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A symmetry analysis of Richard's equation describing flow in porous media

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Abstract
A summary of the classical Lie method is presented as it applies to Richard's equation for water flow in an unsaturated uniform soil. In addition the more general potential symmetries for Richard's equations presented as a system are also given. These results are extended to give a new non-classical symmetry analysis based upon the method of Bluman and Cole. An example of a non-classical symmetry reduction of Richard's equation is presented. Furthermore a new class of potential symmetries is derived.

Keywords: Richard's equation, symmetry analysis, non-classical, potential.
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1. Introduction

The symmetry analysis presented here is motivated by problems associated with water flow in unsaturated soils as described by for example Philip [1]. It is normal to describe such flow by means of Richard's equation [2] which is considered here in the one dimensional form:

$$\Delta\left(x,t,u,u_t,u_x,u_{xx}\right) \equiv u_t - \left(Du_x\right)_x - K_u u_x = 0 \tag{1}$$

and where the soil will be taken to be homogeneous so that both the diffusivity D and hydraulic conductivity K are functions of u alone. In addition a suffix indicates a partial derivative.

The use of symmetry or similarity methods to describe flow in unsaturated soil is not new and has found practical application in the theory of infiltration. For example, in the case of horizontal absorption Philip [3] describes travelling wave solutions based upon an ansatz using the Boltzmann similarity variable ω:

$$u = u\left(\omega\right) \qquad \omega = xt^{-\frac{1}{2}} \tag{2}$$

although the resulting mathematical forms for the diffusivity are not well adapted for fitting empirical data.

In a second example, with application to horizontal flow, with Neuman boundary conditions the infiltration process is described using the ansatz:

$$u(x,t) = \psi(\omega)\, t^{\frac{1}{n+2}} \qquad \omega = x t^{-\frac{n+1}{n+2}} \qquad D(u) = cu^n \tag{3}$$

Similarity solutions of Richard's equation have been described by Sposito [4], Edwards [5] and El-labany *et al* [6] who have conducted a comprehensive classical Lie analysis of this equation and Sophocleous [7] has presented a partial classical analysis of the equation in its potential form. In addition, Gandarias [8] presents potential symmetries for a form of heterogeneous porous media with power law diffusivity and hydraulic conductivity. There is however no detailed analysis of the important non-classical approach of Bluman and Cole [9] applied to equation (1) although Gandarias *et al* [10] has presented a non-classical analysis of the equation in a restricted form. It is the aim here to summarize the main classical symmetries of Richard's equation, excluding obvious cases, for example translation symmetries and furthermore to introduce new symmetries by undertaking both a non-classical and a potential symmetry analysis.

2. Classical results for Richard's equation

In the classical Lie group method one-parameter infinitesimal point transformations,with group parameter ε are applied to the dependent and independent variables (x,t,u) In this case the transformation are

$$\bar{x} = x + \varepsilon\eta_1(x,t,u) + O\left(\xi^2\right) \qquad \bar{t} = t + \varepsilon\eta_2(x,t,u) + O\left(\xi^2\right)$$
$$\bar{u} = u + \varepsilon\phi(x,t,u) + O\left(\xi^2\right) \tag{4}$$

and the Lie method requires invariance of the solution set $\Sigma \equiv \{u(x,t),\ \Delta = 0\}$. This results in a system of overdetermined, linear equations for the infinitesimals η_1, η_2, ϕ. The corresponding Lie algebra of symmetries is the set of vector fields

$$\mathcal{X} = \eta_1(x,t,u)\frac{\partial}{\partial x} + \eta_2(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u} \tag{5}$$

