Hence overall it is natural to take $a, b, \sigma, \tilde{S}, \tilde{T}$ as generators of $\mathcal{M}_{1,2}$.

Next we have to determine what are the relations among the various generators. The braid group relations on punctured Riemann surfaces have been examined in [7]. If we divide by the center we get the following relations among the generators a, b, σ

$$\sigma^{-1}a\sigma^{-1}a = 1 \quad \sigma b\sigma b = 1 \quad aba^{-1}b^{-1} = \sigma^2. \tag{55}$$

Next we need the relations the lifted $SL(2,Z)$ generators satisfy. Since they have to project to the usual $SL(2,Z)$ relations they have to have the form

$$\tilde{T}\tilde{S}\tilde{T}\tilde{S}\tilde{T}\tilde{S}^{-1} = w_1(a,b,\sigma) \quad \tilde{S}^4 = w_2(a,b,\sigma) \tag{56}$$

where $w_1(a,b,\sigma)$, $w_2(a,b,\sigma)$ are words of the indicated elements. To obtain these words it is useful to study the action of the mapping class group on $\pi_1(T-\{p,q\})$. We can choose three generators for $\pi_1(T-\{p,q\})$. Two generators that upon forgetting the insertion points project to an A-cycle, but one generator needs to cross an insertion point to be deformed to the other, and a generator along a B-cycle. Since an element of $\mathcal{M}_{1,2}$ maps the torus to the torus respecting the insertion points we have that it maps the generators of $\pi_1(T-\{p,q\})$ to a different set of generators. This action of $\mathcal{M}_{1,2}$ on $\pi_1(T-\{p,q\})$ permits us to identify the words $w_1(a,b,\sigma)$, $w_2(a,b,\sigma)$. Doing this we finally get the relations

$$\tilde{T}\tilde{S}\tilde{T}\tilde{S}\tilde{T}\tilde{S}^{-1} = 1 \quad \tilde{S}^4\sigma^2 = 1. \tag{57}$$

Next we need the relations among the generators $\tilde{S}, \tilde{T}$ and the generators a, b, σ. Since we know the relations within each group of generators we only need a way to separate each group of generators within a given word. This can be done if we express the words $\tilde{S}^{-1}aS$, $\tilde{S}^{-1}bS$, $\tilde{S}^{-1}\sigma S$, $\tilde{T}^{-1}aT$, $\tilde{T}^{-1}bT$, $\tilde{T}^{-1}\sigma T$ in terms of the braid group generators. This is possible since all the above words project to the identity in $SL(2,Z)$. Using again the action of $\mathcal{M}_{1,2}$ on $\pi_1(T-\{p,q\})$ we get the mixed relations

$$\begin{aligned} \tilde{S}^{-1}a\tilde{S} = \sigma^2 b \quad \tilde{S}^{-1}b\tilde{S} = a^{-1} \quad \tilde{S}^{-1}\sigma\tilde{S} = \sigma \\ \tilde{T}^{-1}a\tilde{T} = a \quad \tilde{T}^{-1}b\tilde{T} = ba \quad \tilde{T}^{-1}\sigma\tilde{T} = \sigma. \end{aligned} \tag{58}$$

The relations (55), (57), (58) give us a presentation of $\mathcal{M}_{1,2}$.

Now we can read off the representation of $\mathcal{M}_{1,2}$ on the vector space of conformal blocks, arising from the action of $\mathcal{M}_{1,2}$ on the integration contours of the conformal block integrals (54) . The result is the following $2k\times 2k$ representation of the above generators

$$a = \tilde{q}^{1/4} diag\left(\tilde{q}^{k/2}, \tilde{q}^{-(k-1)/2-1}, \tilde{q}^{(k-1)/2}, \tilde{q}^{-(k-2)/2-1}, \tilde{q}^{(k-2)/2}, \cdots, \tilde{q}^{-1/2-1}, \tilde{q}^{-1}\right) \tag{59}$$

$$b = \tilde{q}^{-1/4}\begin{pmatrix} 0 & \tilde{q}^{-1/2} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ b_(-k) & 0 & 0 & c(k) & 0 & 0 & \dots & 0 & 0 \\ c(-k) & 0 & 0 & b(k) & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & b(-(k-1)) & 0 & 0 & c(k-1) & \cdots & 0 & 0 \\ 0 & 0 & c(-(k-1)) & 0 & 0 & b(k-1) & \cdots & 0 & 0 \\ \vdots & & & & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & c(2) \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b(2) \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \tilde{q}^{-1/2} & 0 \end{pmatrix} \tag{60}$$

