BULLETIN OF THE
GREEK MATHEMATICAL SOCIETY
Volume 52, 2006 (35–47)

Green's Dyadic for the Three-dimensional Linear Elasticity in Periodic Structures

Constantinos Anestopoulos, Elias Argyropoulos, Drossos Gintides and Kiriakie Kiriaki

Abstract

In this paper, we consider the Green's dyadic in three dimensional linear elasticity for periodic domains. Using the Floquet-Bloch boundary conditions for the elastic case, we give the exact form of the Green's dyadic in terms of sums of images. Applying also the integral representation of the Green's function for the Helmholtz equation with the heat kernel we present a convenient analytic form, which has slow convergence. The main result of this paper is to overcome this difficulty by implementing Ewald's method. This method uses interchanging of integrations and summation processes and specific representations of special functions which in elasticity also involve the dyadic character of the Green's dyadic. Finally, we discuss some details of the method concerning the choice of parameters describing the dimension of the periodic cell or termination for the integration procedures. The potential use of the method in periodic boundary value problems in elasticity is also enlightened.

Keywords: Ewald's method, Green's dyadic, linear elasticity

1. Introduction

In the last few years there is an increasing interest among scientists for wave propagation of acoustic, electromagnetic or elastic waves in periodic media. A lot of new technological applications are justifying this interests. For example, in electromagnetism very promising is the area of photonic crystals. A macroscopic periodic lattice of a dielectric can be considered as photonic crystal [10], [11]. It is well known that photonic crystals exhibit photonic bandgaps, that is, regions of their frequency spectrum over which there are no waves propagating through the lattice. A great variety of technological applications incorporate such type of materials as for example filters, laser , microwave antennas e.t.c.. In elasticity a similar behavior of periodic materials is established [10]. Some applications of periodic materials are elastic wave filters and sensors.

The mathematical formulation of these boundary value problems in integral equations is one of the main methods for the theoretical investigation of well possedness of the problems and also for numerical calculation of solutions. One of the most important problems in this approach is the efficient calculation of periodic Green's functions. In periodic media Green's functions are produced via the method of summing images or eigenfunction expansions [6]. In any case, the resulting series present poor convergence. A lot of techniques are available for accelerating slowly convergence series. One of the most effective methods is due to Ewald, [2]. This method accelerates the convergence of the series of periodic Green's function. It is based on the integral representation of the free space Green's function via the heat kernel. This formula is applied in all free space Green's function entering in the infinite sum of images for the periodic Green's function. Using interchanging of integrations and summation and specific representations via special functions the resulting series exhibit rapid convergence. This procedure is usually described in the literature in the context of two or three dimensional lattice, [7]. In [6] C. M. Linton also uses Ewald's method and reports the results of this method compared with other methods of this direction as Kummer's transformation or integral representations and lattice sum's. In [7] Ewald's method is revisited. It is extended for the Helmholtz equation in certain domains of $\mathbf{R}^d$ with quite general boundary conditions. The goal of this work is except of the derivation of a rapidly converging periodic Green's function to set the basis for studying the boundary value problems for periodic arrays of scatterers. This approach is used by S. Venakides et.al. in two-dimensional electromagnetic waves scattering modelling a Fabry-Perot structure with mirrors which consist a photonic crystals, [11]. A boundary integral formulation is used to solve the problem employing a periodic Green's function rapidly converging according Ewald's representation.

In elasticity, a great variety of materials have periodic structure, for example composite materials, e.t.c., which makes necessary the use of periodic Green's dyadic in any integral formulation of the corresponding wave propagation and scattering problems. We would like to mention that, the vector elastic waves are the superposition of longitudinal and transverse waves which interfere on the boundaries due to the mode conversion mechanism. This character of the waves makes the situation much more complicated. The computation of the very complicated Green's dyadic suffers from poor convergence.

The purpose of this paper is to remedy this lack of fast convergence of the periodic Green's dyadic by employing Ewald's approach in elasticity. In section 2, we give the exact form of the Green's dyadic for periodic structures using the method of images in three-dimensional elasticity with the Floquet-Bloch boundary conditions. We use the heat kernel and a representation of the three-dimensional Green's function in acoustics. In section 3, we apply Ewald's technique for the elastic dyadic. We prove that this representation converges rapidly and consequently is appropriate for numerical computations. At the end we discuss some details of the method. It is underlined that the choice of parameters describing the dimension of the periodic cell does not alter the whole method and that the termination limits of the integration processes can be used to balance the decays of sums and integrals. The potential use of the method in periodic boundary value problems in elasticity is also enlightened.

2. The Green's dyadic for periodic structures

We consider the domain D in $\mathbf{R}^3$ with $D = (0, b_1) \times \cdots \times (0, b_r) \times \mathbf{R}^{3-r}$, $r = 1, 2, 3$, where $b_r > 0,\ r = 1, 2, 3$ and we assume that it consists the fundamental cell of the periodic structure. We also consider that the space is filled up by an isotropic and homogenous elastic medium with Lamé constants λ, μ where $\mu > 0,\ \lambda + 2\mu > 0$ and constant density $\rho = 1$. The propagation of elastic waves in this space is governed by the Navier time-reduce equation

$$(\Delta^* + \omega^2)\mathbf{u}(\mathbf{x}) = \mathbf{0} \tag{1}$$

where the Lamé operator Δ^* is given by the relation

$$\Delta^* = \mu\Delta + (\lambda + \mu)\nabla(\nabla\cdot) \tag{2}$$

and $\mathbf{u}$ is the displacement field and $\omega > 0$ is the angular frequency.

It is well-known [4] that every solution of (1) can be decomposed as a superposition of longitudinal and transverse components given by

$$\mathbf{u}_p = -\frac{1}{k_p^2}\nabla(\nabla \cdot \mathbf{u}), \quad \mathbf{u}_s = \mathbf{u} - \mathbf{u}_p\ . \tag{3}$$

In previous relations, the indices reflect the physical properties and assign the nomination of P-waves (pressure) and S-waves (shear), $k_p = \frac{\omega}{c_p}$, $k_s = \frac{\omega}{c_s}$ are the wave numbers and $c_p = \sqrt{\lambda + 2\mu}$, $c_s = \sqrt{\mu}$ are the phase-velocities of corresponding transverse and longitudinal waves.