The condition for invariance of (1) is the equation:

$$\mathcal{X}_E^{(2)}(\Delta)\,|_{\Delta=0} = 0 \tag{6}$$

where the second prolongation operator $\mathcal{X}_E^{(2)}$ is written in the form

$$\mathcal{X}_E^{(2)} = \mathcal{X} + \phi^{[t]}\frac{\partial}{\partial u_t} + \phi^{[x]}\frac{\partial}{\partial u_x} + \phi^{[xx]}\frac{\partial}{\partial u_{xx}} \tag{7}$$

and where $\phi^{[t]}$, $\phi^{[x]}$ and $\phi^{[xx]}$ are defined through the transformations of the partial derivatives of u. In particular:

$$\bar{u}_{\bar{x}} = u_x + +\varepsilon\phi^{[x]}(x,t,u) + O\left(\xi^2\right) \qquad \bar{u}_{\bar{t}} = u_t + +\varepsilon\phi^{[t]}(x,t,u) + O\left(\xi^2\right)$$
$$\bar{u}_{\bar{x}\bar{x}} = u_{xx} + +\varepsilon\phi^{[xx]}(x,t,u) + O\left(\xi^2\right) \tag{8}$$

Once the infinitesimals are determined the symmetry variables may be found from condition for invariance of surface $u=u\,(x,t)$:

$$\Omega=\phi-\eta_1 u_x - \eta_2 u_t = 0 \tag{9}$$

Throughout the following it has been found convenient to set

$$D = H_u \tag{10}$$

and also, the MACSYMA program `symmgrp.max` due to Champagne *et. al.* [11] has been used to calculate the determining equations.

In the case Richard's equation (1) the nine well known, for example, Sposito [4], Edwards [5], over-determined linear determining equations are :

$$\eta_{2_u} = 0 \qquad \eta_{2_x} = 0 \tag{11}$$

$$\eta_{1_u} H_{u\,u} - \eta_{1_{u\,u}} H_u = 0 \qquad \eta_{2_u} H_{u\,u} + \eta_{2_{u\,u}} H_u = 0 \tag{12}$$

$$\eta_{2_x} H_{u\,u} + \eta_{2_{u\,x}} H_u + \eta_{1_u} = 0 \qquad \phi_x K_u + \phi_{xx} H_u - \phi_t = 0 \tag{13}$$

$$\eta_{2_x} H_u K_u - \phi H_{u\,u} + \eta_{2_{xx}} H_u^2 - \eta_{2_t} H_u + 2\eta_{1_x} H_u = 0 \tag{14}$$

$$2\eta_{1_u} H_u K_u + \phi H_u H_{u\,u\,u} - \phi H_{u\,u}^2 + \phi_u H_u H_{u\,u} + \phi_{u\,u} H_u^2 - 2\eta_{1_{ux}} H_u^2 = 0 \tag{15}$$

$$\phi H_u K_{uu} - \phi H_{uu} K_u + \eta_{1_x} H_u K_u + 2\phi_x H_u H_{uu} + 2\phi_{ux} H_u^2 - \eta_{1_{xx}} H_u^2 + \eta_{1_t} H_u = 0 \tag{16}$$

As may be seen from Table 1 the classical symmetries are given for power and exponential functions of H and K in which the infinesimal η_1 and η_2 are linear functions of x and t and where ϕ is linear in u. Note that each of these symmetries has been used by Edwards [5] to reduce Richard's equation to an ordinary differential equation.

Functions H and K	Symmetries
$H=cu^\lambda$ $K=ku^\mu$	$\eta_1=(\lambda-\mu)x,\ \phi=u$ $\eta_2=(\lambda-2\mu+1)t$
$H=cu^\lambda$ $K=k\ln u$	$\eta_1=\lambda x, \phi=u$ $\eta_2=(\lambda+1)t$
$H=cu^\lambda$ $K=k(u\ln u-u)$	$\eta_1=(\lambda-1)x-kt$ $\eta_2=(\lambda-1)t,\ \phi=u$
$H=ce^{\lambda u}$ $K=ke^{\mu u}$	$\eta_1=(\lambda-\mu)x$ $\eta_2=(\lambda-2\mu)t,\ \phi=1$
$H=ce^{\lambda u},\ K=ku^2$	$\eta_1=\lambda x-2kt,\ \eta_2=\lambda t,\ \phi=1$
$H=cu,\ K=ku^2$	$\eta_1=-x,\ \eta_2=-2t,\ \phi=u$
$H=cu$ $K=ku^2$	$\eta_1=-2kxt,\ \eta_2=-2kt^2$ $\phi=x+2kut$
$H=cu,\ K=ku^2$	$\eta_1=-2kt,\ \eta_2=0,\ \phi=1$

Table 1. *Classical symmetries of Richard's equation*
(based on the comprehensive analysis of Sposito [4], Edwards [5], and excluding obvious cases such as translation symmetries)

3. Non-classical point symmetry

The non-classical method is a generalisation of the classical Lie group approach due to Bluman & Cole [9] that incorporates the invariant surface condition (9) into the condition (6) for form invariance of Richard's equation (1). It follows that non-classical symmetries of Richard's equation may be found by solving the non-linear set of determining equations:

$$\mathcal{X}_E^{(2)}(\Delta)\,|_{\Delta=0,\ \Omega=0} = 0 \tag{17}$$

To apply (17) two cases must be considered as follows.

