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Advertibly Complete Locally m-Convex Topologically Maximally Bilateral Algebras

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Abstract

Many results of two-sided advertibly complete m-convex algebras remain valid for "topologically maximally bilateral" ones; in particular, the commutativity modulo the Jacobson radical. A characterization of an advertibly complete m-convex algebra, which is commutative modulo its Jacobson radical is obtained. Furthermore, several characterizations of *Q-algebras* are given.

I-Introduction and preliminaries

In [3], the authors study the class of advertibly complete and two-sided m-convex algebras, they showed that these algebras are commutative modulo the (Jacobson) radical $RadA$ and share a lot of properties in common with commutative ones. In the context of an *m-convex algebra* (*l.m.c.a.* in brief), we examine here the same questions in a more general case, namely the case of a topological algebra in which all closed (right or left) maximal ideals are two sided. We call such an algebra "Topologically Maximally Bilateral" (a terminology due to A. Mallios), or even, in short, a *t.m.b.* algebra, yet, simply a *t.m.b.a.*. The main result of the second section is a characterization of a unital advertibly complete *l.m.c.t.m.b.a.*; namely: if A is a unital advertibly complete *l.m.c.a.* then A is a *t.m.b.a.* if, and only if, A is commutative modulo its Jacobson radical $RadA$. The third section is devoted, in the context of these class of algebras, to links between the *Q-property*, the compactness of the carrier space , the maximal ideals of codimension 1 of A, the closed maximal ideals of A, the sub-algebra $\mathcal{B}(A)$ of bounded elements of A, the range of the Gelfand transform $\widehat{x}\,(\mathcal{X}(A))$ for every element $x \in A$ and the spectrum $Sp_A\,(x)$ of every element $x \in A$. Within this context, we also refer to a recent paper by H. Arizmendi - V. Valov [1], where the authors deal with similar questions, concerning characterizations of *Q-algebras*.

A locally convex algebra (A, τ), *l.c.a.* in brief, is a complex algebra with a Hausdorff *locally convex topology* for which the product is separately continuous. A *l.c.a.* (A, τ)

is said to be *m-convex* (*l.m.c.a.*) if the origin 0 admits a fundamental system of idempotent neighborhoods [9].

Let $(A,(p_\lambda)_\lambda)$ be a complex *l.m.c.a.*. For every λ, set

$$N_\lambda = \{x : p_\lambda(x) = 0\} \tag{1}$$

Consider $A_\lambda = A/N_\lambda$ endowed with the topology defined by the norm $x_\lambda \longmapsto \|x_\lambda\|_\lambda = p_\lambda(x)$, $x_\lambda = \pi_\lambda(x)$, $x \in A$, where π_λ is the natural quotient map from A onto A/A_λ. Denote by $\widehat{A_\lambda}$ the Banach algebra (completion) of the normed algebra $(A_\lambda, \|.\|_\lambda), \lambda \in \Lambda$. Put $A^2 = \{xy : x, y \in A\}$. If $A^2 = \{0\}$, then A is said to be a zero-algebra. An element $x \in A$, is said to be right (resp. left) quasi- invertible if there is $y \in A$ such that $x \circ y := xy - x - y = 0$ (resp. $y \circ x = 0$); and it is quasi-invertible ($q.i$ in short) if it is left and right quasi-invertible. The set of all quasi-invertible elements of A is denote by $G_q(A)$. The algebra A is said to be a Q-algebra if $G_q(A)$ is open. Given a topological algebra (A,τ), a net $(x_i)_i$ in A is called *advertibly null*, if there exists an element $x \in A$ such that both of the nets $(x \circ x_i)_i$ and $(x_i \circ x)_i$ are null (i. e. they converge to the zero element of A). Now, a topological algebra A is said to be *advertibly complete*, if every Cauchy net in A, which is also advertibly null, converges [8] (the terminology applied herewith, is still due, lately, to A. Mallios). The set of non trivial continuous characters (continuous multiplicative linear functionals) of A, will be designated by $\mathcal{X}(A)$. Also $\mathcal{M}(A)$ will stand for all closed, regular, right or left maximal and two-sided ideals of A; and we put $\Re(A) = \bigcap\limits_{M \in \mathcal{M}(A)} M$. A proper ideal of A is an ideal different from A (it may be $\{0\}$). The classical spectral radius of an element x will be denoted by $\rho_A(x)$; that is $\rho_A(x) = Sup\{|\lambda| : \lambda \in Sp_A(x)\}$, where $Sp_A(x)$ is the spectrum of x. Recall that