The fundamental solution $\widetilde{\mathbf{\Gamma}}(\mathbf{x}, \mathbf{y})$ of Navier's equation (1) in $\mathbf{R}^3$, is given, [4], by

$$\widetilde{\mathbf{\Gamma}}(\mathbf{x}, \mathbf{y}) = -\frac{1}{\omega^2}\left[\nabla_{\mathbf{x}}\nabla_{\mathbf{x}}\left(\frac{e^{\imath k_p|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}\right) - (\nabla_{\mathbf{x}}\nabla_{\mathbf{x}} + k_s^2\widetilde{\mathbf{I}})\left(\frac{e^{\imath k_s|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}\right)\right] \tag{4}$$

with $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3$, $\mathbf{x} \neq \mathbf{y}$ and $\widetilde{\mathbf{I}}$ the identity dyadic.

From the periodicity of the structure in conformance with Bloch's theorem [1], there are constants $a_j :\ |a_j b_j| < \pi,\ j = 1, 2, \ldots, r$, such that the following Floquet-Bloch boundary conditions

$$\mathbf{u}(x_1, \ldots, b_j, \ldots, x_3) = e^{\imath a_j b_j}\mathbf{u}(x_1, \ldots, 0, \ldots, x_3), \tag{5}$$

and

$$\nabla\mathbf{u}(x_1, \ldots, b_j, \ldots, x_3) = e^{\imath a_j b_j}\nabla\mathbf{u}(x_1, \ldots, 0, \ldots, x_3), \tag{6}$$

hold, for $j = 1, 2, \ldots, r,\ r = 1, 2, 3$.

We are now in the position to prove the following theorem.

Theorem 2.1 *The Green's dyadic described by equation (1) with periodicity conditions (5) and (6) is given by the relation*

$$\begin{aligned} \widetilde{\mathbf{G}}(\mathbf{x},\mathbf{y}) &= -\frac{4\pi^2}{\omega^2}\sum_{m\in\mathbf{Z}^r} e^{-\imath\mathbf{a}\cdot(\mathbf{mb})} \\ &\cdot\left\{\int_0^\infty \frac{1}{(4\pi t)^{\frac{7}{2}}}\exp\left(k_p^2 t - \frac{|\mathbf{x}-\mathbf{y}+\mathbf{mb}|^2}{4t}\right)\right. \\ &\cdot\left[(\mathbf{x}-\mathbf{y}+\mathbf{mb})\otimes(\mathbf{x}-\mathbf{y}+\mathbf{mb}) - 2t\widetilde{\mathbf{I}}\right]dt \\ &- \int_0^\infty \frac{1}{(4\pi t)^{\frac{7}{2}}}\exp\left(k_s^2 t - \frac{|\mathbf{x}-\mathbf{y}+\mathbf{mb}|^2}{4t}\right) \\ &\left.\cdot\left[(\mathbf{x}-\mathbf{y}+\mathbf{mb})\otimes(\mathbf{x}-\mathbf{y}+\mathbf{mb}) + 4k_s^2 t^2\widetilde{\mathbf{I}}\right]dt\right\}, \mathbf{x},\mathbf{y}\in\mathbf{R}^3 \end{aligned} \tag{7}$$

where $\mathbf{a} = (a_1, \ldots, a_r, 0, \ldots, 0),\ \mathbf{a} \in \mathbf{R}^3$ *and* $\mathbf{mb} = (m_1b_1, \ldots, m_rb_r, 0, \ldots, 0),\ \mathbf{mb} \in \mathbf{R}^3$, $m_r \in \mathbf{Z}$, $r = 1, 2, 3$, $\widetilde{\mathbf{I}}$ *the unit dyadic and* $t > 0$.

Proof. Let k a complex number. If $\mathbf{G}_n(\mathbf{x},\mathbf{y};k)$ is the Green's function of free space of partial deferential operator $L = -\triangle - kI$ in $\mathbf{R}^n$, $n \geq 1$, where I the identity operator, then [7]

$$\mathbf{G}_n(\mathbf{x},\mathbf{y};k) = \int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}}\exp\left(kt - \frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right)dt \tag{8}$$

with $Rek < 0$ and $t > 0$. The radial solution of equation

$$-\Delta\mathbf{u}(\mathbf{x}) - k\mathbf{u}(\mathbf{x}) = \delta(\mathbf{x}), \quad \mathbf{x}\in\mathbf{R}^n \tag{9}$$

where $\delta(\mathbf{x})$ the Dirac's function with pole at $\mathbf{0} \in \mathbf{R}^n$, is

$$\mathbf{G}_n(\mathbf{x},\mathbf{y};k) = \frac{1}{(2\pi)^{\frac{n}{2}}}(-k)^{\frac{n-2}{4}}|\mathbf{x}-\mathbf{y}|^{\frac{2-n}{2}}K_{\frac{n-2}{2}}(\sqrt{-k}|\mathbf{x}-\mathbf{y}|) \tag{10}$$

where $K_m(\cdot)$ denotes the modified Bessel's function of second kind and m order. From (10), for $n = 3$, taking into account that $K_{\frac{1}{2}}(z) = (\frac{\pi}{2z})^{\frac{1}{2}}e^{-z}$ [5], we find that

$$\mathbf{G}_3(\mathbf{x},\mathbf{y};k) = \frac{e^{-\sqrt{-k}|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}, \tag{11}$$

hence using (8) also for $n = 3$, we obtain

$$\frac{e^{-\sqrt{-k}|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|} = \int_0^\infty \frac{1}{(4\pi t)^{\frac{3}{2}}}\exp\left(kt - \frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right)dt \tag{12}$$

with $Rek < 0$, which is the basic relation of Ewald's method. For $k = k_\alpha^2$, $\alpha = p, s$, especially for $\sqrt{-k} = -\imath k_\alpha$ (12) gives

$$\frac{e^{\imath k_\alpha |\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|} = \int_0^\infty \frac{1}{(4\pi t)^{\frac{3}{2}}} \exp\left(k_\alpha^2 t - \frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right) dt, \ \alpha = p, s, \ \mathbf{x} \neq \mathbf{y}, \ \mathbf{x}, \mathbf{y} \in \mathbf{R}^3. \quad (13)$$