3.1. Case A $\quad \eta_2 = 1 \quad \eta_1 \equiv \eta(x,t,u)$

It is straight forward to show that there are four non-linear determining equations as follows:

$$H_u(\eta_u H_{uu} - \eta_{uu} H_u) = 0 \tag{18}$$

$$\phi_x H_u K_u + \phi^2 H_{uu} + \phi_{xx} H_u^2 - \phi_t H_u - 2\eta_x \phi H_u = 0 \tag{19}$$

$$\begin{aligned} &2\eta_u H_u K_u + \phi H_u H_{uuu} - \phi H_{uu}^2 + \phi_u H_u H_{uu} + \phi_{uu} H_u^2 \\ &- 2\eta_{ux} H_u^2 + 2\eta\eta_u H_u \\ &= 0 \end{aligned} \tag{20}$$

$$\begin{aligned} &\phi H_u K_{uu} - \phi H_{uu} K_u + \eta_x H_u K_u + 2\phi_x H_u H_{uu} - \eta\phi H_{uu} \\ &+ 2\phi_{ux} H_u^2 - \eta_{xx} H_u^2 - 2\eta_u \phi H_u + \eta_t H_u + 2\eta\eta_x H_u \\ &= 0 \end{aligned} \tag{21}$$

Solutions of these equations will be generated below.

3.2. Case B $\quad \eta_2 = 0 \quad \eta_1 \equiv \eta \equiv 1$

In this case the four determining equations are:

$$\phi H_{uu} = 0 \tag{22}$$

$$\phi_x K_u + \phi_{xx} H_u - \phi_t = 0 \tag{23}$$

$$\phi H_u H_{uuu} - \phi H_{uu}^2 + \phi_u H_u H_{uu} + \phi_{uu} H_u^2 = 0 \tag{24}$$

$$\phi H_u K_{uu} - \phi H_{uu} K_u + 2\phi_x H_u H_{uu} + 2\phi_{ux} H_u^2 = 0 \tag{25}$$

There is only one solution of these equations as follows, namely the infinite symmetry

$$\eta = 1 \qquad H = cu \qquad K = ku \qquad \phi = \frac{\partial u}{\partial x} = c_0 u + g \tag{26}$$

where $g = g(x,t)$ and satisfies

$$c g_{xx} + k g_x - g_t = 0 \tag{27}$$

4. Examples of symmetries for Case A

Using (18) and (20) the following explicit forms for $\eta(x,t,u)$ and $\phi(x,t,u)$ may be obtained:

$$\eta(x,t,u) = f(x,t)\,H(u) + g(x,t) \tag{28}$$

$$\begin{aligned}\phi(x,t,u) = H_u^{-1}\big\{&f_x H^2 - 2fZ(u) + 2fgX(u)\\ &+2f^2W(u) + HS(x,t) + R(x,t)\big\}\end{aligned} \tag{29}$$

where f, g, R, S depend on x, t and W, X and Z depend only on u. Examples of explicit non-classical symmetries may now be found by considering two sub-cases $f = 0$ and $f \neq 0$ separately.