$$\begin{aligned}
Sp_A(x) &= \{\lambda \in \mathbb{C} : x - \lambda e \text{ is not invertible}\}, \text{if } A \text{ is unital.}\\
Sp_A(x)\setminus\{0\} &= \left\{\lambda \in \mathbb{C}^* : \frac{x}{\lambda} \text{ is not quasi-invertible}\right\}, \text{if } A \text{ is not unital.}\\
Sp_A(x) &= (Sp_A(x)\setminus\{0\}) \cup \{0\}, \text{if } A \text{ is not unital.}
\end{aligned}$$

Finally recall that if for each $a, b \in A$ there exists $u, v \in A$ such that $ab = va = bu$ (cf. [5]), then A is said to be two-sided or bilateral.

II - Jacobson radical and related topics

All topological algebras considered here are complex and Haussdorff.
Give at first some examples of *t.m.b.a.*.

Examples II-1.

1) Let

$$A = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{C} \right\}. \tag{2}$$

Then
$$Rad(A) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{C} \right\}. \tag{3}$$
The algebra A is Banach, not unitary, not two-sided and its Jacobson radical is the unique (closed) right (and left) maximal ideal of A. So, it is a $t.m.b.a.$

2) Every two-sided nonradical $l.m.c.a.$ is a $t.m.b.a.$

3) The cartesian product of any $t.m.b.a$ by any commutative (or two-sided) nonradical $l.m.c.a.$ is a $t.m.b.a.$ too.

Remark II-2. In a $t.m.b.a$ A, all closed right or left maximal ideals are two-sided. So, for any regular right or left maximal (and closed) ideal M, the algebra A/M is a field. If, in more, the algebra A is a Gelfand-Mazur algebra, then A/M is isomorphic to $\mathbb{C}$. It appears that when A is a Gelfand-Mazur $t.m.b.a$, there is a one to one correspondence between the set $\mathcal{M}(A)$ of closed, regular right or left maximal two-sided ideals of A and the set of kernels of the non-zero continuous characters of A. Consequently one has
$$\Re(A) = \{x \in A : f(x) = 0, \text{ for every } f \in \mathcal{X}(A)\}. \tag{4}$$

The following result is a generalization of Proposition II-3 of [3]; given therein for two-sided algebras.

Proposition II-3. *Let A be a $l.m.c.t.m.b.algebra$. Then the following assertions are equivalent*:

1) *A is advertibly complete.*

2) *$x \in A$ is quasi-invertible if, and only if, $f(x) \neq 1$, for every $f \in X(A)$*

Proof. **1)** $\Longrightarrow$ **2)**. If x is quasi-invertible, then $f(x)$ is quasi-invertible for every $f \in \mathcal{X}(A)$, and so $f(x) \neq 1$ for every $f \in \mathcal{X}(A)$. Conversely, if x is not quasi-invertible then, by Proposition II-6 of [3], there exists a closed, regular and maximal one-sided ideal J of A such that x is a unit element of A modulo J. But J is, by assumption, two-sided. The m-convex algebra A/J, being a field, is isomorphic to $\mathbb{C}$. **2)** $\Rightarrow$ **1)**, is a consequence of ([8], lemma 5.2 and corollary 5.1, chap. III, p. 96). □

Corollary II-4. *Let A be a unital $l.m.c.t.m.b.a.$. Then the following assertions are equivalent*:

1) *A is advertibly complete.*

2) *$x \in A$ is invertible if, and only if, $f(x) \neq 0$, for every $f \in X(A)$.*

Remark II-5. Let A be an advertibly complete and Hausdorff $l.m.c.t.m.b.a.$. Then $RadA = \bigcap_{f \in \mathcal{X}(A)} Ker(f)$ and, in particular, $RadA$ is closed. Indeed, if we denote by K the right side of the previous equality, then one has $RadA \subset K$. Conversely if $x \in K$, then $f(x) \neq 1$ for every $f \in \mathcal{X}(A)$, so x is quasi-invertible. So K is an ideal consisting of quasi-invertible elements. Hence $K \subset RadA$.

