BULLETIN OF THE
GREEK MATHEMATICAL SOCIETY
Volume 52, 2006 (9–23)

On galbed algebras and galbed spaces

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Abstract

Several results about galbed algebras and galbed spaces useful in topological (especially Gelfand-Mazur) algebras are presented.

Keywords: Galbed algebras, galbed spaces

1. Introduction

Let $\mathbb{C}$ be the field of complex numbers, $\mathbb{R}$ the field of real numbers, $\mathbb{K}$ either $\mathbb{C}$ or $\mathbb{R}$, $\mathbb{N} = \{0, 1, 2, ...\}$, $\mathbb{Z}^+ = \{1, 2, ...\}$, $s_{\mathbb{K}}$ the set of all sequences (α_n) with $\alpha_n \in \mathbb{K}$ for each $n \in \mathbb{N}$, $k > 0$, l^k the set of all $(\alpha_n) \in s_{\mathbb{K}}$, for which the series

$$\sum_{v=0}^{\infty} |\alpha_v|^k$$

converges, l^0 the set of all $(\alpha_n) \in s_{\mathbb{K}}$ such that the set $\{k \in \mathbb{N} \ : \alpha_k \neq 0\}$ is finite, $l = l^1 \setminus l^0$ and

$$l^{(0,1]} = \bigcap_{k \in (0,1]} l^k.$$

Let A be an associative topological algebra over $\mathbb{K}$ with separately continuous multiplication (in short, a topological algebra). Through the whole paper we will assume that $A \neq \{\theta_A\}$ and the topology τ of A is not trivial, i.e. $\tau \neq \{\emptyset, A\}$. We will denote by $G(A)$ the set of all $(\alpha_n) \in s_{\mathbb{K}}$ with the following property:

for each neighbourhood O of zero in A there is another neighbourhood U of zero in A such that

$$\Big\{\sum_{k=0}^{n} \alpha_k a_k : a_0, \ldots, a_n \in U\Big\} \subset O$$

for each $n \in \mathbb{N}$.

[1]This paper has been done by the support from IKY and Estonian Science Foundation grant 4514.

Remember, that a topological algebra A is *locally pseudoconvex*, if it has a base of neighbourhoods $\{O_\lambda : \lambda \in \Lambda\}$ such that every O_λ is balanced (that is, $\mu O_\lambda \subset O_\lambda$ for each $\mu \in \mathbb{K}$ with $|\mu| \leq 1$) and there are numbers $p_\lambda \in (0,1]$ such that $O_\lambda + O_\lambda \subset 2^{\frac{1}{p_\lambda}} O_\lambda$. If there exists $p \in (0,1]$ such that $p_\lambda = p$ for every $\lambda \in \Lambda$, then A is a *locally p-convex algebra*. If $p = 1$, then A is a *locally convex algebra*. For every neighbourhood U of zero in A and every $p \in (0,1]$ we will denote the p-convex cover of U by

$$\Gamma_p(U) = \bigcup_{n \in \mathbb{N}} \Big\{ \sum_{k=0}^{n} \alpha_k u_k \ : \ \alpha_0, \ldots, \alpha_n \in \mathbb{K}, \ \sum_{k=0}^{n} |\alpha_k|^p \leq 1, \ u_0, \ldots, u_n \in U \Big\}.$$

Definition. A topological algebra A is called a *galbed algebra*, if there exists a sequence $(\alpha_n) \in l \cap G(A)$.

Furthermore, if there exists a sequence $(\alpha_n) \in l \cap G(A)$ with $\alpha_0 \neq 0$ and

$$\alpha = \inf_{n>0} |\alpha_n|^{\frac{1}{n}} > 0$$

then it is said that A is a *strongly galbed algebra*. Herewith, α is sometimes called the “module of galbness” of A.

In case we have already specified the sequence $(\alpha_n) \in l \cap G(A)$, then we talk of (α_n)-*galbed algebra* and of *strongly* (α_n)-*galbed algebra*.

We can also talk about *galbed spaces, strongly galbed spaces,* (α_n)-*galbed spaces* and *strongly* (α_n)-*galbed spaces* if we consider A only as a topological vector space instead of topological algebra.

The term of “galbed space” and also the notation of $G(A)$ were first used by P. Turpin in the early 1970-s (see [14], [15], [16], [17], [18]). The topic of galbed (metric) spaces was considered also by Rolewicz in [13] (first edition in 1972) and Waelbroeck in [19], [20].

Several articles about galbed algebras and spaces have been published since 1990 (see [3], [4], [8], [9], [10], [11]). The topic of galbed algebras has become more popular among mathematicians studying topological algebras since the class of galbed and especially of strongly galbed algebras allows us to widen the class of Gelfand-Mazur algebras known up to now (see [8], [10]).

