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# Knot parallels as elements of $\pi_1$ of the knot's complement

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**Abstract**

We prove in a purely geometric and sufficiently elementary way that the parallel of a knot as an element of the knot's complement fundamental group is expressed (for any given knot projection) as a product of only the overpass Wirtinger generators, and moreover that for each generator the sum of exponents in this product equals 0.

Our goal will be to prove in a very elementary way using only the very basics the following

**Proposition.** *If $L$ is a knot in $S^3$ and $par(L)$ is a parallel of $L$ through the base point $*$ of $S^3 - L$, then $[par(L)] \in \pi_1(S^3 - L, *)$ can be expressed as a product of the Wirtinger generators of only the overpasses (in any knot projection $proj(L)$, of $L$) so that the sum of the exponents of each generator is equal to* 0.

We first provide a short list of those definitions and properties of knots which will be of interest to us here.

Let $L$ be a smooth oriented knot in $S^3$, and $T^3$ be a tubular neighborhood of $L$ in $S^3$. The knot exterior $S^3 - intT^3$ of $L$ is defined up to isotopy in $S^3$ and it is denoted by $L^c$. Notice that $L^c$ and $T^3$ share a 2-torus as common boundary. A smooth oriented loop $par(L)$ lying in this torus is called a parallel (or else longitude) of $L$, whenever the loops $L$ and $par(L)$ are co-directed with linking number 0.

Co-directed means that the tangent vectors of the two loops at their common points with a plane perpendicular to $L$ (we demand that there exists one such point for each loop) have positive scalar product. Recall also that the linking number $link(L_1, L_2)$ of two smooth directed knots $L_1$ and $L_2$ is defined as the sum of the signs of those crossings in a projection of the link of the two knots, in which $L_1$ goes over $L_2$. Such a crossing has sign $+1$, if the angle formed by the tangent vectors of $L_1$ and $L_2$ (in this order) at the crossing point is positive. Otherwise the crossing has sign $-1$. Linking numbers do not depend on the knot projections or on the order in which we consider the two knots.

Parallels of a knot $L$ are defined up to isotopy in the knot's complement $S^3 - L$ in $S^3$. Notice that the knot exterior $S^3 - intT^3$ of $L$, is a deformation retract of the knot's complement $S^3 - L$, thus the two spaces can be considered essentially indistinguishable. In particular their fundamental groups are isomorphic and the isomorphism is induced by the inclusion. For our purposes it will be more convenient to work with the knot's complement (as opposed to the knot's exterior). We shall denote the base point of $S^3 - L$ by $*$.

Our arguments will be based on the notion of knot projections. A knot projection $proj(L)$ of $L$ is an actual projection of $L$ into a plane, containing only finitely many multiple points, all of them being double points. Such projections exist (and moreover they are dense in the set of all projections of the knot [1]). Then the knot $L$ can be viewed in a new position inside $S^3$ as follows: Consider an axis perpendicular to the projection plane with its origin on the plane. Next choose any crossing point $p$ of $proj(L)$ and consider the arcs $p_1$ and $p_2$ of $proj(L)$ inside a small neighborhood $V$ of $p$. Note that there exist exactly two disjoint arcs $c_1$ and $c_2$ of $L$ that project onto $p_1$ and $p_2$. Push under the plane and inside $V$ either $p_1$ or $p_2$ (fixing its endpoints), depending on whose point over $p$ among $c_1$ and $c_2$ was in a lower altitude (measured by the axis). Finally perform this push off for all crossing points of $proj(L)$. We shall consider this new position of $L$ as its actual position.

Finally, the Wirtinger loops corresponding to a projection of $L$, are just unknots in $S^3 - L$, one for each arc of the projection, linking once with this arc and linking with no other arc of the projection. Their homotopy classes generate $\pi_1(S^3 - L, *)$. The Wirtinger loops are considered smooth and oriented (arbitrarily but in a fixed way).

We will be working with projections of the link of $L$ and $par(L)$, thus projecting both of them onto the same plane. Since we are going to consider the homotopy class representing $par(L)$ in $\pi_1(S^3 - L, *)$, we need to think of $par(L)$ as going through the base point $*$. Thus $par(L)$ will mean a parallel of $L$ joined to $*$ via a simple arc in $S^3 - L$ whose only common points with the projection plane are those in $par(L)$ (i.e. so that it does not coil around $L$).

