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On a certain nonlinear retarded Volterra integrodifferential equation

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Abstract
The aim of this paper is to study the behavior of solutions of a certain nonlinear retarded Volterra integrodifferential equation by using the integral inequality recently established by the present author.

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1991 *Mathematics subject classification*: 34K05, 34K10, 34K40

1. Introduction

Consider the initial value problem (IVP for short) for a retarded Volterra integrodifferential equation of the form

$$x'(t) = A(t)\,x(t) + F\left(t, x(t-h(t)), \int_0^t k(t,s,x(s-h(s)))\,ds\right), \tag{1.1}$$

$$x(0) = x_0, \tag{1.2}$$

where $t \in I = [0, \beta]$, x, k, F are the elements of R^n an n-dimensional Euclidean space, prime denote differentation with respect to t and $A(t)$ is a continuous $n \times n$ matrix. We assume that for $0 \le s \le t$, $k \in C\left(I^2 \times R^n, R^n\right)$, $F \in C\left(I \times R^n \times R^n, R^n\right)$ and $h \in C\left(I, R_+\right)$ be nonincreasing, $h(0) = 0$, $t - h(t) \ge 0$, $t - h(t) \in C^1(I, I)$, $h'(t) < 1$ for $t \in I$.

In [4,5] the authors have studied the global existence of solutions of the slightly different versions of equation (1.1) by using the Leray-Schauder Alternative. The same method can be used to establish the global existence of IVP (1.1)-(1.2) on I (see also [1-3]). It frequently happeans that a method which works very effectively to establish the existence of solutions does yield other properties of the solutions in ready fashion and one needs some new ideas and methods in analysis. The main purpose of this paper is to study the boundedness, uniqueness and continuous dependence of the solutions of IVP (1.1)-(1.2) via the integral inequality recently established by the present author in [6].

2. Main Results

Let $R_+ = [0,\infty)\,, J = [t_0, d)\,, I = [0,\beta]$ be the given subsets of R , the set of real numbers. The symbol $\|.\|$ will be used to denote any convenient norm on R^n as well as a corresponding consistent matrix norm. We regard (1.1) as a perturbed equation of the linear system

$$y'(t) = A(t)\, y(t)\,, \quad y(0) = x_0. \tag{2.1}$$

Let $Y(t)$ be the fundamental matrix of (2.1) which is the identity matrix at $t = 0$. It is known that the solution of IVP (1.1)-(1.2) is equivalent to the integral equation

$$\begin{aligned} x(t) =& Y(t)\, Y^{-1}(0)\, x_0 \\ &+ \int_0^t Y(t)\, Y^{-1}(s)\, F\left(s, x(s-h(s))\,, \int_0^s k(s,\tau, x(t-h(\tau)))\, d\tau\right) ds. \end{aligned} \tag{2.2}$$

Throughout, we assume that the fundamental matrix $Y(t)$ of the linear system (2.1) satisfies the estimate

$$\left\|Y(t)\, Y^{-1}(t)\right\| \le M e^{-\omega(t-s)}, \;\; 0 \le s \le t \le \beta, \tag{2.3}$$

where M, ω are positive constants.

The following inequality established in [6] will be used in the proofs of our results.

Lemma (Pachpatte [6]). *Let $u(t)\,, a(t) \in C(J, R_+)\,,\ b(t,s) \in C\left(J^2, R_+\right)\,,\ t_0 \le s \le t \le d$ and $\alpha(t) \in C^1(J,J)$ be nondecreasing with $\alpha(t) \le t$ on J and $c \ge 0$ be a constant. If*

$$u(t) \le c + \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s)\, u(s) + \int_{\alpha(t_0)}^{s} b(s,\sigma)\, u(\sigma)\, d\sigma\right] ds$$

for $t \in J$, then

$$u(t) \le c\ \exp\left(\int_{\alpha(t_0)}^{\alpha(t)} \left[a(s) + \int_{\alpha(t_0)}^{s} b(s,\sigma)\, d\sigma\right] ds\right),$$

for $t \in J$.

Our first result deals with the estimate on the solution of IVP (1.1)-(1.2).