We use the relations

$$\nabla_{\mathbf{x}} \exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right) = -\frac{1}{2t} \exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right)(\mathbf{x}-\mathbf{y}), \ \ \mathbf{x}, \mathbf{y} \in \mathbf{R}^3, \quad (14)$$

$$\nabla_{\mathbf{x}}\nabla_{\mathbf{x}} \exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right) = \frac{1}{4t^2} \exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right)\left[(\mathbf{x}-\mathbf{y}) \otimes (\mathbf{x}-\mathbf{y}) - 2t\widetilde{\mathbf{I}}\right], \quad (15)$$

the fact that we can interchange the order of differentiation and integration in (13) and we conclude to

$$\nabla_{\mathbf{x}}\nabla_{\mathbf{x}} \exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right) = 4\pi^2 \int_0^\infty \frac{1}{(4\pi t)^{\frac{7}{2}}} \exp\left(k_\alpha^2 t - \frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right) \cdot\left[(\mathbf{x}-\mathbf{y}) \otimes (\mathbf{x}-\mathbf{y}) - 2t\widetilde{\mathbf{I}}\right] dt. \quad (16)$$

Substituting the last relation to (4), we obtain

$$\begin{aligned} \widetilde{\mathbf{\Gamma}}(\mathbf{x}, \mathbf{y}) \quad = \quad & -\frac{4\pi^2}{\omega^2}\left\{ \int_0^\infty \left[\frac{1}{(4\pi t)^{\frac{7}{2}}} \exp\left(k_p^2 t - \frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right)\right.\right. \\ & \left.\cdot\left[(\mathbf{x}-\mathbf{y}) \otimes (\mathbf{x}-\mathbf{y}) - 2t\widetilde{\mathbf{I}}\right]\right] dt \\ & -\int_0^\infty \left[\frac{1}{(4\pi t)^{\frac{7}{2}}} \exp\left(k_s^2 t - \frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right)\right. \\ & \left.\left.\cdot\left[(\mathbf{x}-\mathbf{y}) \otimes (\mathbf{x}-\mathbf{y}) + 4k_s^2 t^2 \widetilde{\mathbf{I}}\right]\right] dt \right\}, \mathbf{x}, \mathbf{y} \in \mathbf{R}^3. \end{aligned} \quad (17)$$

We are now in the position to handle the problem governed by (1) and satisfying the Floquet-Bloch boundary conditions. The method of images gives that the Green's dyadic of this problem is

$$\widetilde{\mathbf{G}}(\mathbf{x}, \mathbf{y}) = \sum_{m \in \mathbf{Z}^r} e^{-\imath \mathbf{a}\cdot(\mathbf{mb})} \widetilde{\mathbf{\Gamma}}(\mathbf{x} + \mathbf{mb}, \mathbf{y}) \quad (18)$$

where $\mathbf{a} = (a_1, \ldots, a_r, 0, \ldots, 0)$ and $\mathbf{mb} = (m_1 b_1, \ldots, m_r b_r, 0, \ldots, 0)$, $m_r \in \mathbf{Z}$, $r = 1, 2, 3$.

Substituting expressions given by (16) and (17) in (18) we derive formula (7). □

The series in (7) converges absolutely provided $\mathbf{x} \neq \mathbf{y}$, with $\mathbf{x}, \mathbf{y} \in D$.

3. The Ewald's representations

Now, following Ewald's method, we derive a series representation of $\widetilde{\mathbf{G}}(\mathbf{x},\mathbf{y}),\ \mathbf{x}\neq\mathbf{y}$, which rapidly converges for all complex wave numbers except a discrete countable set. Ewald's idea is to split $\widetilde{\mathbf{G}}$ in two parts

$$\widetilde{\mathbf{G}}(\mathbf{x},\mathbf{y}) = \widetilde{\mathbf{G}}_1(\mathbf{x},\mathbf{y}) + \widetilde{\mathbf{G}}_2(\mathbf{x},\mathbf{y}) \tag{19}$$

where

$$\begin{aligned}\widetilde{\mathbf{G}}_1(\mathbf{x},\mathbf{y}) \;=\; & -\frac{4\pi^2}{\omega^2}\sum_{m\in\mathbf{Z}^r} e^{-\imath\mathbf{a}\cdot(\mathbf{mb})} \\ & \cdot\left\{\int_0^{E_p}\frac{1}{(4\pi t)^{\frac{7}{2}}}\exp\left(k_p^2 t-\frac{|\mathbf{x}-\mathbf{y}+\mathbf{mb}|^2}{4t}\right)\right. \\ & \cdot\left[(\mathbf{x}-\mathbf{y}+\mathbf{mb})\otimes(\mathbf{x}-\mathbf{y}+\mathbf{mb})-2t\widetilde{\mathbf{I}}\right]dt \\ & -\int_0^{E_s}\frac{1}{(4\pi t)^{\frac{7}{2}}}\exp\left(k_s^2 t-\frac{|\mathbf{x}-\mathbf{y}+\mathbf{mb}|^2}{4t}\right) \\ & \left.\cdot\left[(\mathbf{x}-\mathbf{y}+\mathbf{mb})\otimes(\mathbf{x}-\mathbf{y}+\mathbf{mb})+4k_s^2t^2\widetilde{\mathbf{I}}\right]dt\right\},\ \mathbf{x},\mathbf{y}\in\mathbf{R}^3,\end{aligned} \tag{20}$$