The aim of this article is to collect together the main results about galbed algebras obtained by different people and to give detailed (in most cases new) proofs (see also Remark). Since the authors are mainly working with (topological) algebras then the results will be given and proved for galbed algebras. Nevertheless, everything below is also true if we use the term “galbed space” instead of the term “galbed algebra”. For the applications of galbed spaces and algebras we suggest to take a look at the [14], [15], [16], [18] (where galbed spaces are used for studying the properties of Orlitz spaces) or [1], [2], [3], [4], [5], [6], [7], [9], [11] (where the galbed or exponentially galbed algebras are used for obtaining a description of the ideal structure of topological algebras).

2. Properties of $G(A)$

In this section we give several results necessary for galbed algebras (and spaces) in general without trying to fix a concrete sequence (α_n) for which the algebra A is (α_n)-galbed.

Lemma 1. *Let A be a topological algebra and $(\alpha_n), (\beta_n) \in s_{\mathbb{K}}$ such that $|\beta_k| \leq |\alpha_k|$ for every $k \in \mathbb{N}$. If $(\alpha_n) \in G(A)$, then also $(\beta_n) \in G(A)$.*

Proof. Let $(\alpha_n) \in G(A)$ and let O be any neighbourhood of zero in A. Then there is another balanced neighbourhood U of zero in A such that

$$\Big\{\sum_{k=0}^{n} \alpha_k a_k : a_0, \ldots, a_n \in U\Big\} \subset O$$

for each $n \in \mathbb{N}$. Since $|\frac{\beta_k}{\alpha_k}| \leq 1$ then $\beta_k a_k = \alpha_k \frac{\beta_k}{\alpha_k} a_k \in \alpha_k U$ for every $k \in \mathbb{N}$. Thus

$$\Big\{\sum_{k=0}^{n} \beta_k a_k : a_0, \ldots, a_n \in U\Big\} \subset \Big\{\sum_{k=0}^{n} \alpha_k u_k : u_0, \ldots, u_n \in U\Big\} \subset O$$

for each $n \in \mathbb{N}$. Since O was any neighbourhood of zero in A, we have $(\beta_n) \in G(A)$.

The following result could be found in [18] (without proof) and partly in [13] (for metric vector spaces). For the sake of completeness we present here complete proof different from the one on [13].

Lemma 2. *Let A be a topological algebra. Then $l^0 \subseteq G(A) \subseteq l^1$.*

Proof. Let $(\alpha_n) \in l^0$. Then there are $m \in \mathbb{N}$ and $k_0, \ldots, k_m \in \mathbb{N}$ such that $\alpha_n = 0$ if $n \notin \{k_0, \ldots, k_m\}$ and otherwise $\alpha_n \neq 0$. Let O be a neighbourhood of zero in A. By the properties of topological vector spaces, there is another neighbourhood U of zero such that

$$\alpha_{k_0} U + \cdots + \alpha_{k_m} U \subset O.$$

Since

$$\sum_{v=0}^{n} \alpha_v u_v = \begin{cases} \theta_A & \text{if } n < k_0 \\ \sum_{v=0}^{s} \alpha_{k_v} u_{k_v} & \text{if } k_s \leq n < k_{s+1} \text{ with } s \in \{0, \ldots, m-1\} \\ \sum_{v=0}^{m} \alpha_{k_v} u_{k_v} & \text{if } k_m \leq n \end{cases}$$

for each $n \in \mathbb{N}$ and $u_0, \ldots, u_n \in U$, then

$$\Big\{\sum_{v=0}^{n} \alpha_v u_v : u_o, \ldots, u_n\Big\} \subset O$$

for each $n \in \mathbb{N}$. So $l^0 \subseteq G(A)$ for every topological vector space A.

Suppose that there exists a sequence $(\alpha_n) \in G(A) \setminus l^1$. Then $(|\alpha_n|) \in G(A)$, by Lemma 1 and for each $m \in \mathbb{N}$ there exists $n_m \in \mathbb{N}$ such that

$$m \leq \sum_{k=0}^{n_m} |\alpha_k|.$$

Let O be a neighbourhood of zero in A. Then there exists another balanced neighbourhood U of zero in A such that

$$\Big\{\sum_{k=0}^{n} |\alpha_k| u_k : u_0, \ldots, u_n \in U\Big\} \subset O$$

for each $n \in \mathbb{N}$. Fix $u \in U$. Now, for every $m \in \mathbb{N}$ we have

$$\frac{m}{\sum_{i=0}^{n_m} |\alpha_i|} u \in U$$

and

$$mu = \sum_{k=0}^{n_m} \Big(|\alpha_k| \frac{m}{\sum_{i=0}^{n_m} |\alpha_i|} u\Big) \in \Big\{\sum_{k=0}^{n_m} |\alpha_k| u_k : u_0, \ldots, u_{n_m} \in U\Big\} \subset O,$$

which means that $mu \in O$. Hence

$$\bigcup_{m \in \mathbb{N}} mU \subset O.$$

Since every neighbourhood of zero absorbs all elements of A, then every $a \in A$ defines $\rho > 0$ such that $a \in \rho U = m\frac{\rho}{m} U \subset mU$, where $m > [\rho] + 1$. It means that

$$A = \bigcup_{m \in \mathbb{N}} mU.$$

Thus, A is the only neighbourhood of zero in A, but it is impossible, since the topology of A is not trivial. Therefore, $l^0 \subseteq G(A) \subseteq l^1$.