Theorem 1 *Assume that*

$$\|k(t,s,x)\| \le b(t,s)\, \|x\|\,, \tag{2.4}$$

$$\|F(t,x,u)\| \le a(t)\, \|x\| + \|u\|\,, \tag{2.5}$$

where $a(t) \in C(I, R_+), b(t,s) \in C(I^2, R_+)$ *for* $0 \le s \le t \le \beta$ *and let*

$$L = \begin{matrix} \max \\ t \in I \end{matrix} \quad \frac{1}{1 - h'(t)}. \tag{2.6}$$

If $x(t)$ *is any solution of IVP (1.1)-(1.2), then*

$$\|x(t)\| \le M \|x_0\| \exp\left(-\omega t + \int_0^{t-h(t)} \left[\bar{a}(\xi) + \int_0^{\xi} \bar{b}(\xi,\sigma)\, d\sigma \right] d\xi \right), \tag{2.7}$$

where

$$\bar{a}(\xi) = M\, L\, e^{\omega\, h(s)} a(\xi + h(s)),$$
$$\bar{b}(\xi,\sigma) = M\, L^2\, e^{w(\xi - \sigma + h(s))} b(\xi + h(s), \sigma + h(\tau)),$$

for $\xi, \sigma, s, \tau \in I, 0 \le \sigma \le \xi$ *and* M, ω *are as in (2.3).*

Proof. Let $x(t)$ be a solution of IVP (1.1)-(1.2), then it satisfies the equation (2.2). From (2.2), (2.3), (2.4) and (2.5) we have

$$\|x(t)\| \le M\, e^{-\omega\, t} \|x_0\| + \int_0^t M\, e^{-\omega(t-s)} \Bigg[a(s) \|x(s - h(s))\|$$
$$+ \int_0^s b(s,\tau) \|x(\tau - h(\tau))\| d\tau \Bigg] ds. \tag{2.8}$$

Rewriting (2.8), making the change of variables and using (2.6) we have

$$e^{\omega\, t} \|x(t)\| \le M\ \|x_0\| + \int_0^{t-h(t)} \left[\bar{a}(\xi)\, e^{\omega\, \xi}\, \|x(\xi)\| + \int_0^{\xi} \bar{b}(\xi,\sigma)\, e^{\omega\, \sigma}\, \|x(\sigma)\|\, d\sigma \right] d\xi. \tag{2.9}$$

Now a suitable application of the inequality given in Lemma to (2.9) yields

$$e^{\omega\, t} \|x(t)\| \le M\ \|x_0\| \exp\left(\int_0^{t-h(t)} \left[\bar{a}(\xi) + \int_0^{\xi} \bar{b}(\xi,\sigma) d\sigma \right] d\xi \right). \tag{2.10}$$

Rewriting (2.10) we get the required estimate in (2.7). The proof is complete.

Next, we establish the uniqueness of solutions of IVP (1.1)-(1.2).

Theorem 2 *Suppose that the functions* k, F *in (1.1) satisfy the conditions*

$$\|k(t,s,x) - k(t,s,y)\| \le b(t,s)\, \|x - y\|, \tag{2.11}$$

$$\|F(t,x,\bar{x}) - F(t,y,\bar{y})\| \le a(t)\, \|x - y\| + \|\bar{x} - \bar{y}\|, \tag{2.12}$$

where $a(t), b(s,t)$ *are as defined in Theorem 1. Let* $L, \bar{a}, \bar{b}$ *be as in Theorem 1. Then IVP (1.1)-(1.2) has at most one solution on I.*

Proof. Let $x_1(t)$ and $x_2(t)$ be two solutions of IVP (1.1)-(1.2) on I, then we have

$$x_1(t) - x_2(t) = \int_0^t Y(t)\, Y^{-1}(s) \left[F\left(s, x_1(s - h(s)), \int_0^s k(s, \tau, x_1(\tau - h(\tau)))\, d\tau \right) \right.$$

$$\left. - F\left(s, x_2(s - h(s)), \int_0^s k(s, \tau, x_2(\tau - h(\tau)))\, d\tau \right) \right] ds. \tag{2.13}$$

From (2.13), (2.3), (2.11), (2.12) we have

$$\|x_1(t) - x_2(t)\| \le M\, e^{-\omega\, t} \int_0^t e^{\omega\, s} \left[a(s)\, \|x_1(s - h(s)) - x_2(s - h(s))\| \right.$$

$$\left. + \int_0^s b(s, \tau)\, \|x_1(\tau - h(\tau)) - x_2(\tau - h(\tau))\|\, d\tau \right] ds. \tag{2.14}$$