4.1. Sub-case $f = 0$

When $f = 0$ equations (28) and (29) become:

$$\eta = g \qquad \phi = H_u^{-1}\{HS(x,t) + R(x,t)\} \tag{30}$$

and the determining equations (19) and (21) are now

$$\begin{aligned}&(H\,H_{u\,u}K_u - HH_uK_{u\,u} + g\,H_{u\,u})\,(R + HS)\\ &+ H_u^3\,(g_{xx} - 2\,S_x) - H_u^2\,(g_xK_u + g_t + 2\,g\,g_x)\\ &= 0\end{aligned} \tag{31}$$

and

$$\begin{aligned}&H_u^3\,(R_{xx} + HS_{xx}) + H_{u\,u}\,(R + HS)^2 + H_u^2K_u\,(R_x + H\,S_x)\\ &- 2\,g_x\,H_u^2\,(R + H\,S) - H_u^2\,(R_t + H\,S_t)\\ &= 0\end{aligned} \tag{32}$$

The following three solutions of these equations have found. In the first case:

$$f = 0 \qquad H = \frac{c}{u} \qquad K = \frac{k}{u} \tag{33}$$

where c and k are constants and where the infinitesimals are

$$\eta = c_2e^{-\frac{kx}{c}} \qquad \phi = \frac{c_2ku}{c}e^{-\frac{kx}{c}} \tag{34}$$

To generate a similarity solution of Richard's equation these infinitesimals may be substituted into the surface invariant condition (9) and the method of characteristics employed to determine the following ansatz for $u(x,t)$:

$$u(x,t) = \psi(\omega)\,e^{\frac{kx}{c}} \tag{35}$$

The similarity variable $\omega(x,t)$ is given by:

$$e^{\frac{k\omega}{c}} = e^{\frac{kx}{c}} - \frac{kc_2 t}{c} \tag{36}$$

Substitution of these relationships into Richard's equation (1) gives the ordinary differential equation:

$$\frac{4c\psi_{\omega\omega}}{\psi^2} - \frac{8c\psi_\omega^2}{\psi^3} - \frac{2k\psi_\omega}{\psi^2} - c_2 e^{\frac{k\omega}{c}}\psi_\omega = 0 \tag{37}$$

Clearly the equations (36) and (37) together define $\psi(\omega)$ and $\omega(x,t)$ to give the final form of the solution (35).

For the second symmetry the following solutions of (31) and (32) have also been found:

$$f = 0 \qquad H = \text{arbitrary} \qquad K = \text{arbitrary} \tag{38}$$

with infinitesimals:

$$\eta = \eta(t) \qquad \phi = 0 \tag{39}$$

and in a third case solutions of (31) and (32) may be obtained when:

$$f = 0 \qquad H = cu \qquad K = ku \tag{40}$$

This gives rise to the infinitesimals:

$$\eta = \frac{x - kt + c_2}{2t + c_0} \qquad \phi = \frac{c_4 + c_1 u + c_6 e^{c_3(cc_3 t - kt - x)}}{2t + c_0} \tag{41}$$

The similarity solutions corresponding to the second and third symmetry follows as for the first symmetry. However the details will not be presented here.

4.2. The sub-case $f \neq 0$

For this sub-case substitution of (28) and (29) into the determining equation (20) gives rise to the following condition for f and g:

$$a_0 - g a_2 - f a_4 = 0 \tag{42}$$

where a_i are constants for which the functions Z, X and W satisfy:

$$\frac{Z_u}{H_u} - K = a_0 u + a_1 \qquad \frac{X_u}{H_u} + u = a_2 u + a_3 \tag{43}$$

$$\frac{W_u}{H_u} + \int H du = a_4 u + a_5 \tag{44}$$

The remaining two determining equations (19) and (21) have the lengthy form:

$$\begin{aligned}
&f_x\left(4H_u^2Z_u - H^2H_uK_{uu} + H^2K_uH_{uu} - H\,H_u^2K_u\right)\\
&+2f\,Z\left(H_uK_{uu} - K_uH_{uu}\right)\\
&+2f^2\left(-H\,H_{uu}Z - 2\,H_u^2Z - H_uK_{uu}W + K_uH_{uu}W\right)\\
&+2f\,g\left(-H_{uu}Z - H_uK_{uu}X + K_uH_{uu}X\right)\\
&+f_x\,g\left(-4H_u^2X_u + H^2H_{uu} - 2H\,H_u^2\right) + 2f\,g_x\left(-2H_u^2X_u - HH_u^2\right)\\
&+2f^2\,g\left(H\,H_{uu}X + 2\,H_u^2X + H_{uu}W\right) + 2f\,g^2H_{uu}X\\
&+f f_x\left(H^3H_{uu} - 8H_u^2W_u\right) + 2f^3W\left(HH_{uu} + 2H_u^2\right) - 2H_u^3S_x\\
&+\left(K_uH_{uu} - H_uK_{uu}\right)(R+HS) + f\left(HH_{uu} + 2H_u^2\right)(R+HS)\\
&+g\,H_{uu}(R+HS) - g_xH_u^2K_u - 3\,f_{xx}H\,H_u^3 + g_{xx}H_u^3\\
&-f_t\,H\,H_u^2 - g_t\,H_u^2 - 2gg_x\,H_u^2\\
&=0
\end{aligned} \tag{45}$$