Lemma 3. *The set $G(A)$ is a linear subspace of l^1 for every topological algebra A.*

Proof. Let $(\alpha_n), (\beta_n) \in G(A)$, $\lambda \in \mathbb{K}$ and O be any neighbourhood of zero in A. Then there is a neighbourhood U of zero such that $U + U \subset O$ and neighbourhoods V_1 and V_2 of zero in A such that

$$\Big\{\sum_{k=0}^{n} \alpha_k a_k : a_0, \ldots, a_n \in V_1\Big\} \subset U$$

and

$$\Big\{\sum_{k=0}^{n} \beta_k b_k : b_0, \ldots, b_n \in V_2\Big\} \subset U$$

for each $n \in \mathbb{N}$. Let $V = V_1 \cap V_2$. Then

$$\Big\{\sum_{k=0}^{n}(\alpha_k + \beta_k)v_k : v_0, \dots, v_n \in V\Big\} \subset$$

$$\subset \Big\{\sum_{k=0}^{n} \alpha_k a_k : a_0, \dots, a_n \in V_1\Big\} + \Big\{\sum_{k=0}^{n} \beta_k b_k : b_0, \dots, b_n \in V_2\Big\} \subset U + U \subset O$$

for each $n \in \mathbb{N}$ which means that $(\alpha_n) + (\beta_n) \in G(A)$.

Analogically we get that $\lambda(\alpha_n) \in G(A)$. Thus, $G(A)$ is a linear subspace of l^1.

From Lemma 3 we have the following

Corollary 1. *Let A be a topological algebra and $(\alpha_n), (\beta_n) \in l^1$ such sequences that $(\gamma_n) \in G(A)$, where $\gamma_k = \alpha_k - \beta_k$ for each $k \in \mathbb{N}$. Then $(\alpha_n) \in G(A)$ if and only if $(\beta_n) \in G(A)$.*

Lemma 4. *Let B be a subspace of a topological algebra A. Then $G(A) \subseteq G(B)$.*

Proof. Let $(\alpha_n) \in G(A)$ and let O_B be any neighbourhood of zero of B in the subspace topology. Then there exists a neighbourhood O_A of zero in A such that $O_B = O_A \cap B$. Now there is a neighbourhood U_A of zero in A such that

$$\Big\{\sum_{k=0}^{n} \alpha_k a_k : a_0, \dots, a_n \in U_A\Big\} \subset O_A$$

for each $n \in \mathbb{N}$. Let $U_B = U_A \cap B$. Then

$$\Big\{\sum_{k=0}^{n} \alpha_k b_k : b_0, \dots, b_n \in U_B\Big\} \subset O_A \cap B = O_B$$

for each $n \in \mathbb{N}$. Thus, $(\alpha_n) \in G(B)$. Hence, $G(A) \subseteq G(B)$.

Lemma 5. *Let T be a continuous linear map from a topological algebra A onto a topological algebra B. Then $G(A) \subseteq G(T(A)) = G(B)$.*

Proof. The proof is exactly the same as in [13], proof of Proposition 3.9.9, p. 160-161. Herewith, the proof does not depend neither on the fact that A and B are not metric vector spaces nor on the fact that A and B are topological algebras in our case.

3. Properties of galbed algebras and galbed spaces

In this chapter we give the results which depend more on the concrete sequence (α_n) for which the algebra A is (α_n)-galbed.

Lemma 6. *Let A be a topological algebra, $(\alpha_n) \in s_{\mathbb{K}}$ and (α_{k_n}) a subsequence of (α_n). If $(\alpha_n) \in G(A)$, then $(\alpha_{k_n}) \in G(A)$.*

Proof. Let $(\alpha_n) \in G(A)$ and O be any neighbourhood of zero in A. Then there is another neighbourhood U of zero such that

$$\Big\{\sum_{v=0}^{n} \alpha_v a_v : a_0, \dots, a_n \in U\Big\} \subset O$$

for each $n \in \mathbb{N}$. If (α_{k_n}) is a subsequence of (α_n), then $(\alpha_{k_n}) \in l^1$. Since

$$\sum_{v=0}^{n} \alpha_{k_v} a_v = \sum_{k=0}^{k_n} \alpha_k c_k,$$

where $c_{k_v} = a_v$ for each $v \in \{1, \dots, n\}$ and $c_k = \theta_A$ otherwise, then

$$\Big\{\sum_{v=0}^{n} \alpha_{k_v} a_v : a_0, \dots, a_n \in U\Big\} \subset \Big\{\sum_{v=0}^{k_n} \alpha_v c_v : c_0, \dots, c_{k_n} \in U\Big\} \subset O$$

for each $n \in \mathbb{N}$. Consequently, $(\alpha_{k_n}) \in G(A)$.