$$\begin{aligned}\widetilde{\mathbf{G}}_2(\mathbf{x},\mathbf{y}) \;=\; & -\frac{4\pi^2}{\omega^2}\sum_{m\in\mathbf{Z}^r} e^{-\imath\mathbf{a}\cdot(\mathbf{mb})} \\ & \cdot\left\{\int_{E_p}^{\infty}\frac{1}{(4\pi t)^{\frac{7}{2}}}\exp\left(k_p^2 t-\frac{|\mathbf{x}-\mathbf{y}+\mathbf{mb}|^2}{4t}\right)\right. \\ & \cdot\left[(\mathbf{x}-\mathbf{y}+\mathbf{mb})\otimes(\mathbf{x}-\mathbf{y}+\mathbf{mb})-2t\widetilde{\mathbf{I}}\right]dt \\ & -\int_{E_s}^{\infty}\frac{1}{(4\pi t)^{\frac{7}{2}}}\exp\left(k_s^2 t-\frac{|\mathbf{x}-\mathbf{y}+\mathbf{mb}|^2}{4t}\right) \\ & \left.\cdot\left[(\mathbf{x}-\mathbf{y}+\mathbf{mb})\otimes(\mathbf{x}-\mathbf{y}+\mathbf{mb})+4k_s^2t^2\widetilde{\mathbf{I}}\right]dt\right\},\ \mathbf{x},\mathbf{y}\in\mathbf{R}^3,\end{aligned} \tag{21}$$

and the positive numbers $E_p,\ E_s$ are chosen appropriately, as we discuss in section 4, to accelerate the convergence of the series which involved in $\widetilde{\mathbf{G}}_1$ and $\widetilde{\mathbf{G}}_2$.

Firstly, we consider the full periodic case in $\mathbf{R}^3$, which corresponds to the choice $r=3$. We assume that $Rek_a^2<0, a=p,s$. In this case, we can interchange summation and integration in (20), so we obtain

$$\begin{aligned}\widetilde{\mathbf{G}}_1(\mathbf{x},\mathbf{y}) \;=\; & -\frac{4\pi^2}{\omega^2}\left\{\int_0^{E_p}\frac{\exp(k_p^2 t)}{(4\pi t)^{\frac{7}{2}}}\left[\sum_{m\in\mathbf{Z}^3}\exp\left(-\frac{|\mathbf{x}-\mathbf{y}+\mathbf{mb}|^2}{4t}-\imath\mathbf{a}\cdot(\mathbf{mb})\right)\right.\right. \\ & \left.\cdot\left[(\mathbf{x}-\mathbf{y}+\mathbf{mb})\otimes(\mathbf{x}-\mathbf{y}+\mathbf{mb})-2t\widetilde{\mathbf{I}}\right]\right]dt\end{aligned}$$

$$- \int_0^{E_s} \frac{\exp(k_s^2 t)}{(4\pi t)^{\frac{7}{2}}} \Bigg[\sum_{m \in \mathbf{Z}^3} \exp\left(- \frac{|\mathbf{x}-\mathbf{y}+\mathbf{mb}|^2}{4t} - \imath \mathbf{a} \cdot (\mathbf{mb}) \right)$$
$$\cdot \left[(\mathbf{x}-\mathbf{y}+\mathbf{mb}) \otimes (\mathbf{x}-\mathbf{y}+\mathbf{mb}) + 4k_s^2 t^2 \widetilde{\mathbf{I}} \right] \Bigg] dt \Bigg\}, \mathbf{x}, \mathbf{y} \in \mathbf{R}^3. \tag{22}$$

In (22), both series are already rapidly convergent, for $\mathbf{x} \neq \mathbf{y}$, since its general term decays in $\mathbf{m}$ like

$$C|\mathbf{mb}|^{-2} \exp\left(- \frac{|\mathbf{mb}|^2}{4E} \right), \tag{23}$$

where $E = \min\{E_s, E_p\}$.

Interchanging summation and integration in (21) since $\widetilde{\mathbf{G}}_2(\mathbf{x}, \mathbf{y})$ converges uniformly, we obtain

$$\widetilde{\mathbf{G}}_2(\mathbf{x}, \mathbf{y}) = -\frac{4\pi^2}{\omega^2} \Bigg\{ \int_{E_p}^{\infty} \frac{\exp(k_p^2 t)}{(4\pi t)^{\frac{7}{2}}} \Bigg[\sum_{m \in \mathbf{Z}^3} \exp\left(- \frac{|\mathbf{x}-\mathbf{y}+\mathbf{mb}|^2}{4t} - \imath \mathbf{a} \cdot (\mathbf{mb}) \right)$$
$$\cdot \left[(\mathbf{x}-\mathbf{y}+\mathbf{mb}) \otimes (\mathbf{x}-\mathbf{y}+\mathbf{mb}) - 2t \widetilde{\mathbf{I}} \right] \Bigg] dt$$
$$- \int_{E_s}^{\infty} \frac{\exp(k_s^2 t)}{(4\pi t)^{\frac{7}{2}}} \Bigg[\sum_{m \in \mathbf{Z}^3} \exp\left(- \frac{|\mathbf{x}-\mathbf{y}+\mathbf{mb}|^2}{4t} - \imath \mathbf{a} \cdot (\mathbf{mb}) \right)$$
$$\cdot \left[(\mathbf{x}-\mathbf{y}+\mathbf{mb}) \otimes (\mathbf{x}-\mathbf{y}+\mathbf{mb}) + 4k_s^2 t^2 \widetilde{\mathbf{I}} \right] \Bigg] dt \Bigg\}, \mathbf{x}, \mathbf{y} \in \mathbf{R}^3. \tag{24}$$

From Lemma1 of [7], we have that if $s > 0$ and $z, \gamma \in \mathbf{C}$, then

$$\sum_{n \in \mathbf{Z}} e^{-s^2(z+n) - \imath\gamma n} = \frac{\sqrt{\pi}}{s} e^{\imath\gamma z - \frac{\gamma^2}{4s^2}} \sum_{n \in \mathbf{Z}} e^{-\frac{\pi^2 n^2}{s^2} - \frac{\pi\gamma n}{s^2} + 2\pi\imath z n}. \tag{25}$$

Using the last relation, whose proof is based in Poisson Summation Formula, applied to the function $f(x) = e^{-Ax^2 + Bx}$, $A > 0, B \in \mathbf{C}$, we obtain

$$\sum_{m \in \mathbf{Z}^3} \exp\left(- \frac{|\mathbf{x}-\mathbf{y}+\mathbf{mb}|^2}{4t} - \imath \mathbf{a} \cdot (\mathbf{mb}) \right) = \frac{(4\pi t)^{\frac{3}{2}}}{|D|} \exp[\imath \mathbf{a} \cdot (\mathbf{x}-\mathbf{y}) - |\mathbf{a}|^2 t]$$
$$\cdot \sum_{m \in \mathbf{Z}^3} \exp\left[- \left(4\pi^2 |\frac{\mathbf{m}}{\mathbf{b}}|^2 + 4\pi \mathbf{a} \cdot (\frac{\mathbf{m}}{\mathbf{b}}) \right) t + 2\pi\imath (\mathbf{x}-\mathbf{y}) \cdot (\frac{\mathbf{m}}{\mathbf{b}}) \right], \tag{26}$$

where $|D| = b_1 b_2 b_3$ is the volume of D and $\frac{\mathbf{m}}{\mathbf{b}} = (\frac{m_1}{b_1}, \frac{m_2}{b_2}, \frac{m_3}{b_3})$.