$$\begin{aligned}
&4\,f^2H_{uu}Z^2 + f\,f_x\left(-4\,H^2H_{uu}Z + 4\,H\,H_u^2Z + 4\,H_u^2K_uW\right)\\
&+f_{xx}\left(H^2H_u^2K_u - 2H_u^3Z\right) + f\,g_x\left(4H_u^2Z + 2H_u^2K_uX\right)\\
&-8\,f^2g\,H_{uu}X\,Z - 8\,f^3H_{uu}W\,Z - 4\,f\,H_{uu}Z\,(R+HS)\\
&-2\,f_xH_u^2K_uZ + 2\,f_tH_u^2Z + 4\,f^2g^2H_{uu}X^2\\
&+f\,f_x\,g\left(4\,H^2H_{uu}X - 4\,H\,H_u^2X\right) + f_xg_x\left(4\,H_u^3X - 2\,H^2H_u^2\right)\\
&+8\,f^3g\,H_{uu}W\,X + 4\,f\,gH_{uu}X\,(R+HS) + 2f_x\,gH_u^2K_uX + 2\,f\,g_{xx}H_u^3X\\
&+2\,f_{xx}g\,H_u^3X - 2\,f\,g_tH_u^2X - 4\,f\,g\,g_xH_u^2X - 2\,f_t\,g\,H_u^2X + 4\,f^4H_{uu}W^2\\
&+f^2f_x\left(4\,H^2H_{uu}W - 4\,H\,H_u^2W\right) + f_x^2\left(4\,H_u^3W + H^4H_{uu} - 2\,H^3H_u^2\right)\\
&+4\,f^2H_{uu}W(R+HS) + 4\,f\,f_{xx}H_u^3W - 4\,f^2g_xH_u^2W\\
&-4\,f\,f_tH_u^2W - H_u^2(R_t + HS_t) + H_u^3(R_{xx} + HS_{xx}) + H_u^2K_u(R_x + HS_x)\\
&+H_{uu}(R+HS)^2 + f_x\left(2\,H^2H_{uu} - 2\,H\,H_u^2\right)(R+HS)\\
&-2g_xH_u^2(R+HS) + f_{xxx}H^2H_u^3 - f_{xt}H^2H_u^2\\
&=0
\end{aligned} \tag{46}$$

The following three symmetry solutions have been found have been found from these equations. In the first sub-case:

$$f = c_0 \qquad K_u = -c_0H \tag{47}$$

where c_0 is constant and the corresponding infinitesimals are:

$$\eta = c_0H \qquad \phi = 0 \tag{48}$$

Secondly the sub-case

$$f = \frac{k}{c} \qquad H = cu \qquad K = ku^2 \tag{49}$$

gives rise to the infinitesimals:

$$\eta = k\,(u+h) \qquad \phi = -\frac{k^2u^2\,(u+h)}{c} \tag{50}$$

where $h = h\,(x,t)$ such that

$$h_t = ch_{xx} + 2khh_x \tag{51}$$

and finally the sub-case:

$$f = -\frac{c_1\lambda}{c_0 + c_1x + c_2t} \qquad H = ce^{\lambda u} \qquad K = ke^{\lambda u} \tag{52}$$

yields:

$$\eta = -\frac{cc_1\lambda e^{\lambda u}}{c_0 + c_1x + c_2t} \qquad \phi = \frac{kc_1\lambda e^{\lambda u} - c_2}{\lambda\,(c_0 + c_1x + c_2t)} \tag{53}$$

The corresponding similarity solutions for these sub-cases giving the solutions of Richard's equations will be considered elsewhere.