Lemma 7. *Let A be a topological algebra and $(\alpha_n) \in G(A) \setminus l^0$. Then there exists a sequence $(\beta_n) \in G(A) \setminus l^0$ such that $\beta_k \in \mathbb{R}$ and $\beta_k > 0$ for each $k \in \mathbb{N}$.*

Proof. Let A be an (α_n)-galbed algebra. Define

$$i_0 := \min\{k : \alpha_k \neq 0\}$$

and

$$i_j := \min\{k : k > i_{j-1}, \alpha_k \neq 0\}$$

for each $j \in \mathbb{N}\backslash\{0\}$. As $(\alpha_n) \notin l^0$, then (α_n) has infinite number of nonzero elements. Let us define

$$\gamma_k := \alpha_{i_k}$$

for each $k \in \mathbb{N}$. Since (α_{i_k}) is a subsequence of (α_n), then $(\gamma_n) = (\alpha_{i_n}) \in G(A)$ by Lemma 6. Obviously $(\gamma_k) \neq 0$ for each $k \in \mathbb{N}$. Thus, $(\gamma_n) \in G(A) \setminus l^0$.

We construct the sequence (β_n) as follows:

$$\beta_k := |\gamma_k| \text{ for every } k \in \mathbb{N}.$$

Then $\beta_k > 0$ for each $k \in \mathbb{N}$.

Let O be any neighbourhood of zero in A. Since $(\gamma_n) \in G(A)$, then there exists a neighbourhood U of zero such that

$$\Big\{\sum_{k=0}^{n} \gamma_k u_k : u_0, \dots, u_n \in U\Big\} \subset O$$

for each $n \in \mathbb{N}$. Let V be a balanced neighbourhood of zero in A such that $V \subset U$. Fix some $n \in \mathbb{N}$ and let $b_0, \ldots, b_n$ be arbitrary elements of V. Let

$$a_k := \frac{|\gamma_k|}{\gamma_k} b_k$$

for each $k \in \{0, \ldots, n\}$. Then $a_k \in V$ for each $k \in \mathbb{N}$, because V is balanced. Now

$$\Big\{\sum_{k=0}^{n} \beta_k b_k : b_0, \ldots, b_n \in V\Big\} = \Big\{\sum_{k=0}^{n} \gamma_k a_k : a_0, \ldots, a_n \in V\Big\} \subset$$

$$\subset \Big\{\sum_{k=0}^{n} \gamma_k u_k : u_0, \ldots, u_n \in U\Big\} \subset O$$

since $V \subset U$. As O, n and $b_0, \ldots, b_n \in V$ were chosen arbitrarily, then $(\beta_n) \in G(A)\backslash l^0$.

The following result gives us a posibility of using more concrete sequence (α_n) in case $G(A) \setminus l^0 \neq \emptyset$.

Lemma 8. *Let A be a topological algebra such that $G(A) \setminus l^0 \neq \emptyset$. Then there is a sequence $(\alpha_n) \in G(A)$ such that*

$$\alpha_n = \frac{1}{k_n^n}$$

for each $n \in \mathbb{N}\backslash\{0\}$ where $k_n \in \mathbb{N}\backslash\{0\}$ and $k_i \leq k_j$, if $i \leq j$.

Proof. Let $(\gamma_n) \in G(A) \setminus l^0$. By Lemma 7, there exists a sequence $(\beta_n) \in G(A) \setminus l^0$ such that $\beta_k \in \mathbb{R}$ and $\beta_k > 0$ for each $k \in \mathbb{N}$.

Let us define $i_0 := 0$ and

$$i_k := \min\{m \in \mathbb{N}\backslash\{0\} : \beta_m < \frac{1}{k^m}\}$$

for each $k \in \mathbb{N}\backslash\{0\}$. If for some $k_0 \in \mathbb{N}$ holds $\beta_m \geq \frac{1}{k_0^m}$ for every $m \in \mathbb{N}\backslash\{0\}$, then we take $i_k := \infty$ for every $k \geq k_0$. Then, for example, $i_4 = 18$ means that

$$\beta_m \geq \frac{1}{4^m}$$

for all $m \in \{1, 2, \ldots, 16, 17\}$ but

$$\beta_{18} < \frac{1}{4^{18}}.$$

Next, let $\alpha_0 = \beta_0$ and for every $m \in \mathbb{N}\backslash\{0\}$ let[1]

$$\alpha_m := \frac{1}{s^m} \text{ if } i_{s-1} \leq m < i_s.$$

[1] If $i_s = i_{s-1} \neq \infty$ for some s, then increase s until $i_s \neq i_{s-1}$.

Then (α_n) is a sequence such that[2]

$$\alpha_n = \frac{1}{k_n^n}$$

for every $n \in \mathbb{N}\backslash\{0\}$ where $k_n = s \in \mathbb{N}\backslash\{0\}$, if $i_{s-1} \leq n < i_s$ and $k_i \leq k_j$, if $i \leq j$.

According to the definitions of i_m and α_m we have

$$\alpha_m = \frac{1}{s^m} \leq \beta_m$$

for every $m > 0$. Hence $(\alpha_n) \in G(A)$, by Lemma 1.