Thus (24) takes the form

$$\widetilde{\mathbf{G}}_2(\mathbf{x}, \mathbf{y}) = -\frac{4\pi^2}{\omega^2 |D|} \Bigg\{ \int_{E_p}^{\infty} \Bigg[\frac{\exp(k_p^2 t)}{(4\pi t)^2} \exp[\imath \mathbf{a} \cdot (\mathbf{x}-\mathbf{y}) - |\mathbf{a}|^2 t]$$

$$
\begin{aligned}
&\cdot\Big(\sum_{m\in\mathbf{Z}^3}\exp\Big[-\Big(4\pi^2\big|\frac{\mathbf{m}}{\mathbf{b}}\big|^2+4\pi\mathbf{a}\cdot\big(\frac{\mathbf{m}}{\mathbf{b}}\big)\Big)t+2\pi\imath(\mathbf{x}-\mathbf{y})\cdot\big(\frac{\mathbf{m}}{\mathbf{b}}\big)\Big]\Big)\\
&\cdot\big[(\mathbf{x}-\mathbf{y}+\mathbf{mb})\otimes(\mathbf{x}-\mathbf{y}+\mathbf{mb})-2t\widetilde{\mathbf{I}}\big]\Big]dt\\
&-\int_{E_s}^{\infty}\Big[\frac{\exp(k_s^2t)}{(4\pi t)^2}\exp[\imath\mathbf{a}\cdot(\mathbf{x}-\mathbf{y})-|\mathbf{a}|^2t]\\
&\cdot\Big(\sum_{m\in\mathbf{Z}^3}\exp\Big[-\Big(4\pi^2\big|\frac{\mathbf{m}}{\mathbf{b}}\big|^2+4\pi\mathbf{a}\cdot\big(\frac{\mathbf{m}}{\mathbf{b}}\big)\Big)t+2\pi\imath(\mathbf{x}-\mathbf{y})\cdot\big(\frac{\mathbf{m}}{\mathbf{b}}\big)\Big]\Big)\\
&\cdot\big[(\mathbf{x}-\mathbf{y}+\mathbf{mb})\otimes(\mathbf{x}-\mathbf{y}+\mathbf{mb})+4k_s^2t^2\widetilde{\mathbf{I}}\big]\Big]dt\Big\},\mathbf{x},\mathbf{y}\in\mathbf{R}^3. \qquad (27)
\end{aligned}
$$

Now, we interchange integration and summation in (27) because we have uniform convergence for this relation. If we set

$$A_m^p=-k_p^2+|\mathbf{a}|^2+4\pi^2\big|\frac{\mathbf{m}}{\mathbf{b}}\big|^2+4\pi\mathbf{a}\cdot\big(\frac{\mathbf{m}}{\mathbf{b}}\big) \qquad (28)$$

$$A_m^s=-k_s^2+|\mathbf{a}|^2+4\pi^2\big|\frac{\mathbf{m}}{\mathbf{b}}\big|^2+4\pi\mathbf{a}\cdot\big(\frac{\mathbf{m}}{\mathbf{b}}\big), \qquad (29)$$

with $A_m^p > A_m^s$, since $k_p < k_s$, we have

$$
\begin{aligned}
\widetilde{\mathbf{G}}_2(\mathbf{x},\mathbf{y}) \;=\; &-\frac{1}{\omega^2|D|}\Big\{\sum_{m\in\mathbf{Z}^3}\Big[\exp\big[\imath\big(\mathbf{a}+2\pi\frac{\mathbf{m}}{\mathbf{b}}\big)\cdot(\mathbf{x}-\mathbf{y})\big]\\
&\cdot\Big(\frac{1}{4}(\mathbf{x}-\mathbf{y}+\mathbf{mb})\otimes(\mathbf{x}-\mathbf{y}+\mathbf{mb})\int_{E_p}^{\infty}\frac{\exp(-A_m^pt)}{t^2}dt\\
&-\frac{1}{2}\widetilde{\mathbf{I}}\int_{E_p}^{\infty}\frac{\exp(-A_m^pt)}{t}dt\Big)\Big]\\
&-\sum_{m\in\mathbf{Z}^3}\Big[\exp\big[\imath\big(\mathbf{a}+2\pi\frac{\mathbf{m}}{\mathbf{b}}\big)\cdot(\mathbf{x}-\mathbf{y})\big]\\
&\cdot\Big(\frac{1}{4}(\mathbf{x}-\mathbf{y}+\mathbf{mb})\otimes(\mathbf{x}-\mathbf{y}+\mathbf{mb})\int_{E_s}^{\infty}\frac{\exp(-A_m^st)}{t^2}dt\\
&+k_s^2\widetilde{\mathbf{I}}\int_{E_s}^{\infty}\exp(-A_m^st)dt\Big)\Big]\Big\},\quad \mathbf{x},\mathbf{y}\in\mathbf{R}^3. \qquad (30)
\end{aligned}
$$

If $A_m^s > 0$, then

$$k_s^2<|\mathbf{a}|^2+4\pi^2\big|\frac{\mathbf{m}}{\mathbf{b}}\big|^2+4\pi\mathbf{a}\cdot\big(\frac{\mathbf{m}}{\mathbf{b}}\big), \qquad (31)$$

namely in the tail of the series, the last integral in the right term of (30) becomes

$$\int_{E_s}^{\infty}\exp(-A_m^st)dt=\frac{e^{-A_m^sE_s}}{A_m^s} \qquad (32)$$

and using the limit comparison test follows that the integral

$$\int_{E_s}^{\infty} \frac{e^{-A_m^s t}}{t^2} dt < \int_{E_s}^{\infty} \frac{1}{t^2} dt = \frac{1}{E_s} \tag{33}$$

which means that the above integrals converge. Then, using the same arguments we infer that the other two integrals of (30) exist. Therefore, the terms of four series in (30) decay in $\mathbf{m}$ like

$$C|\frac{\mathbf{m}}{\mathbf{b}}|^2 \exp\left(-4\pi^2|\frac{\mathbf{m}}{\mathbf{b}}|^2 E\right), \tag{34}$$

where $E = \min\{E_s, E_p\}$.