A line of argument towards a proof of our Proposition is the following: Express $[par(L)]$ as a product of Wirtinger generators in any given knot projection $proj(L)$ of $L$, in one of the known ways. For example, using the notion of a Seifert surface of $L$, one can show that $[par(L)] = (a_1 b_1 a_1^{-1} b_1^{-1})(a_2 b_2 a_2^{-1} b_2^{-1}) \cdots (a_n b_n a_n^{-1} b_n^{-1})$ for some elements $a_i, b_i$ of $\pi_1(S^3 - L, *)$ ([1, p.37], [4, p.140]). Then express each $a_i, b_i$ in terms of the Wirtinger generators of $proj(L)$. Finally, observe that any Wirtinger generator of an arc that remains always an underpass in $proj(L)$ can be expressed as a product of the adjacent overpass generators, and you have a proof of the Proposition.

The promised proof relying only to the basics is the following:

*Proof.* For a knot projection with no crossing points (i.e. for the standard projection of the unknot) the result is trivial.

For all other cases we first fix a projection $proj(L)$ of $L$ and note that if the result holds for this projection, then trivially, it also holds if we slightly isotope $L$ (together with its tubular neighborhood, parallels and projection). Note that it is also trivially true that if the result holds for some specific parallel of $L$, then it also holds for any parallel of $L$.

Now our plan for the proof will be (I) to slightly isotope $L$ (together with its tubular neighborhood, parallels and projection), and (II) to construct a "nice" parallel of $L$, so that (III) the result would follow relatively easily by observing a picture of the situation:

(I) It is convenient to think of $S^3$ as $\mathbb{R}^3 U\{\infty\}$. It is also convenient to choose our point of view at some point of the positive $z$-axis and choose $xOy$ as the projection plane.

Let $c_1, c_2, ..., c_n$ be the crossing points of $proj(L)$, and $B_1^3, B_2^3, ..., B_n^3$ be some small disjoint 3-balls around these points.

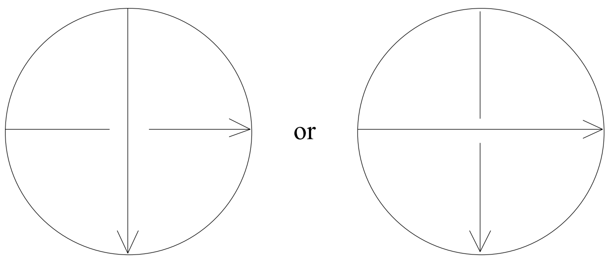


Figure 1:

We choose these balls small enough so that the part of $proj(L)$ inside each one of them consists of just two intersecting arcs. We can assume (after performing some small isotopies if necessary) that inside each ball, one of these arcs is horizontal (parallel to $Ox$), and the other vertical (parallel to $Oy$), and moreover that the vertical segment is an overpass. The orientation of $proj(L)$ induces an orientation to each one of these arcs. We can assume even further (again by performing small isotopies if necessary) that the orientation of the vertical arc runs always opposite to the positive orientation of the axis $Oy$. Then the part of $proj(L)$ inside each $B_i^3$ looks (from our point of view) like in Figure 1.

(II) Since $proj(L)$ is an immersed circle in the plane $xOy$, it possesses a trivial normal vector bundle; a proof of this can be trivially deduced from [2] (Lem 4.1 and 4.3) and [3] (p. 30, Cor.3.5), since the tangent bundles of the circle and of the plane are trivial. We can choose one, so that at each point of $proj(L)$, the tangent vector of $proj(L)$, followed by the normal vector at that point, form a positive angle. In order to produce a parallel copy $proj'(L)$ of $proj(L)$, we consider a section of this vector bundle. We can choose it so close to $proj(L)$ so that its part outside the $B_i^3$'s does not share any common points with $proj(L)$. Note that $proj'(L)$ gets an induced orientation from $proj(L)$ (as a cross section of the normal vector bundle).

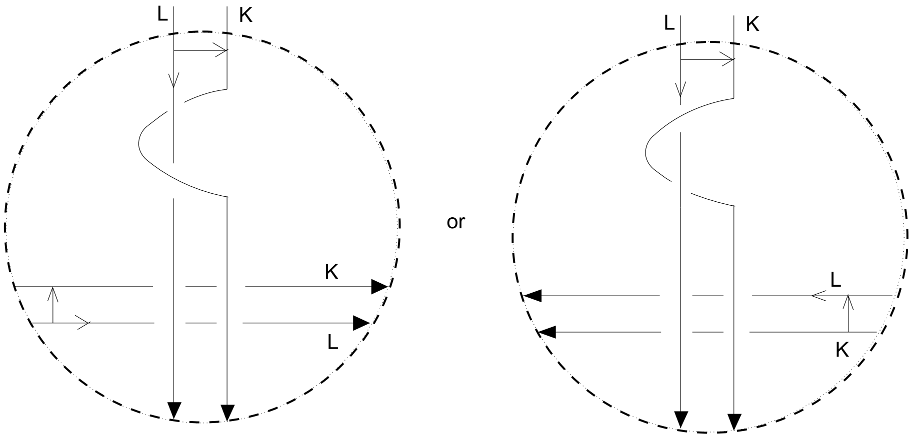


Figure 2:

The part of $proj(L)$ and $proj'(L)$ outside the $B_i^3$'s consists of disjoint simple arcs; to each arc of $proj(L)$ corresponds a parallel arc of $proj'(L)$ (which can be chosen arbitrarily close).