$$\sigma = \tilde{q}^{-3/4}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & b_0(k) & c_0(k) & 0 & 0 & \cdots & 0 \\ 0 & c_0(-k) & b_0(-k) & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & b_0(k-1) & c_0(k-1) & \cdots & 0 \\ 0 & 0 & 0 & c_0(-(k-1)) & b_0(-(k-1)) & \cdots & 0 \\ \vdots & & & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \tag{61}$$

where we have set $\tilde{q} = e^{\frac{2\pi i}{k+2}}$ and $b(n) = \tilde{q}^{n-1/2}\frac{\tilde{q}-1}{\tilde{q}^n-1}$, $c(n) = \tilde{q}^{-1/2}\frac{\tilde{q}^{n+1}-1}{\tilde{q}^n-1}$, $b_0(n) = \frac{\tilde{q}-1}{\tilde{q}^n-1}$ and $c_0(n) = \tilde{q}^{-n}\frac{\tilde{q}^{n+1}-1}{\tilde{q}^n-1}$. The lifted $SL(2,Z)$ generators are represented as

$$\tilde{T} = \tilde{q}^{-\frac{k+2}{8}} diag\left(\tilde{q}^{(k+1)^2/4}, \tilde{q}^{k^2/4}, \tilde{q}^{k^2/4}\cdots, \tilde{q}^{2^2/4}, \tilde{q}^{2^2/4}, \tilde{q}^{1^2/4}\right) \tag{62}$$

and

$$\tilde{S} = K\begin{pmatrix} (C_{nm}) & ((-1)^n C_{nm}) \\ ((-1)^m C_{nm}) & ((-1)^{k+2+n+m} C_{nm}) \end{pmatrix} K^T \tag{63}$$

where

$$C_{nm} = \frac{i}{\sqrt{2(k+2)}} \tilde{q}^{-nm/2} \frac{\tilde{q}^{m-n} - \tilde{q}^{2m-n} + \tilde{q}^{nm+2m} - \tilde{q}^{m} + \tilde{q}^{2m+1} - \tilde{q}^{nm+2m+1}}{(1-\tilde{q}^{m})(1-\tilde{q}^{m+1})}$$

and K stands for the change of basis matrix from the basis

$$\{F^-_{k+1}(Z|\tau), F^+_k(Z|\tau), F^-_k(Z|\tau), \cdots, F^+_1(Z|\tau)\}$$

to the basis

$$\{F^+_1(Z|\tau), F^+_2(Z|\tau), \cdots F^+_k(Z|\tau), F^-_{k+1}(Z|\tau), \cdots, F^-_2(Z|\tau)\}.$$

5. Conclusion

There are at least two practical ways to determine a basis of horizontal sections of a torus conformal block bundle. One is to obtain the Fuchsian differential equation that they satisfy and the other is through a free field representation. The differential equation method is more general, since the very structure of the conformal blocks as traces over irreducible algebra modules guarrantees that such Fuchsian differential equations exist for any rational conformal field theory (see e.g. [13]). Nevertheless given a particular conformal field theory it is rather difficult to find which particular differential equations are satisfied by the various associated vector spaces of conformal blocks. Furthermore, given a particular Fuchsian differential equation there is not a good way to tell if it corresponds to a vector space of conformal blocks of some conformal field theory. In addition to these problems, if one wants to read the monodromies of the horizontal sections of the conformal block bundle then one has to solve these equations, or at least obtain integral representation of the solutions. This can be a cumbersome procedure. If one chooses the free field representation technique, (s)he is restricted to the theories that admit a free field representation. Although this class of theories is a large class, it is far from clear which theories admit such a representation. Nevertheless, using a free field representation one can compute directly a basis of horizontal sections of the conformal block bundle. This means that this method is more appropriate if one wants to read monodromies. Another advantage of this method is that it is more easily extendible to higher genus Riemann surfaces, since we know the conformal blocks of free field vertex operators on higher genus Riemann Surfaces [18]. Furthermore, calculations can be carried out for arbitrary level, unlike the differential equation technique.

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◇ Ioannis Smyrnakis
University of Crete,
Department of Applied Mathematics,
L. Knosou-Ambelokipi,
71409 Iraklio Crete,
Greece
smyrnaki@tem.uoc.gr