From (32) follows that the only singularities of $\widetilde{\mathbf{G}}_2(\mathbf{x},\mathbf{y})$ are

$$|\mathbf{a}|^2 + 4\pi^2\left|\frac{\mathbf{m}}{\mathbf{b}}\right|^2 + 4\pi\mathbf{a}\cdot\left(\frac{\mathbf{m}}{\mathbf{b}}\right) \tag{35}$$

which are the poles, in k_s^2, of the left term of (32). Hence we should have $k_s^2 \neq |\mathbf{a}|^2 + 4\pi^2\left|\frac{\mathbf{m}}{\mathbf{b}}\right|^2 + 4\pi\mathbf{a}\cdot\left(\frac{\mathbf{m}}{\mathbf{b}}\right)$ which holds from (31).

Now, let us discuss the case $r < 3$. For $\mathbf{x} = (x_1, x_2, x_3)$ we denote

$$\mathbf{x} = \mathbf{x}^1 + \overline{\mathbf{x}}^1, \quad \mathbf{x}^1 = (x_1, 0, 0), \quad \overline{\mathbf{x}}^1 = (0, x_2, x_3) \tag{36}$$

or

$$\mathbf{x} = \mathbf{x}^2 + \overline{\mathbf{x}}^2, \quad \mathbf{x}^2 = (x_1, x_2, 0), \quad \overline{\mathbf{x}}^2 = (0, 0, x_3) \tag{37}$$

for

$$\mathbf{x} = \mathbf{x}^r + \overline{\mathbf{x}}^r, \quad r = 1, 2 \tag{38}$$

then, from the orthogonality of $\mathbf{x}^r - \mathbf{y}^r + \mathbf{m}\mathbf{b}^r$ and $\overline{\mathbf{x}}^r - \overline{\mathbf{y}}^r$, we obtain

$$|\mathbf{x} - \mathbf{y} + \mathbf{m}\mathbf{b}|^2 = |\overline{\mathbf{x}}^r - \overline{\mathbf{y}}^r|^2 + |\mathbf{x}^r - \mathbf{y}^r + \mathbf{m}\mathbf{b}|^2, \tag{39}$$

substituting in (21), we have

$$\begin{aligned}
\widetilde{\mathbf{G}}_2(\mathbf{x},\mathbf{y}) \;=\; & -\frac{4\pi^2}{\omega^2} \sum_{m\in\mathbf{Z}^r} e^{-\imath\mathbf{a}\cdot(\mathbf{m}\mathbf{b})} \\
& \cdot\left\{ \int_{E_p}^{\infty} \frac{1}{(4\pi t)^{\frac{7}{2}}} \exp\left(k_p^2 t - \frac{|\overline{\mathbf{x}}^r - \overline{\mathbf{y}}^r|^2 + |\mathbf{x}^r - \mathbf{y}^r + \mathbf{m}\mathbf{b}|^2}{4t}\right)\right. \\
& \cdot\left[(\overline{\mathbf{x}}^r - \overline{\mathbf{y}}^r + \mathbf{x}^r - \mathbf{y}^r + \mathbf{m}\mathbf{b}) \otimes (\overline{\mathbf{x}}^r - \overline{\mathbf{y}}^r + \mathbf{x}^r - \mathbf{y}^r + \mathbf{m}\mathbf{b}) - 2t\widetilde{\mathbf{I}}\right] dt \\
& - \int_{E_s}^{\infty} \frac{1}{(4\pi t)^{\frac{7}{2}}} \exp\left(k_s^2 t - \frac{|\overline{\mathbf{x}}^r - \overline{\mathbf{y}}^r|^2 + |\mathbf{x}^r - \mathbf{y}^r + \mathbf{m}\mathbf{b}|^2}{4t}\right) \\
& \cdot\left[(\overline{\mathbf{x}}^r - \overline{\mathbf{y}}^r + \mathbf{x}^r - \mathbf{y}^r + \mathbf{m}\mathbf{b}) \otimes (\overline{\mathbf{x}}^r - \overline{\mathbf{y}}^r + \mathbf{x}^r - \mathbf{y}^r + \mathbf{m}\mathbf{b})\right. \\
& \left.\left. + 4k_s^2 t^2 \widetilde{\mathbf{I}}\right] dt \right\}, \mathbf{x},\mathbf{y} \in \mathbf{R}^3.
\end{aligned} \tag{40}$$

We set

$$\widetilde{\mathbf{A}}_m(\mathbf{x},\mathbf{y}) = (\overline{\mathbf{x}}^r - \overline{\mathbf{y}}^r) \otimes (\mathbf{x}^r - \mathbf{y}^r + \mathbf{mb}) + (\mathbf{x}^r - \mathbf{y}^r + \mathbf{mb}) \otimes (\overline{\mathbf{x}}^r - \overline{\mathbf{y}}^r), \tag{41}$$