4. Galbed algebras and convexity

Here we give some results which connect the properties of being galbed with the properties on convexity.

Proposition 1. *Let $p \in (0,1]$. If A is a locally p-convex algebra, then $l^p \subset G(A)$.*

Proof. Let $p \in (0,1]$, A be a locally p-convex algebra, $\{q_\lambda : \lambda \in \Lambda\}$ a saturated family of p-homogeneous seminorms on A which defines the topology of A and O a neighbourhood of zero in A. Then there exist $\epsilon > 0$ and $\lambda \in \Lambda$ such that

$$O_{\lambda,\epsilon} = \{a \in A : q_\lambda(a) < \epsilon\} \subset O.$$

Let $(\alpha_n) \in l^p$,

$$M = \sum_{v=0}^{\infty} |\alpha_v|^p,$$

$\delta \in (0, \epsilon M^{-1})$, $n \in \mathbb{N}$ and $u_0, \ldots, u_n \in O_{\lambda,\delta}$. Since

$$q_\lambda\Big(\sum_{v=0}^{n} \alpha_v u_v\Big) \leq \sum_{v=0}^{n} |\alpha_v|^p q_\lambda(u_v) < \delta \sum_{v=0}^{\infty} |\alpha_v|^p = \delta M < \epsilon,$$

then

$$\Big\{\sum_{v=0}^{n} \alpha_v u_v : u_0, \ldots, u_n \in O_{\lambda,\delta}\Big\} \subset O_{\lambda,\epsilon} \subset O$$

for each $n \in \mathbb{N}$. Hence, $l^p \subset G(A)$.

From the Proposition 1 follows easily

Corollary 2. *Let A be a locally convex algebra. Then $G(A) = l^1$.*

Corollary 3. *Let A be a topological algebra. Then $G(G(A)) = l^1$.*

[2] There are several sequences (β_n) with positive elements for which $k_n = s \neq n$. For example, (β_n) with $\beta_k = \frac{1}{(k^2+1)^k}$ for each $k \in \mathbb{N}$.

Proof. Let A be a topological algebra. Then $G(A) \subset l^1$, by Lemma 2. Now, by Lemma 4, $G(l^1) \subset G(G(A))$. Since l^1 is locally convex (since it is normed space), then $G(l^1) = l^1$, by Corollary 2. Thus, $l^1 \subset G(G(A))$. By Lemma 3 we know that $G(A)$ is a topological vector space. Hence, by Lemma 2 (for topological vector spaces) we have $G(G(A)) \subset l^1$. Therefore, $G(G(A)) = l^1$.

Corollary 4. *If A is a locally convex algebra (or locally convex space), then $G(G(A)) = G(A)$.*

Proposition 2. *If A is a locally pseudoconvex algebra, then $l^{(0,1]} \subset G(A)$.*

Proof. Let A be a locally pseudoconvex algebra, $\{q_\lambda : \lambda \in \Lambda\}$ a saturated family of p_λ-homogeneous seminorms on A, with $p_\lambda \in (0, 1]$ for each $\lambda \in \Lambda$, which defines the topology of A and O a neighbourhood of zero in A. Then there exist $\epsilon > 0$ and $\lambda \in \Lambda$ such that

$$O_{\lambda,\epsilon} = \{a \in A : q_\lambda(a) < \epsilon\} \subset O.$$

Let $(\alpha_n) \in l^{(0,1]}$,

$$M_\lambda = \sum_{v=0}^{\infty} |\alpha_v|^{p_\lambda},$$

$\delta = \delta(\lambda) \in (0, \epsilon M_\lambda^{-1})$, $n \in \mathbb{N}$ and $u_0, \ldots, u_n \in O_{\lambda,\delta}$. Since

$$q_\lambda\Big(\sum_{v=0}^{n} \alpha_v u_v\Big) \leq \sum_{v=0}^{n} |\alpha_v|^{p_\lambda} q_\lambda(u_v) < \delta \sum_{v=0}^{\infty} |\alpha_v|^{p_\lambda} = \delta M_\lambda < \epsilon,$$

then

$$\Big\{\sum_{v=0}^{n} \alpha_v u_v : u_0, \ldots, u_n \in O_{\lambda,\delta}\Big\} \subset O_{\lambda,\epsilon} \subset O$$

for each $n \in \mathbb{N}$. Hence, $l^{(0,1]} \subset G(A)$.