Functions H and K	Symmetries $\phi = \eta u_x + u_t$
$H=\frac{c}{u}$ $K=\frac{k}{u}$	$\eta=c_2e^{-\frac{kx}{c}}$ $\phi=\frac{c_2ku}{c}e^{-\frac{kx}{c}}$
$H,\ K$ arbitrary	$\eta=\eta(t)$ $\phi=0$
$H=H(u)$ $K_u=-c_0H$	$\eta=c_0H$ $\phi=0$
$H=cu$ $K=ku^2$	$\eta=k(u+h)$ $\phi=-\frac{k^2u^2(u+h)}{c}$ $h_t=ch_{xx}+2khh_x$
$H=ce^{\lambda u}$ $K=ke^{\lambda u}$	$\eta=-\frac{cc_1\lambda e^{\lambda u}}{c_0+c_1x+c_2t}$ $\phi=\frac{kc_1\lambda e^{\lambda u}-c_2}{\lambda(c_0+c_1x+c_2t)}$

Table 2: *Examples of non-classical symmetries of Richard's equation*

5. Potential symmetries

Richard's equation may also be written as the potential system $\mathbf{\Delta} \equiv (\Delta_1, \Delta_2) = \mathbf{0}$ where:

$$\Delta_1 = v_x - u = 0 \qquad \Delta_2 = v_t - H_uu_x - K = 0 \tag{54}$$

and this form gives rise to many new important symmetries. In this case the classical Lie analysis is based upon the infinitesimal transformations of the four variables:

$$\bar{x} = x + \varepsilon\eta_1(x,t,u,v) + O(\xi^2) \qquad \bar{t} = t + \varepsilon\eta_2(x,t,u,v) + O(\xi^2)$$
$$\bar{u} = u + \varepsilon\phi_1(x,t,u,v) + O(\xi^2) \qquad \bar{v} = v + \varepsilon\phi_2(x,t,u,v) + O(\xi^2) \tag{55}$$

Note that since $u = v_x$ these define contact transformations for v. The associated generator:

$$\mathcal{X} = \eta_1\frac{\partial}{\partial x} + \eta_2\frac{\partial}{\partial t} + \phi_1\frac{\partial}{\partial u} + \phi_2\frac{\partial}{\partial v} \tag{56}$$

and the condition for form invariance of (54) is found by applying the first prolongation so that now

$$\mathcal{X}_E^{(1)}(\mathbf{\Delta})\,|_{\Delta_1=0,\Delta_2=0} = 0 \tag{57}$$

It may be shown that the seven linear determining equations are:

$$\eta_{2_u} = 0 \qquad \eta_{2_v}H_u - \eta_{1_u} = 0 \tag{58}$$
$$u\eta_{2_v} + \eta_{2_x} = 0 \qquad \phi_{2_u} - u\eta_{1_u} = 0 \tag{59}$$

$$2\eta_{1_u}K + \phi_1 H_{u\,u} - \phi_{2_v}H_u + \phi_{1_u}H_u + \eta_{2_t}H_u - \eta_{1_x}H_u = 0 \tag{60}$$

$$\phi_{2_u}K - u\eta_{1_u}K - u\phi_{2_v}H_u - \phi_{2_x}H_u + \phi H_u + u^2\eta_{1_v}H_u + u\eta_{1_x}H_u$$
$$= 0 \tag{61}$$

$$\phi_1 H_u K_u - \eta_{1_u}K^2 - \phi_1 H_{u\,u}K - \phi_1 H_u K + u\eta_{1_v}H_u K$$
$$+ \eta_{1_x}H_u K + u\phi_{1_v}H_u^2 + \phi_{1_x}H_u^2 - \phi_{2_t}H_u + u\eta_{1_t}H_u$$
$$= 0 \tag{62}$$

There are two main cases. to be considerd. In the first $\eta_{1_v} \neq 0$ whilst in the second, (considered in detail by Sophocleous [7]) $\eta_{1_v} = 0$.