Rewriting (2.14), making the change of variables on the right hand side and using (2.6) we get

$$e^{\omega\, t} \|x_1(t) - x_2(t)\| \le \int_0^{t-h(t)} [\bar{a}(\xi) e^{\omega\, \xi} \|x_1(\xi) - x_2(\xi)\|$$

$$+ \int_0^{\xi} \bar{b}(\xi, \sigma) e^{\omega\, \sigma} \|x_1(\sigma) - x_2(\sigma)\| d\sigma] d\xi. \tag{2.15}$$

A suitable application of the inequality in Lemma (with $c = 0$) to (2.15) yields

$$e^{\omega\, t}\, \|x_1(t) - x_2(t)\| \le 0.$$

Therefore $x_1(t) = x_2(t)$, i.e. there is at most one solution of IVP (1.1)-(1.2) on I.

The following theorem shows the continuous dependence of solutions of IVP (1.1)-(1.2) on given initial conditions.

Theorem 3 *Let $x_1(t)$ and $x_2(t)$ be the solutions of (1.1) with the given initial conditions*

$$x_1(0) = x_{10}, \tag{2.16}$$

and

$$x_2(0) = x_{20}, \tag{2.17}$$

respectively. Suppose that the functions k and F satisfy the conditions (2.11) and (2.12) in Theorem 2. Let $L, \bar{a}, \bar{b}, M, \omega$ be as in Theorem 1. Then

$$\|x_1(t) - x_2(t)\| \le M \|x_{10} - x_{20}\| \exp\left(-\omega t + \int_0^{t-h(t)} \left[\bar{a}(\xi) + \int_0^{\xi} \bar{b}(\xi,\sigma)\, d\sigma\right] d\xi\right), \tag{2.18}$$

for $t \in I$.

Proof. By using the facts that $x_1(t)$ and $x_2(t)$ are the solutions of IVP (1.1)-(1.16) and IVP (1.1)-(1.17) respectively, we have

$$x_1(t) - x_2(t) = Y(t)\, Y^{-1}(0)\{x_{10} - x_{20}\}$$

$$+ \int_0^t Y(t)\, Y^{-1}(s) \left[F\left(s, x_1(s-h(s)), \int_0^s k(s,\tau,x_1(\tau - h(\tau)))\, d\tau\right)\right.$$

$$\left. -F\left(s, x_2(s-h(s)), \int_0^s k(s,\tau,x_2(\tau - h(\tau)))\, d\tau\right)\right] ds. \tag{2.19}$$

From (2.19), (2.3), (2.11), (2.12) we have

$$\|x_1(t) - x_2(t)\| \le M e^{-\omega t} \|x_{10} - x_{20}\|$$

$$+ e^{-\omega t} \int_0^t M e^{\omega s} \left[a(s) \|x_1(s-h(s)) - x_2(s-h(s))\|\right.$$

$$\left. + \int_0^s b(s,\tau) \|x_1(\tau - h(\tau)) - x_2(\tau - h(\tau))\|\, d\tau\right] ds. \tag{2.20}$$

Rewriting (2.20), making the change of variables on the right hand side and using (2.6) we have

$$e^{\omega t} \|x_1(t) - x_2(t)\| \le M \|x_{10} - x_{20}\| + \int_0^{t-h(t)} \left[\bar{a}(\xi)\, e^{\omega \xi} \|x_1(\xi) - x_2(\xi)\|\right.$$

$$\left. + \int_0^{\xi} \bar{b}(\xi,\sigma)\, e^{\omega \sigma} \|x_1(\sigma) - x_2(\sigma)\|\, d\sigma\right] d\xi. \tag{2.21}$$

Now a suitable application of the inequality given in Lemma to (2.21) yields

$$e^{\omega t}\left\|x_1(t)-x_2(t)\right\| \le M\left\|x_{10}-x_{20}\right\|\exp\left(\int_0^{t-h(t)}\left[\bar{a}(\xi)+\int_0^{\xi}\bar{b}(\xi,\sigma)\,d\sigma\right]d\xi\right). \tag{2.22}$$

Rewriting (2.22) we get the required estimate in (2.18), which shows the continuous dependence of solutions of equation (1.1) on given initiai conditions.

We next consider the following retarded integrodifferential equations

$$x'(t)=A(t)\,x(t)+F\left(t,x(t-h(t)),\int_0^t k(t,s,x(s-h(s)))\,ds,\mu\right), \tag{2.23}$$

$$x'(t)=A(t)\,x(t)+F\left(t,x(t-h(t)),\int_0^t k(t,s,x(s-h(s)))\,ds,\mu_0\right), \tag{2.24}$$

with the given initial condition (1.2), where A,k,h are as in equation (1.1), $F \in C(I\times R^n\times R^n\times R,R^n)$ and μ,μ_0 are real parameters.