Proposition 3. *Let A be a metrizable topological algebra and $p \in (0, 1]$. If $l^p \subset G(A)$, then A is locally p-convex.*

Proof. Let A be a metrizable topological algebra and $p \in (0, 1]$. Then there exists a countable base $\{U_n : n \in \mathbb{N}\}$ of neighbourhoods of zero of A. Suppose, that A is not locally p-convex. Then there exists a neighbourhood O of zero in A such that $\Gamma_p(2^{-\frac{m}{p}} U_m) \not\subset O$ for every $m \in \mathbb{N}$. Thus, for each $m \in \mathbb{N}$ there exist $n_m \in \mathbb{N}$, $\alpha_0^m, \ldots, \alpha_{n_m}^m \in \mathbb{K}$ and $u_0^m, \ldots, u_{n_m}^m \in U_m$ such that

$$\sum_{k=0}^{n_m} |\alpha_k^m|^p \leq 1, \qquad \text{and} \qquad 2^{-\frac{m}{p}} \sum_{k=0}^{n_m} \alpha_k^m u_k^m \notin O.$$

For every $n \in \mathbb{N}$, let

$$S_n = \sum_{k=0}^{n} (n_k + 1).$$

We define the sequence (β_n) as follows:

$$\beta_k = \begin{cases} \alpha_k^0 & \text{if } S_{-1} = 0 \leq k \leq n_0 = S_0 - 1 \\ 2^{-\frac{m}{p}} \alpha^m_{k-S_{m-1}} & \text{if } S_{m-1} \leq k < S_m \end{cases}$$

for each $m = 1, 2, \ldots$ It means that

$$\sum_{k=0}^{\infty} |\beta_k|^p = \sum_{k=0}^{n_0} |\alpha_k^0|^p + \sum_{k=S_0}^{S_1-1} |2^{-\frac{1}{p}} \alpha^1_{k-S_0}|^p + \cdots + \sum_{k=S_{m-1}}^{S_m-1} |2^{-\frac{m}{p}} \alpha^m_{k-S_{m-1}}|^p + \cdots =$$

$$= \sum_{k=0}^{n_0} |\alpha_k^0|^p + 2^{-1} \sum_{k=0}^{n_1} |\alpha_k^1|^p + 2^{-2} \sum_{k=0}^{n_2} |\alpha_k^2|^p + \ldots \leq \sum_{k=0}^{\infty} 2^{-k} = 2,$$

because

$$\sum_{k=0}^{n_m} |\alpha_k^m|^p \leq 1$$

for each $m \in \mathbb{N}$. Thus, $(\beta_n) \in l^p$.

Let V be an arbitrary neighbourhood of zero in A. Then there exists $m = m_V \in \mathbb{N}$ such that $U_m \subset V$. Let $v_k = \theta_A$, if $k < S_{m-1}$ and $v_k = u^m_{k-S_{m-1}}$, if $S_{m-1} \leq k < S_m$. Then

$$2^{-\frac{m}{p}} \sum_{k=0}^{n_m} \alpha_k^m u_k^m = \sum_{k=0}^{S_m-1} \beta_k v_k \in \bigcup_{n \in \mathbb{N}} \Big\{ \sum_{k=0}^{n} \beta_k b_k : b_0, \ldots, b_n \in V \Big\}.$$

Thus,

$$\bigcup_{n \in \mathbb{N}} \Big\{ \sum_{k=0}^{n} \beta_k b_k : b_0, \ldots, b_n \in V \Big\} \not\subset O.$$

As V was chosen arbitrarily, we see that $(\beta_n) \in l^p \setminus G(A)$ or $l^p \not\subset G(A)$. Consequently, A is locally p-convex.

From the Proposition 3 follows easily

Corollary 5. *If A is a metrizable topological algebra with $G(A) = l^1$, then A is locally convex.*

Proposition 4. *Let A be a metrizable topological algebra. If $l^{(0,1]} \subset G(A)$, then A is locally pseudoconvex.*

Proof. Let A be a metrizable topological algebra. Then there exists a countable base $\{U_n : n \in \mathbb{N}\}$ of neighbourhoods of zero of A. For each $m \in \mathbb{N}$ let

$$W_m = \bigcup_{n \in \mathbb{N}} \Big\{ \sum_{k=0}^{n} \alpha_k u_k \ : \alpha_0, \ldots, \alpha_n \in \mathbb{K}, \sup_{p \in (0,1]} \sum_{k=0}^{\infty} |\alpha_k|^p \leq 1, u_0, \ldots, u_n \in U_m \Big\}.$$

Then it is easy to check that $\mu W_m \subset W_m$ for each $|\mu| \leq 1$ and $W_m + W_m \subset 2W_m$. Thus W_m is balanced and pseudoconvex for each $m \in \mathbb{N}$. If for every neighbourhood of zero O in A there exists such $m(O) \in \mathbb{N}$ that $W_{m(O)} \subset O$, then A is locally pseudoconvex algebra. It is also easy to see that $2^{-m^2}U_m \subset 2^{-m^2}W_m \subset \Gamma_p(2^{-m^2}U_m)$ for each $m \in \mathbb{N}$ and each $p \in (0,1]$.