Proposition II-6. *Let A be an advertibly complete $l.m.c.t.m.b.a.$ and $x \in A$. Then,*

a) *In the non unital case,* $Sp_A(x) = \{f(x) : f \in \mathcal{X}(A)\} \cup \{0\}$.

b) *In the unital case,*

$Sp_A(x) = \{f(x) : f \in \mathcal{X}(A)\} \cup \{0\}$, *if* x *is not invertible.*

$Sp_A(x) = \{f(x) : f \in \mathcal{X}(A)\}$, *if* x *is invertible.*

Proposition II-7. *Let* A *be a unital advertibly complete l.m.c.t.m.b.a. Let* M *be a two-sided right or left maximal ideal of* A. *Then there are only two possibilities.*

i) M *is the kernel of a non vanishing continuous character of* A.

ii) M *is everywhere dense in* A.

Proof. Since every ideal of A is regular, A/M is a field. By Remark II-2, M is closed, if, and only if, it is the kernel of a non vanishing continuous character of A. If M is not closed, then $\overline{M}$ is an ideal containing M; so $\overline{M} = A$. □

Remark II-8. Let A be an algebra. Then M is a right or left maximal two-sided ideal of A if, and only if, $S(M)$ is of the same type in $B := A/RadA$, where S is the canonical map from A onto B. Indeed, let Ψ be the set of all left or right maximal ideals of A and $\widetilde{\Psi}$ the set of all right or left maximal ideals of B. Then, the correspondence

$$\begin{array}{ccc} \Psi & \longrightarrow & \widetilde{\Psi} \\ M & \longrightarrow & S(M) \end{array}$$

is a one to one. Suppose now that A is a Gelfand-Mazur algebra. Then M is a closed right or left maximal and two-sided ideal of A if, and only if, $S(M)$ is a closed right or left maximal and two-sided ideal of B. Indeed, if M is such an ideal in A, there exists a continuous character f of A such that $M = Kerf$. So, there exists a continuous character g of B such that $f = g \circ S$. Hence $S(M)$ is closed. Conversely, if $\widehat{M}$ is a closed left or right maximal ideal of B, then $S^{-1}\left(\widehat{M}\right)$ is closed left or right maximal ideal of A such that $S\left(S^{-1}\left(\widehat{M}\right)\right) = \widehat{M}$.

Proposition II-9. *Let* A *be a unital and advertibly complete l.m.c.a. Then* A *is a t.m.b.a if, and only if, the quotient algebra* $B := A/RadA$ *is commutative.*

Proof. Since the (Jacobson) radical of A is closed, $A/RadA$ is a *l.m.c.a.* Suppose that A is a *t.m.b.a.* Then $RadA = \bigcap_{f \in \mathcal{M}(A)} Ker(f)$. So one has $xy - yx \in RadA$ for every $x, y \in A$. Hence, the commutativity of B. Conversely, suppose that B is a commutative *l.m.c.a.*. By Corollary II-7 of [3], B is normal. Hence B is a *t.m.b.a.* By the last remark, A is a *t.m.b.a* too.

Recall that, by Corollary II-3 of [3], a unital *l.m.c.a.* is always normal. Besides, we have the following property.

(R): Let A be a unital and advertibly *l.m.c.a.* with $x \in A$. Then

$$x \in G(A) \Longleftrightarrow \overline{Ax} = \overline{xA} = A. \tag{5}$$

Proposition II-10. *Let A be a unital topological algebra. Moreover, consider the following assertions.*

1) *$RadA$ is closed and $A/RadA$ is advertibly complete.*

2) *A is advertibly complete.*

Then **1)** $\Longrightarrow$ **2)**. *If, in more, A is l.m.c.a., then* **2)** $\Longrightarrow$ **1)**.

Proof. **1)** $\Longrightarrow$ **2)**. Let $(x_\lambda) \subset A$ be a generalized Cauchy sequence and $x \in A$ such that $xx_\lambda \longrightarrow e$ and $x_\lambda x \longrightarrow e$. Then $\widetilde{x}\widetilde{x_\lambda} \longrightarrow \widetilde{e}$ and $\widetilde{x_\lambda}\widetilde{x} \longrightarrow \widetilde{e}$. In more, $(\widetilde{x_\lambda})$ is a generalized Cauchy sequence. Hence, there exists $\widetilde{y} \in A/RadA$ such that $\widetilde{x_\lambda} \longrightarrow \widetilde{y}$. So, $\widetilde{x}\widetilde{y} = \widetilde{y}\widetilde{x} = \widetilde{e}$. That is $\widetilde{x}$ is invertible in $A/RadA$. So, x is invertible in A. Hence, the sequence (x_λ) is convergent (to x^{-1}).