The following theorem shows the dependency of solutions of IVP (2.23)-(1.2) and (2.24)-(1.2) on parameters.

Theorem 4 *Suppose that*

$$\left\|k(t,s,x_1)-k(t,s,x_2)\right\| \le b(t,s)\left\|x_1-x_2\right\|, \tag{2.25}$$

$$\left\|F(t,z_1,z_2,\mu)-F(t,w_1,w_2,\mu)\right\| \le a(t)\left\|z_1-w_1\right\|+\left\|z_2-w_2\right\|, \tag{2.26}$$

$$\left\|F(t,z_1,z_2,\mu)-F(t,w_1,w_2,\mu_0)\right\| \le m(t)\left\|\mu-\mu_0\right\|, \tag{2.27}$$

where $a(t),b(t,s)$ are as in Theorem 1 , $m\in C(I,R_+)$ and

$$\int_0^t Me^{\omega s}m(s)\,ds \le N, \tag{2.28}$$

where $N>0$ is a constant. Let $L,\bar{a},\bar{b},M,\omega$ be as in Theorem 1. If $x_1(t)$ and $x_2(t)$ are the solutions of IVP (2.23)-(1.2) and IVP (2.24)-(1.2) respectively, then

$$\left\|x_1(t)-x_2(t)\right\| \le N\left|\mu-\mu_0\right|\exp\left(-\omega t+\int_0^{t-h(t)}\left[\bar{a}(\xi)+\int_0^{\xi}\bar{b}(\xi,\sigma)d\sigma\right]d\xi\right), \tag{2.29}$$

for $t\in I$.

Proof. Let $z(t) = x_1(t) - x_2(t)$ for $t \in I$. Since $x_1(t)$ and $x_2(t)$ are the solutions of IVP (2.23)-(1.2) and (2.24)-(1.2) respectively, we have

$$z(t) = x_1(t) - x_2(t)$$

$$= \int_0^t Y(t) Y^{-1}(s) \left[F\left(s, x_1(s - h(s)), \int_0^s k(s, \tau, x_1(\tau - h(\tau))) d\tau, \mu \right) \right.$$

$$- F\left(s, x_2(s - h(s)), \int_0^s k(s, \tau, x_2(\tau - h(\tau))) d\tau, \mu \right)$$

$$+ F\left(s, x_2(s - h(s)), \int_0^s k(s, \tau, x_2(\tau - h(\tau))) d\tau, \mu \right)$$

$$\left. - F\left(s, x_2(s - h(s)), \int_0^s k(s, \tau, x_2(\tau - h(\tau))) d\tau, \mu_0 \right) \right] ds. \tag{2.30}$$

From (2.30), (2.3), (2.25), (2.26), (2.27) we have

$$\|z(t)\| \le e^{-\omega t} \int_0^t M e^{\omega s} \left[a(s) \|z(s - h(s))\| + \int_0^s b(s, \tau) \|z(\tau - h(\tau))\| d\tau \right] ds$$

$$+ |\mu - \mu_0| e^{-\omega t} \int_0^t M e^{\omega s} m(s) ds. \tag{2.31}$$

Rewriting (2.31), making the change of variables on the right hand side and using (2.6) we get

$$e^{\omega t} \|z(t)\| \le N |\mu - \mu_0| + \int_0^{t-h(t)} \left[\bar{a}(\xi) e^{\omega \xi} \|z(\xi)\| + \int_0^{\xi} \bar{b}(\xi, \sigma) e^{\omega \sigma} \|z(\sigma)\| d\sigma \right] d\xi. \tag{2.32}$$

By a suitable application of the inequality given in Lemma to (2.32) yields

$$e^{\omega t} \|z(t)\| \le N |\mu - \mu_0| \exp\left(\int_0^{t-h(t)} \left[\bar{a}(\xi) + \int_0^{\xi} \bar{b}(\xi, \sigma) d\sigma \right] d\xi \right). \tag{2.33}$$

Rewriting (2.33) we get the required estimate in (2.29) , which shows the dependency of solutions of IVP (2.23)-(1.2) and IVP (2.24)-(1.2) on parameters.

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