Suppose, that A is not locally pseudoconvex. Then there exists a neighbourhood O of zero in A such that $2^{-m^2}W_m \not\subset O$ for every $m \in \mathbb{N}$. Now, for each $m \in \mathbb{N}$ there exist $n_m \in \mathbb{N}$, $\alpha_0^m, \dots, \alpha_{n_m}^m \in \mathbb{K}$ and $u_0^m, \dots, u_{n_m}^m \in U_m$ such that

$$\sup_{p\in(0,1]} \sum_{k=0}^{n_m} |\alpha_k^m|^p \le 1, \qquad \text{and} \qquad 2^{-m^2}\sum_{k=0}^{n_m} \alpha_k^m u_k^m \notin O.$$

For each $n \in \mathbb{N}$ let

$$S_n = \sum_{k=0}^{n}(n_k+1).$$

We define the sequence (β_n) as follows:

$$\beta_k = \begin{cases} \alpha_k^0 & \text{if } S_{-1} = 0 \le k \le n_0 = S_0 - 1 \\ 2^{-m^2}\alpha_{k-S_{m-1}}^m & \text{if } S_{m-1} \le k < S_m \end{cases}$$

for each $m = 1, 2, \dots$

Choose an arbitrary $p \in (0,1]$. Then there exists $r = r(p) \in \mathbb{N}$ such that $tp > 1$ whenever $t \ge r$. Clearly, $2^{-t^2 p} \le 2^{-t}$ whenever $t \ge r$. Denote

$$R = R(p) = \sum_{k=0}^{S_{r-1}-1} |\beta_k|^p.$$

Then

$$\sum_{k=0}^{\infty}|\beta_k|^p = \sum_{k=0}^{S_{r-1}-1}|\beta_k|^p + \sum_{k=S_{r-1}}^{S_r-1}\left|2^{-r^2}\alpha_{k-S_{r-1}}^r\right|^p + \sum_{k=S_r}^{S_{r+1}-1}\left|2^{-(r+1)^2}\alpha_{k-S_r}^{r+1}\right|^p + \cdots =$$

$$= R + 2^{-r^2 p}\sum_{k=0}^{n_r}|\alpha_k^r|^p + 2^{-(r+1)^2 p}\sum_{k=0}^{n_{r+1}}|\alpha_k^{r+1}|^p + 2^{-(r+2)^2 p}\sum_{k=0}^{n_{r+2}}|\alpha_k^{r+2}|^p + \ldots \le$$

$$\le R + 2^{-r}\sum_{k=0}^{n_r}|\alpha_k^r|^p + 2^{-(r+1)}\sum_{k=0}^{n_{r+1}}|\alpha_k^{r+1}|^p + 2^{-(r+2)}\sum_{k=0}^{n_{r+2}}|\alpha_k^{r+2}|^p + \ldots \le$$

$$\le R + \sum_{k=r}^{\infty} 2^{-k} \le R + 2,$$

because

$$\sum_{k=0}^{n_m}|\alpha_k^m|^p \le 1$$

for each $m \in \mathbb{N}$ and each $p \in (0,1]$. Thus, $(\beta_n) \in l^p$. Since p was chosen arbitrarily from $(0,1]$ and (β_n) does not depend on the selection of p, then $(\beta_n) \in l^{(0,1]}$.

Let V be an arbitrary neighbourhood of zero in A. Then there exists $m = m_V \in \mathbb{N}$ such that $U_m \subset V$. Let $v_k = \theta_A$, if $k < S_{m-1}$ and $v_k = u^m_{k-S_{m-1}}$, if $S_{m-1} \leq k < S_m$. Then

$$2^{-m^2}\sum_{k=0}^{n_m}\alpha_k^m u_k^m = \sum_{k=0}^{S_m-1}\beta_k v_k \in \bigcup_{n\in\ \mathbb{N}}\Big\{\sum_{k=0}^{n}\beta_k b_k : b_0,\dots,b_n \in V\Big\}.$$

Thus,

$$\bigcup_{n\in\ \mathbb{N}}\Big\{\sum_{k=0}^{n}\beta_k b_k : b_0,\dots,b_n \in V\Big\} \not\subset O.$$

As V was chosen arbitrarily, we see that $(\beta_n) \in l^{(0,1]} \setminus G(A)$ or $l^{(0,1]} \not\subset G(A)$. Consequently, A is locally pseudoconvex.

Open problem 1. *Does there exist a topological algebra A such that $l^p \subset G(A)$ for some $p \in (0,1]$ but A is not locally p-convex?*

Open problem 2. *Does there exist a topological algebra A such that $l^{(0,1]} \subset G(A)$ but A is not locally pseudoconvex?*

5. Topological algebra which is not a galbed algebra

Let (p_n) be a sequence with $p_n \in (0,1]$ for each $n \in \mathbb{N}$ and $l^{(p_n)}$ the set of all sequences (x_n) such that

$$||(x_n)|| = \sum_{n=0}^{\infty}|x_n|^{p_n} < \infty.$$

All algebraic operations in $l^{(p_n)}$ we define (as often in case of sequence algebras) coordinate-wise that is, $(x_n)+(y_n) = (x_n+y_n)$, $\lambda(x_n) = (\lambda x_n)$ and $(x_n)\cdot(y_n) = (x_n y_n)$ for each $(x_n), (y_n) \in l^{(p_n)}$ and $\lambda \in \mathbb{K}$. Then $||\ ||$ is a submultiplicative F-norm[3] on $l^{(p_n)}$. Hence $l^{(p_n)}$ is a metrizable algebra (topological algebra, the underlying topological vector space of which is a metrizable space (see [12], p. 163)).