2) $\Longrightarrow$ **1)**. By Corollary II-7 of [3], $RadA$ is closed. Now, let $(\widetilde{x}_\lambda) \subset A/RadA$ be a generalized Cauchy sequence and $x \in A$ such that $\widetilde{x}\widetilde{x_\lambda} \longrightarrow \widetilde{e}$ and $\widetilde{x_\lambda}\widetilde{x} \longrightarrow \widetilde{e}$. Then $\overline{\widetilde{x}\,(A/RadA)} = \overline{(A/RadA)\,\widetilde{x}} = A/RadA$, that is, $\overline{(xA+RadA)\,/RadA} = \overline{(Ax+RadA)\,/RadA} = A/RadA$. Hence, we have $\overline{xA+RadA} = \overline{Ax+RadA} = A$. Suppose that x is not invertible in A. By (R), we can suppose that $\overline{Ax} \neq A$. But A is normal, so there exists a left maximal and closed ideal M of A that contains $\overline{Ax}$. By the fact that M contains $RadA$, it contains also $\overline{Ax+RadA} = A$. Which is impossible. Hence, x is invertible in A. So, $\widetilde{x}$ is invertible in $A/RadA$. Thereby, the sequence $(\widetilde{x}_\lambda)$ is convergent (to $\widetilde{x}^{-1}$). □

In [3], we have already mention that several authors, as, for instance, A. Mallios, E. A. Michael, S. Warner and W. Zelazko, have worked on Q-*algebras*. They considered characterizations of this sort of algebras, in certain particular classes of topological algebras. We generalize here previous results of the aforementioned authors in a more general framework.

III - Structure of some Q-algebras

We will need the following standard notations.

$$\begin{array}{rcl} \mathcal{B}(A) & = & \{x \in A : \rho_A(x) < \infty\}\,, \\ \mathcal{S}(A) & = & \{x \in A : \rho_A(x) \leq 1\}\,. \end{array}$$

Recall that if E is a locally convex space and $E^{'}$ its topological dual, the polar F° of a subset F of E is, by definition,

$$F^\circ = \left\{x^{'} \in E^{'} : Sup\left\{\left|< x, x^{'} >\right| : x \in F\right\} \leq 1\right\}, \tag{6}$$

Notice that if A is an advertibly complete *l.m.c.t.m.b.a.*, then, by Proposition II-6, $\mathcal{B}(A)$ is a subalgebra of A and $\mathcal{S}(A)$ is m-convex i.e., convex and idempotent.

The following result generalizes Remark III-1 of [3], given, therein, for a two-sided algebra.

Proposition III-1. Let A be a *l.m.c.t.m.b.a.*. *Then the following assertions are equivalent:*

1) *A is advertibly complete.*

2) $Sp_A(x) \cup \{0\} = \widehat{x}\,(\mathcal{X}(A) \cup \{0\})$, *for every* $x \in A$.

Proof. It is proved in [8] (Theorem 6.2, p. 104) that **2**) $\Longrightarrow$ **1**). Besides, if A is an advertibly complete *l.m.c.t.m.b.a* then, by Proposition II-6, we know that **1**) $\Longrightarrow$ **2**).□

Corollary III-2. *If A is a unital l.m.c.t.m.b.a., then, A is advertibly complete if, and only if*

$$\widehat{x}\,(\mathcal{X}(A)) = Sp_A(x) \neq \emptyset, \text{ for every } x \text{ in } A. \tag{7}$$

Remark III-3. Let $(A,(p_i)_{i\in I})$ be a unital advertibly complete and Hausdorff *l.m.c.t.m.b.a.* such that the Gelfand map g_A of A is continuous, then the Gelfand map g_B of $B = A/RadA$ is also continuous. Indeed, let $x \in A$ and S be the natural map from A onto B. We have

$$\|g_B(S(x))\| = Sup\left\{\left|g_B(S(x))(\dot{f})\right| : \dot{f} \in \mathcal{X}(B)\right\} \tag{8}$$

$$= Sup\left\{\left|\dot{f}(S(x))\right| : \dot{f} \in \mathcal{X}(B)\right\} \tag{9}$$

$$\leq Sup\left\{|f(x)| : f \in \mathcal{X}(A)\right\} = \|g_A(x)\| = \|g_A(y)\| \tag{10}$$

for every $y \in S^{-1}(x)$. But there exists $i \in I$ such that $\|g_A(y)\| \leq \|g_A\|\, p_i(y)$ for every $y \in A$. So

$$\|g_B(S(x))\| \leq \|g_A\|\, inf\left\{p_i(y) : y \in S^{-1}(x)\right\} = \|g_A\|\, \underset{i}{\dot{p}}(S(x)) \tag{11}$$

for every $x \in A$. Hence g_B is continuous.