using (26) in (40) and after some calculations

$$\begin{aligned}
\widetilde{\mathbf{G}}_2(\mathbf{x},\mathbf{y}) \;=\; & -\frac{4\pi^2}{\omega^2}\Bigg\{\frac{1}{b_1\dots b_r}\sum_{m\in\mathbf{Z}^r} e^{\imath\left(\mathbf{a}+2\pi\frac{\mathbf{m}}{\mathbf{b}}\right)\cdot(\mathbf{x}^r-\mathbf{y}^r)}\widetilde{\mathbf{A}}_m(\mathbf{x},\mathbf{y}) \\
& \cdot\int_{E_p}^{\infty}\frac{1}{(4\pi t)^{\frac{7-r}{2}}}\exp\left(-A_p t - \frac{|\overline{\mathbf{x}}^r-\overline{\mathbf{y}}^r|^2}{4t}\right) \\
& \cdot\left[(\overline{\mathbf{x}}^r-\overline{\mathbf{y}}^r)\otimes(\overline{\mathbf{x}}^r-\overline{\mathbf{y}}^r) - 2t\widetilde{\mathbf{I}}\right]dt \\
& -\frac{1}{b_1\dots b_r}\sum_{m\in\mathbf{Z}^r} e^{\imath\left(\mathbf{a}+2\pi\frac{\mathbf{m}}{\mathbf{b}}\right)\cdot(\mathbf{x}^r-\mathbf{y}^r)}\widetilde{\mathbf{A}}_m(\mathbf{x},\mathbf{y}) \\
& \cdot\int_{E_s}^{\infty}\frac{1}{(4\pi t)^{\frac{7-r}{2}}}\exp\left(-A_s t - \frac{|\overline{\mathbf{x}}^r-\overline{\mathbf{y}}^r|^2}{4t}\right) \\
& \cdot\left[(\overline{\mathbf{x}}^r-\overline{\mathbf{y}}^r)\otimes(\overline{\mathbf{x}}^r-\overline{\mathbf{y}}^r) + 4k_s^2t^2\widetilde{\mathbf{I}}\right]dt\Bigg\},\mathbf{x},\mathbf{y}\in\mathbf{R}^3.
\end{aligned} \tag{42}$$

The terms of the series in (42) decays in $\mathbf{m}$ like

$$C|\frac{\mathbf{m}}{\mathbf{b}}|^{-2}\exp\left(-4\pi^2|\frac{\mathbf{m}}{\mathbf{b}}|^2E_a\right), \quad a=p,s \tag{43}$$

respectively, for a fixed $k_a^2,\ a=p,s$. In the tail of the series we can have any $k_a,\ a=p,s$. There seems to be a problem, for the convergence of the integrals, though with the first few terms of the series, where we may have $k_a^2 > |\mathbf{a}|^2+4\pi^2\left|\frac{\mathbf{m}}{\mathbf{b}}\right|^2+4\pi\mathbf{a}\cdot\left(\frac{\mathbf{m}}{\mathbf{b}}\right),\ a=p,s$. In (42) appear integrals in the form of

$$\int_E^{\infty}\frac{1}{(4\pi t)^{\frac{n}{2}}}\exp\left[-(b-k^2)t-\frac{c^2}{4t}\right]dt, \tag{44}$$

where $b,c,E>0$ and n positive integer. Lemma 2 in [7] say us that the only singularity of the function

$$g(\lambda) = \int_E^{\infty}\frac{1}{(4\pi t)^{\frac{n}{2}}}\exp\left[-(b-\lambda)t-\frac{c^2}{4t}\right]dt, \tag{45}$$

with $\mathrm{Re}\lambda>0, \lambda\in\mathbf{C}$, is the branch point at $\lambda=b$. Thus, if $r<d$ the only singularities of $\widetilde{\mathbf{G}}_2(\mathbf{x},\mathbf{y})$, as a function of $\lambda=k_a^2,\ a=p,s$, are the branch points at $k_a^2=|\mathbf{a}|^2+4\pi^2\left|\frac{\mathbf{m}}{\mathbf{b}}\right|^2+4\pi\mathbf{a}\cdot\left(\frac{\mathbf{m}}{\mathbf{b}}\right),\ a=p,s$. These are also the singularities of $\widetilde{\mathbf{G}}(\mathbf{x},\mathbf{y})$.

For $r=2$ relation (42) becomes

$$\widetilde{\mathbf{G}}_2(\mathbf{x},\mathbf{y}) \;=\; -\frac{4\pi^2}{\omega^2}\Bigg\{\frac{1}{b_1b_2}\sum_{m\in\mathbf{Z}^2} e^{\imath\left(\mathbf{a}+2\pi\frac{\mathbf{m}}{\mathbf{b}}\right)\cdot(\mathbf{x}^2-\mathbf{y}^2)}\widetilde{\mathbf{A}}_m(\mathbf{x},\mathbf{y})$$

$$
\begin{aligned}
&\cdot \int_{E_p}^{\infty} \frac{1}{(4\pi t)^{\frac{5}{2}}} \exp\left(-A_p t - \frac{|\overline{\mathbf{x}}^2 - \overline{\mathbf{y}}^2|^2}{4t}\right) \\
&\cdot \left[(\overline{\mathbf{x}}^2 - \overline{\mathbf{y}}^2) \otimes (\overline{\mathbf{x}}^2 - \overline{\mathbf{y}}^2) - 2t\widetilde{\mathbf{I}} \right] dt \\
&- \frac{1}{b_1 b_2} \sum_{m \in \mathbf{Z}^2} e^{\imath \left(\mathbf{a} + 2\pi \frac{\mathbf{m}}{\mathbf{b}}\right) \cdot (\mathbf{x}^2 - \mathbf{y}^2)} \widetilde{\mathbf{A}}_m(\mathbf{x}, \mathbf{y}) \\
&\cdot \int_{E_s}^{\infty} \frac{1}{(4\pi t)^{\frac{5}{2}}} \exp\left(-A_s t - \frac{|\overline{\mathbf{x}}^2 - \overline{\mathbf{y}}^2|^2}{4t}\right) \\
&\cdot \left[(\overline{\mathbf{x}}^2 - \overline{\mathbf{y}}^2) \otimes (\overline{\mathbf{x}}^2 - \overline{\mathbf{y}}^2) + 4k_s^2 t^2 \widetilde{\mathbf{I}} \right] dt \Big\}, \mathbf{x}, \mathbf{y} \in \mathbf{R}^3, \qquad (46)
\end{aligned}
$$

with $\mathbf{x}^2, \overline{\mathbf{x}}^2$ as in (37). The integrals appear in the last relation are of the form

$$
\int_{E}^{\infty} \frac{1}{(4\pi t)^{\frac{q}{2}}} \exp\left[-(b-\lambda)t - \frac{c^2}{4t} \right] dt, \qquad (47)
$$

with $q = 1, 3, 5$. The integral for $q = 1$ can be computed in terms of the special function $erfc(z)$ defined by

$$
erfc(z) = 1 - erf(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-s^2} ds. \qquad (48)
$$