We shall now modify $proj'(L)$ so that we get a parallel of $L$ (Figure 2):

Inside each $B_i^3$ we i) push the horizontal arc of $proj'(L)$ into $\mathbb{R}^3_-$ fixing its endpoints and ii) discard the vertical part of $proj'(L)$ replacing it by a new arc with the same endpoints and linking inside $B_i^3$ once with the vertical part of $L$ (preferably in the half of $B_i^3$ with positive $y$ coordinates.

Figure 2 shows the orientations of the two new arcs inside each $B_i^3$. The orientations are chosen so that the new arcs together with the oriented arcs of $proj'(L)$ outside the $B_i^3$'s form an oriented knot $K$ in $S^3$. The linking of $K$ and $L$ inside each $B_i^3$ was chosen so that each crossing point of their projections inside each such ball contributes 0 to their linking number.

Since all crossing points of $L, K$ lie inside these balls we conclude that $link(L, K) = 0$.

Clearly, the knots $L, K$ are co-directed.

Thus $K$ is a parallel of $L$.

(III) In order for $K$ to be considered as an element of $\pi_1(S^3 - L, *)$, we need to homotope it inside $S^3 - L$ so that it goes through the base point $*$, which chosen inside $\mathbb{R}^3_+$.

We now need a way to compare $K$ with the Wirtinger loops corresponding to the projection $proj(L)$ of $L$.

We achieve this by pushing into $R^3_+$ all the arcs of $K$ which lie inside the balls $B^3_i$, in the way shown in Figure 3. Let's call this homotoped version of $K$ as $par(L)$.

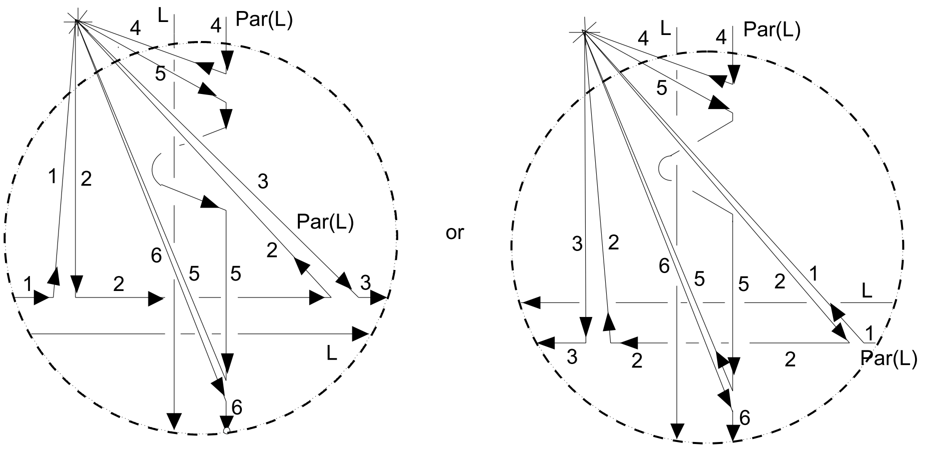


Figure 3:

Observe that the homotopy classes of the loops 2 and 5 are inverses of each other in $\pi_1(S^3 - L, *)$, and that one of these two classes is exactly the homotopy class of the Wirtinger loop corresponding to the overpass arc of $L$ at the crossing point $c_i$.

Observe also that each one of the "half-loops" $1, 3, 4, 6$, meet another "half-loop" arising from another (not necessarily distinct) crossing point of $proj(L)$. Then these two half-loops, form an oriented closed loop (call it 2-halves) through $*$ representing thus an element of $\pi_1(S^3 - L, *)$. Clearly these loops are contractible to a point in $S^3 - L$, thus representing the identity element of $\pi_1(S^3 - L, *)$.

Finally observe that a journey once around $par(L)$, starting at $*$ and following its orientation, gives an expression of $[par(L)]$ as a product of the various classes $[loop2]$, $[loop5]$, $[2 - halves]$. The exponent of any one of these classes in the product equals of course to 1.

The last three observations imply the required result. □

## References

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