Proposition III-4. *Let A be a l.m.c.t.m.b.a. such that the set $\mathcal{X}(A)$ is equicontinuous. Then the next assertions are equivalent:*

1) *An element $x \in A$ is $quasi-invertible$ if, and only if, $f(x) \neq 1$, for every $f \in X(A)$.*

2) *A is advertibly complete.*

3) *A is Q-algebra.*

4) *Every regular maximal ideal of A is closed.*

Proof. Without the concept of two-sidedness and independently, S. Warner and A. Mallios showed that **1**) $\Longrightarrow$ **3**) $\Longrightarrow$ [**2**) and **4**)]. By Proposition II-3, **2**) $\Longrightarrow$ **1**).Let us prove that **4**) $\Longrightarrow$ **1**). Since the necessary condition of **1**) is evident, suppose that $f(x) \neq 1$, for every $f \in \mathcal{X}(A)$ and x is not advertible. Then $x \in M$, with M a (two-sided) regular maximal ideal of A; but x is also a unit modulo M. Since, by hypothesis, M is closed , there exists a continuous character f such that $M = ker(f)$. Since $yx - y \in M$, for every $y \in A$, one has $f(x) = 1$; a contradiction with $x \in M$. □

Remark III-5. By Proposition II-6, in an advertibly complete $l.m.c.$ $t.m.b.a.$ A, we have $\rho_A(x) = sup\{|f(x)| : f \in \mathcal{X}(A)\}$. Then, the following assertions are equivalent

1) A is Q-algebra.

2) $\mathcal{X}(A)$ is equicontinuous.

Corollary III-6. *Let A be a unital $l.m.c.t.m.b.a.$ such that $\mathcal{X}(A)$is equicontinuous. Then the following assertions are equivalent:*

1) *A is Q-algebra.*

2) *Every regular maximal ideal of A is closed.*

3) *$Sp_A(x) \cup \{0\} = \widehat{x}\,(\mathcal{X}(A) \cup \{0\})$,for every $x \in A$.*

4) *$\rho_A(x) = sup\{|f(x)| : f \in \mathcal{X}(A)\}$.*

5) *A is advertibly complete.*

Proof. It is classical that **1**) $\Longrightarrow$ **2**). Let us prove that **2**) $\Longrightarrow$ **3**). If $\lambda \in Sp_A(x)$, then $x - \lambda e$ is not, for example, right invertible. Then there exists a right maximal ideal M such that $x - \lambda e \in M$. By hypothesis M is two-sided and closed. Then there exists a character f of A such that $f(x - \lambda e) = 0$. Consequently $Sp_A(x) = \{f(x) : f \in \mathcal{X}(A)\}$. The implication **3**) $\Longrightarrow$ **4**) is evident. The implication **4**) $\Longrightarrow$ **1**) is ensured by the equicontinuity of $\mathcal{X}(A)$. Since it is known that **1**) $\Longrightarrow$ **5**), it is enough to show that **5**) $\Longrightarrow$ **1**). By Proposition II-6, **5**) implies **4**). But $\mathcal{X}(A)$ is equicontinuous; so ρ_A is continuous at zero (so everywhere). Whence **1**). □

The next corollary is an improvement of Corollary II-6 of [3].

Corollary III-7. If the completion of a unital Q-$l.m.c.a.$ is a $t.m.b.a$, then it is Q-$l.m.c.a.$