In fact holds

$$
\begin{aligned}
\int_{E}^{\infty} \exp\left[-(b-\lambda)t - \frac{c^2}{4t} \right] \frac{dt}{\sqrt{4\pi t}} = \frac{1}{4\sqrt{b-\lambda}} \Big[& e^{c\sqrt{b-\lambda}} erfc\Big(\sqrt{E(b-\lambda)} + \frac{c}{2\sqrt{E}}\Big) \\
& + e^{-c\sqrt{b-\lambda}} erfc\Big(\sqrt{E(b-\lambda)} - \frac{c}{2\sqrt{E}}\Big) \Big], \qquad (49)
\end{aligned}
$$

this last relation can be proved substituting $t = s^2$, differentiating with respect to $\sqrt{E}$ and finally taking into account that both sides vanish when $E \to \infty$.

4. Discussion

From the previous analysis we conclude that:

• When $r = 3$, the dyadic $\widetilde{\mathbf{G}}$ like in (19) with $\widetilde{\mathbf{G}}_1$ and $\widetilde{\mathbf{G}}_2$ which are given by (20) and (30), is expressed by a rapidly convergent series that is very convenient in numerical calculations, from the series which gives the $\widetilde{\mathbf{\Gamma}}$ by relation (17) as certify (23) and (34). By these last two relations is demonstrated the role of E_p and E_s in the velocity of convergence of the series. Moreover, these relations show the different effect of E_a, $a = p, s$ in convergence of $\widetilde{\mathbf{G}}_1$ from this of $\widetilde{\mathbf{G}}_2$.

• If we want to balance the decays of the terms of the series for $\widetilde{\mathbf{G}}_1$ and $\widetilde{\mathbf{G}}_2$, as referred in [7], a reasonable choice is

$$
E_a = \frac{1}{4\pi} \left[\frac{b_1^2 + \ldots + b_r^2}{b_1^{-2} + \ldots + b_r^{-2}} \right]^{\frac{1}{2}}, \quad a = p, s, \quad r = 1, 2, 3. \qquad (50)
$$

This choice is obtained equating (23) and (34) or (23) and (43) and taking into account linear approximations for the exponentials.

• Since especially in elasticity there are a lot of periodic materials, for example composite filters, the Ewald's method can be applied to the solution of boundary value problems. In the case of Dirichlet or Neumann boundary conditions we can apply the Ewald's method, since the corresponding Green's dyadics $\widetilde{\mathbf{\Gamma}}^D$ and $\widetilde{\mathbf{\Gamma}}^N$ is a sum of free space Green's dyadic $\widetilde{\mathbf{\Gamma}}$ and another dyadic $\widetilde{\mathbf{U}}$ of a similar form with appropriate constant coefficients.

• In the case $r = 2$ everything holds as in the case for $r = 3$. Moreover one of the integrals which appear in the expression of $\widetilde{\mathbf{G}}_2$, as is given by (46), can be computed in terms of the special function, error function $erf(z)$, which is an entire function, in contradiction with acoustic case, where the terms of both $\widetilde{\mathbf{G}}_1$ and $\widetilde{\mathbf{G}}_2$ can be computed in terms of the special function $erf(z)$.

Acknowledgements

The authors would like to express their thanks to V. Papanicolaou, for bringing to their attention the new interest in Ewald's method, and the fruitful discussions in the details of the method.

References

[1] Ashcroft and Merhin, *Solid State Physics.* Saunders College Publishing, (1976).

[2] P. P. Ewald, *Die Berechnung optischer and elektrostatischer Gitterspotentiale.* Ann.Phys., **64**, 253-287, (1921).

[3] A. D. Klironomos and E. N. Economou, *Elastic wave band gaps and single scattering.* Solid State Commun., **105**(5), 327-322, (1998).

[4] V. D. Kupradze, *Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity* . North-Holland Publishing Company, Amsterdam, (1979).

[5] N. N. Lebedev, *Special functions and their applications.* Dover Publications,Inc., New York, (1972).

[6] C. M. Linton, *The Green's function for the two-dimensional Helmholtz equation in periodic domains.* J. Engin. Math., **33**, 377-402, (1998).

[7] V. Papanicolaou, *Ewald's method revisited: Rapidly convergent series representations of certain Green's-functions.* J. Comp. Anal. Appl., **1**, 105-114, (1999).

[8] J. A. F. Santiago and L. C. Wrobel, *2D modeling of shallow water acoustic wave propagation using subregions technique.* Boundary Element Techniques-An International Conference. Queen Mary and Westfield College, 415-424, U.K.,(1999).

[9] J. A. F. Santiago and L. C. Wrobel, *A boundary element model for underwater acoustics in shallow waters.* Comp. Model. Engin. Sci., **1**, 73-80, (2000).

[10] M. M. Sigalas and E. N. Economou, *Elastic waves in plates with periodically placed inclusions.* J. Appl. Phys., **75**(6), 2845-2850,(1994).

[11] S. Venakides, M. A. Haider and V. Papanicolaou, *Boundary integral calculations of 2-D electromagnetic scattering by photonic crystal Fabry-Perot structures.* SIAM J. Appl. Math., **60**, 1686-1706, (2000).

◇ Constantinos Anestopoulos
National Technical University of Athens,
Department of Applied Mathematics and Physics,
GR-15780 Zografou Campus, Athens, Greece
kanesto@mail.ntua.gr

◇ Elias Argyropoulos
Technological Education Institute,
Department of Electrical Engineering,
GR-35100 Lamia, Greece
protepkste@stellad.pde.sch.gr

◇ Drossos Gintides
Hellenic Naval Academy.
Department of Mathematics,
Pireus, Greece
dgindi@math.ntua.gr

◇ Kiriakie Kiriaki
National Technical University of Athens,
Department of Applied Mathematics and Physics,
GR-15780 Zografou Campus, Athens, Greece
kkouli@math.ntua.gr