5.1. The case $\eta_{1_v} \neq 0$

The determining equations have the following new symmetry solution:

$$H = c\left(\frac{u}{c_0 + uc_1}\right)^{\frac{1}{c_0}-1} \qquad K = ku\left(\frac{u}{c_0 + uc_1}\right)^{\frac{1}{c_0}-1} \qquad \text{when} \qquad c_0 \neq 0, 1 \tag{63}$$

$$H = ce^{-\frac{1}{c_1 u}} \qquad K = kue^{-\frac{1}{c_1 u}} \qquad \text{when} \qquad c_0 = 0 \tag{64}$$

$$H = c\ln\left(\frac{u}{1 + c_1 u}\right) \qquad K = ku \qquad \text{when} \qquad c_0 = 1 \tag{65}$$

with infinitesimals

$$\eta_1 = c_0 x + c_1 v \qquad \eta_2 = c_3 + t$$
$$\phi_1 = -u\,(c_0 + c_1 u) \qquad \phi_2 = 0 \tag{66}$$

A detailed discussion of the corresponding similarity solutions of Richard's equation is the subject of on going research.

5.1. The case $\eta_{1_v} = 0$

In this case it follows that $\eta_2 = \eta_2\,(t)$ and the corresponding results have been analysed by Sophocleous [7]. Table 3 is based upon his comprehensive work in which all the known symmetries for this case are presented excluding the obvious translational symmetries.

Functions H and K	Symmetries
$H=cu$, $K=ku^{\mu}+k_0u$	$\eta_1=(1-\mu)(x-k_0t)$, $\phi_1=u$, $\eta_2=2(1-\mu)t$, $\phi_2=(2-\mu)v$
$H=cu$, $K=ke^{\lambda u}+k_0u$	$\eta_1=\mu(k_0t-x)$, $\eta_2=-2\mu t$, $\phi_1=1$, $\phi_2=-\mu v+x+k_0t$
$H=cu$, $K=k\ln u+k_0u$	$\eta_1=-kt$, $\eta_2=2t$, $\phi_1=u$, $\phi_2=2v+kt$
$H=cu$, $K=ku\ln u+k_0u$	$\eta_1=x-k_0t$, $\eta_2=0$, $\phi_1=u$, $\phi_2=v$
$H=cu$, $K=ku^2$	$\eta_1=-2kt$, $\eta_2=0$, $\phi_1=1$, $\phi_2=x$
$H=cu$, $K=ku^2$	$\eta_1=-2kxt$, $\eta_2=-2kt^2$, $\phi_1=x+2kut$, $\phi_2=ct+\frac{t^2}{2}$
$H=ce^{\lambda u}$, $K=ku^2$	$\eta_1=\lambda x-2kt$, $\eta_2=\lambda t$, $\phi_1=1$, $\phi_2=x+\lambda v$
$H=ce^{\lambda u}$, $K=ke^{\mu u}+k_0u$	$\eta_1=(\lambda-\mu)x-k_0\mu t$, $\eta_2=(\lambda-2\mu)t$, $\phi_1=1$, $\phi_2=x+k_0t+(\lambda-\mu)v$
$H=cu^{\lambda}$, $K=ku^{\mu}+k_0u$	$\eta_1=(\lambda-\mu)x+k_0(\mu-1)t$, $\phi_1=u$, $\eta_2=(\lambda-2\mu+1)t$, $\phi_2=(\lambda-\mu+1)v$
$H=cu^{\lambda}$, $K=ku$	$\eta_1=\frac{(\lambda-1)}{2}(x+kt)$, $\eta_2=0$, $\phi_1=u$, $\phi_2=\frac{(\lambda+1)}{2}v$
$H=cu^{\lambda}$, $K=ku\ln(u)+k_0u$	$\eta_1=(\lambda-1)x-kt$, $\phi_1=u$, $\eta_2=(\lambda-1)t$, $\phi_2=\lambda v$
$H=cu^{\lambda}$, $K=k\ln(u)+k_0u$	$\eta_1=\lambda x-k_0t$, $\phi_1=u$, $\eta_2=(\lambda+1)t$, $\phi_2=(\lambda+1)v+kt$

Table 3 *Potential symmetries of Richard's equation* (based upon the comprehenvive analysis of the case $\eta_2 = \eta_2\,(t)$ by Sophocleous [7])

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