Proof. Let A be a Q-$l.m.c.a.$. To show that its completion is a Q-$algebra$, it is enough, by Remark III-5, to show that the set of continuous characters of $\widehat{A}$ is equicontinuous. In a Q-$algebra$ we know that the set $\mathcal{X}(A)$ of continuous characters, of A, is equicontinuous. Then, for every $\varepsilon > 0$, there exists a neighborhood V of zero in A such that $|f(x)| \leq \varepsilon$, for every $x \in V$ and for every $f \in \mathcal{X}(A)$. Let $f \in \mathcal{X}(A)$ and $M = Ker(f)$. Then the closed ideal M is, for example, right maximal. Since $\widehat{A}$ is normal, by Lemma B.11 of [9], there exists a closed and right maximal ideal J of $\widehat{A}$ that contains M. By assumption, J is two-sided. So, there exists a continuous character $\widehat{f}$ of $\widehat{A}$ such that $J = Ker(\widehat{f})$ (conversely, every character $\widehat{f}$ of $\widehat{A}$ define a character f of A such that $f = \widehat{f}_{/A}$). Whence $\left|\widehat{f}(x)\right| \leq \varepsilon$, for every $x \in \overline{V}^{\widehat{A}}$ and for every $\widehat{f} \in \mathcal{X}(\widehat{A})$, where $\overline{V}^{\widehat{A}}$ is the closure of V in $\hat{A}$. Besides $\overline{V}^{\widehat{A}}$ is a neighborhood of zero in $\hat{A}$ ([6]). Then $\mathcal{X}(\widehat{A})$ is equicontinuous. □

Corollary III-8. *Let $(A, (p_\alpha)_\alpha)$ be a unital and advertibly complete $l.m.c.t.m.b.a.$, such that its completion $\hat{A}$ is a $t.m.b.a$ too and such that $X(A)$ equicontinuous. Then* *the following assertions are equivalent.*

- **a**) *A is Q-algebra.*
- **b**) *The closure in $\hat{A}$ of any subset of A not containing any invertible element in A, does not contain any invertible element in $\hat{A}$.*
- **c**) *The closure in $\hat{A}$ of any proper ideal of A is a proper ideal of $\hat{A}$.*

Proof. **a**) $\Rightarrow$ **b**). Let B be a subset of A not containing any invertible element of A. Let us suppose that the closure $\overline{B}^{\hat{A}}$, of B in $\hat{A}$, contains an invertible element x in $\hat{A}$. There is a generalized sequence, $(x_\alpha)_\alpha \subset B$ converging to x. By Corollary III-7, the assertion **a**) entails that $G(\hat{A})$ is open. Hence, $x_\alpha \in G(\hat{A}) \cap B$, for α large enough. So $0 \notin Sp_{\hat{A}}(x_\alpha)$. Now, by Proposition III-1, $Sp_{\hat{A}}(x_\alpha) = Sp_A(x_\alpha)$, for A is inverse closed in its completion. So, x_α should be invertible in A; a contradiction.

b) $\Rightarrow$ **c**). The product, in A, being continuous, the implication results from the fact that the closure, in $\hat{A}$, of any ideal of A is an ideal of $\hat{A}$.

c) $\Rightarrow$ **a**). Let us show that A is inverse closed in its completion. Let x be an element of A, which is not invertible in A. There is a maximal ideal M of A such that $x \in M$. So $x \in \overline{M}^{\hat{A}}$. By **c**), $\overline{M}^{\hat{A}}$ is a proper ideal of $\hat{A}$. Consequently x is not invertible in $\hat{A}$. The algebra A being inverse closed in its completion it is advertibly complete. One concludes by Corollary III-6. □

Now, we are going to generalize Proposition III-9, Corollary III-11, given in [3] for the case of spectrally barrelled algebras, to the more general one, that one has, just, Gelfand map continuous (validating thus an initial proposal of A Mallios).

Proposition III-9. *Let A be an advertibly complete l.m.c.t.m.b.a., having the Gelfand map continuous. Then the following assertions are equivalent:*

- **1)** *A is Q-algebra.*
- **2)** *$\mathcal{X}(A)$ is bounded.*
- **3)** *$\mathcal{X}(\mathcal{A})$ is compact.*
- **4)** *$A = \mathcal{B}(\mathcal{A})$.*
- **5)** *$\mathcal{X}(A)$ is equicontinuous.*

Proof.

1) $\Longleftrightarrow$ **5**). Results from Remark III-5.

1) $\Longrightarrow$ **2**) $\Longrightarrow$ **3**), for every advertibly complete and *l.m.c.a.*.See S. Warner [10].