Proposition 5. *Let (p_n) be a sequence with $p_n \in (0,1]$ for each $n \in \mathbb{N}$. If (p_n) converges to 0, then $G(l^{(p_n)}) = l^0$.*

Proof. Let $X = l^{(p_n)}$. Suppose that $G(X)\setminus l^0$ is not empty. Then there is a sequence $(\lambda_n) \in G(X) \setminus l^0$. Hence, for each neighbourhood O of zero in X there is another neighbourhood U of zero such that

$$\bigcup_{n\in\ \mathbb{N}}\Big\{\sum_{k=0}^{n}\lambda_k u_k : u_0,\dots,u_n \in U\Big\} \subset O$$

[3]That is, for each $x = (x_k) \in l^{(p_n)}$ and $y = (y_k) \in l^{(p_n)}$ hold $||x|| \geq 0$; $||x|| = 0$ if and only if x is the zero sequence; $||\lambda x|| \leq ||x||$ if $|\lambda| \leq 1$; $||x+y|| \leq ||x|| + ||y||$; $||xy|| \leq ||x||\,||y||$; $(||\lambda x_n||)$ converges to 0 if $(||x_n||)$ converges to 0 for each $\lambda \in \mathbb{K}$ and $(||\lambda_n x||)$ converges to 0 if (λ_n) converges to 0.

and $\lambda_n > 0$ for infinitely many n. Let $\{\lambda_{k_n} : n \in \mathbb{N}\}$ be the set of such values of λ_n. Since $p_n \to 0$, then for each k_n there is a number $m(n) \in \mathbb{N}$ such that $\lambda_{k_n}^{p_m} > \frac{1}{2}$ whenever $m \geq m(n)$ and $m(n) \leq m(n+1)$ for each $n \in \mathbb{N}$. Now for each $n \in \mathbb{N}$ let

$$v_n = \{0, 0 \ldots, 0, \frac{1}{(n+1)^{\frac{1}{p_{m(n)}}}}, 0, \ldots\}$$

(only the $m(n)$-th entry is different from zero here). Since $||v_n||_X = \frac{1}{(n+1)}$ for each $n \in \mathbb{N}$, then (v_n) converges to zero element in X. Therefore there is a number $n_0 \in \mathbb{N}$ such that $v_n \in U$ whenever $n \geq n_0$.

Let

$$y_l = \sum_{r=0}^{l} \lambda_{k_r} v_{n_0+r}$$

for each $l \in \mathbb{N}$. Since

$$y_l = \sum_{r=0}^{k_l} \lambda_r w_r,$$

where $w_r = \theta_A$ if $r \neq \{k_0, \ldots, k_l\}$ and $w_{k_r} = v_{n_0+r} \in U$ for each $r \in \{0, \ldots, l\}$, then $y_l \in O$ for each $l \in \mathbb{N}$. Hence (y_l) is a bounded sequence in X. On the other hand, $y_l = (z_n)$, where $z_n = 0$ if $n \notin \{m(n_0), \ldots, m(n_0 + l)\}$ and

$$z_{m(n_0+r)} = \frac{\lambda_{k_r}}{(n_0 + r + 1)^{\frac{1}{p_{m(n_0+r)}}}}$$

for each $r \in \{0, \ldots, l\}$. Therefore from

$$\| y_l \| = \sum_{r=0}^{l} |z_{m(n_0+r)}|^{p_{m(n_0+r)}} = \sum_{r=0}^{l} \frac{\lambda_{k_r}^{p_{m(n_0+r)}}}{(n_0 + r + 1)} > \frac{1}{2} \sum_{r=n_0}^{n_0+l} \frac{1}{r+1}$$

for all $l \in \mathbb{N}$ follows that

$$\lim_{l \to \infty} \| y_l \| \geq \frac{1}{2} \sum_{r=n_0}^{\infty} \frac{1}{r+1} = \infty.$$

Thus $G(X) = l^0$.

Corollary 6. *There are topological algebras and topological vector spaces which are not galbed.*

By Corollary 6, the class of all galbed algebras (galbed spaces) does not coincide with the class of all topological algebras (respectively topological vector spaces). On the other hand, Corollary 2 gives us another “extreme” example, where $G(A) = l^1$ is the largest possible.

Remark. Notice that Lemma 1 for sequences (α_n) and (β_n) with $0 \leq \beta_n \leq \alpha_n$ for each $n > 0$; Corollary 2 for locally convex F-spaces; Lemma 4 and Propositions 1 - 4 for metric vector spaces with different proofs and Proposition 5 with different proof could be found in [13], pp. 158 - 165. Moreover, Propositions 1 - 4 and Corollaries 2 and 5 without proofs could be also found in [18], pp. $51 - 58$ and [14], pp. $458 - 459$.

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