3) $\Longrightarrow$ **4**), by Proposition II-6. Indeed, $\mathcal{X}(A)$ being compact,

$$\rho_{_A}(x) = sup\,\{|f(x)| : f \in \mathcal{X}(A)\} = sup\,\{|\widehat{x}(f)| : f \in \mathcal{X}(A)\} < \infty. \tag{12}$$

4) $\Longrightarrow$ **1**). By Proposition II-6, we have $\rho_{_A}(x) = sup\,\{|f(x)| : f \in \mathcal{X}(A)\} = sup\,\{|\widehat{x}(f)| : f \in \mathcal{X}(A)\}$. As $A = \mathcal{B}(A)$ then $\rho_{_A}$ is real-valued. So, $\rho_{_A}$ is, obviously, a (submultiplicative) semi-norm. Since the Gelfand map is continuous, it is thus for $\rho_{_A}$. So $\mathcal{S}(A)$ is a zero neighborhood. Whence **1**). □

By the previous proposition, a result of A. Mallios ([7], Theorem 3.5, p. 466) remains valid, if we replace the fact that A is commutative by the fact that A is a $t.m.b.a$ (which englobe the case of two-sided algebras). Before, it is easy to prove the next lemma.

Lemma III-10. *Let A be a topological algebra. Then A is Q-algebra if, and only if $RadA$ is closed and $A/RadA$ is Q-algebra.*

Corollary III-11. *Let A be a unital advertibly complete l.m.c.t.m.b.a., having the respective Gelfand map continuous. Then, the next assertions are equivalent:*

1) *A is Q-algebra.*
2) *$\mathcal{X}(\mathcal{A})$ is bounded.*
3) *$\mathcal{X}(\mathcal{A})$ is compact.*
4) *$A = \mathcal{B}(\mathcal{A})$.*
5) *$\mathcal{X}(\mathcal{A})$ is equicontinuous.*
6) *Every left or right maximal and two-sided ideal of A is of codimension 1.*
7) *Every left or right maximal ideal of A is closed.*
8) *$Sp_A(x)$ is compact, for every $x \in A$.*

Proof. By Proposition III-9, the assertions $\mathbf{1}), \mathbf{2}), \mathbf{3}), \mathbf{4}), \mathbf{5})$ are equivalent. It is classical that $\mathbf{1}) \Longrightarrow \mathbf{6})$. In all of the next we note, by $\mathbf{2}) \Longrightarrow \mathbf{1})$ of Proposition II-10, that $B = A/RadA$ is (commutative) advertibly complete $l.m.c.a.$ with closed radical and by Remark III-5, the Gelfand map of B is continuous. Prove that $\mathbf{6}) \Longrightarrow \mathbf{1})$. Let $B := A/RadA$ and S the canonical map from A onto B. Let Ψ be the set of all left or right maximal and two-sided ideals of A and $\widetilde{\Psi}$ the set of all left or right maximal and two-sided ideals of B. By Remark II-8, the correspondence

$$\begin{array}{ccc} \Psi & \longrightarrow & \widetilde{\Psi} \\ M & \longrightarrow & S(M) \end{array}$$

is a one to one correspondence; and M is of codimension 1 if, and only if $S(M)$ is of codimension 1. So, all left or right maximal and two-sided ideals of B are of codimension 1. By $\mathbf{1}) \Longrightarrow \mathbf{5})$ of Theorem 3.5 (p. 466 of [7]), B is Q-*algebra*. Hence, by Lemma III-10, A is Q-*algebra*. It is well known that $\mathbf{1}) \Longrightarrow \mathbf{7})$. Prove that $\mathbf{7}) \Longrightarrow \mathbf{1})$. By Remark II-8, all right or left maximal and two-sided ideals of A are closed if, and only if, all right or left maximal and two-sided ideals of B are closed. By $\mathbf{2}) \Longrightarrow \mathbf{5})$ of Theorem 3.5 (p. 466 of [7]), B is Q-*algebra*. Hence, by Lemma III-10, A is Q-*algebra*. It is obvious that $\mathbf{1}) \Longrightarrow \mathbf{8})$. Prove that $\mathbf{8}) \Longrightarrow \mathbf{1})$. Suppose that, for every $x \in A$ $Sp_A(x) = \{f(x) : f \in \mathcal{X}(A)\}$, is a compact subset of $\mathbb{C}$. Thereby, for every $x \in A$, $Sp_B(S(x)) = Sp_A(x)$, is also a compact subset of $\mathbb{C}$. By $\mathbf{6}) \Longrightarrow \mathbf{5})$ of Theorem 3.5 (p. 466 of [7]), B is $Q-algebra$. Hence, by Lemma III-10, A is Q-*algebra*. □

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