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Relative numerical ranges in lmc algebras and pseudoisometries

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Abstract

In this paper we extend on lmc algebras without unit, the notion of the numerical range. We prove that some basic properties of the classical numerical range (which is defining on unital algebras) are valid and for the *relative numerical range* on algebras without unit. Moreover, we extend and strengthen on lmc algebras results relatively with the relative numerical range on Banach algebras (without unit) the definition of which has given by A.K. Gaur and T. Husain in their paper "relative numerical ranges", Math. Japonica 36(1991), 127-135.

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1. Introduction

By a *locally m-convex (: lmc) algebra* $(E, \Gamma \equiv \{p_\alpha\}_{\alpha \in I})$ we mean a locally convex space E, the topology of which is defined by an upper directed family of submultiplicative seminorms $\Gamma \equiv \{p_\alpha\}_{\alpha \in I}$ (see [1]).

Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha \in I})$ be an lmc algebra with a unit 1_E and let E' be the topological dual space of E. We consider the following sets,

$$(\mathcal{U}_\alpha(1))^0 = \{f \in E' : |f(x)| \leq 1,\ x \in \mathcal{U}_\alpha(1)\} \tag{1}$$

$$\begin{aligned} D_\alpha(E, p_\alpha, 1_E) &= \{f \in (\mathcal{U}_\alpha(1))^0 : f(1_E) = 1\}, \\ D(E, 1_E) &= \bigcup_{\alpha \in I} D_\alpha(E, p_\alpha, 1_E), \end{aligned} \tag{2}$$

where $\mathcal{U}_\alpha(1) = \{x \in E : p_\alpha(x) \leq 1\}$, $\alpha \in I$. The set $D(E, 1_E)$ is called the set of *normalized states* of E.
On the other hand, we call *numerical range* of an element $a \in E$, the set

$$V(E, a) := \hat{a}(D(E, 1_E)) \tag{3}$$

where $\hat{a}$ is the generalized Gel'fand transform of the a (cf. [2]).

Clearly, the definition of $V(E,a)$ is dependent on the identity 1_E of E. There are many lmc algebras which do not possess an identity. It is therefore of some interest, to make the notion of numerical range identity-free. This is precisely what we wish to do in this paper. For a similar approach on Banach algebra, see [3], [4] and [5].
Now, let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha\in I})$ be an lmc algebra (not necessary with a unit). Let b be a fixed element of E. We introduce (Definition 2.1) the notion of relative numerical range $V_b(E,a)$ of an element $a \in E$ relative to b.
If the lmc algebra E has a unit 1_E and $b = 1_E$, $p_\alpha(1_E) = 1$, $\alpha \in I$, then $V_b(E,a)$ coinside with $V(E,a)$.
Among the results it is show that for $a, b \in E$ the set $V_b(E,a)$ is bounded if and only if $\sup_{\alpha\in I_\alpha} p_\alpha(ab) < +\infty$ and when $p_\alpha(b) = 1$, $\alpha \in I$ then $V_b(E,a)$ is moreover convex (Proposition 2.2).
Furthermore, if $ab = b$ then $V_b(E,a) = \{1\}$ (Proposition 2.4(i)). A partial converse of this also holds (Proposition 2.4(ii)). On the other hand, we give on lmc algebras, the notion of pseudoisometry and we prove that the elements $a \in E$ and $\phi(a) \in F$, where $\phi : E \to F$ is a pseudoisometric algebraic omomorphism between lmc algebras E and F, have the same relative numerical range, relative b and $\phi(b)$ respectively (Theorem 3.1).

Throughout this paper all algebras are complex and the underlying topological spaces are always assumed to be Hausdorff.

2. Relative numerical ranges

Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha\in I})$ be an lmc algebra (not necessary with a unit), the topology of which is defined by an upper directed family of submultiplicative semi-norms $\Gamma \equiv \{p_\alpha\}_{\alpha\in I}$. Let $b \in E$. We consider the following sets.

$$\begin{aligned} D_\alpha(E, p_\alpha, b) &= \{f \in (\mathcal{U}_\alpha(1))^0 : f(b) = p_\alpha(b)\} \\ &= (\mathcal{U}_\alpha(1))^0 \cap \hat{b}^{-1}(\{p_\alpha(b)\}) \end{aligned} \tag{4}$$

$$D(E,b) \equiv D(E, \{p_\alpha\}_{\alpha\in I}, b) = \bigcup_{\alpha\in I} D_\alpha(E, p_\alpha, b). \tag{5}$$

where $(\mathcal{U}_\alpha(1))^0$ is the set (1) and $\hat{b}$ the generalized Gel'fand transform of the element b.

The elements of the set $D(E,b)$ are called *the relative states of E at b*.
Now, we call *relative numerical range* of an element $a \in E$ relative to b, the set

$$V_b(E,a) := \widehat{ab}(D(E,b)) = \{f(ab) : f \in D(E,b)\}. \tag{6}$$

On the other hand, the number

$$v_b(E,a) \equiv v_b(a) := \sup_{\alpha\in I}\{|\lambda| : \lambda \in V_b(E,a)\} \tag{7}$$

is called *relative numerical radius* of a, relative to b.
Now, for the above lmc algebra E, let $(E_\alpha)_{\alpha\in I}$ be the family of normed algebras in the Arens-Michael decomposition of E (cf. [6; p. 91]) and let the sets

$$D(E_\alpha, a_\alpha) \equiv D(E_\alpha, \|\cdot\|_\alpha, b_\alpha) = \{F \in E'_\alpha : \|F\|_\alpha = 1, F(b_\alpha) = \|b_\alpha\|_\alpha \equiv p_\alpha(b)\} \quad (8)$$

where $E_\alpha = E/\mathcal{N}_\alpha$, $\mathcal{N}_\alpha = \ker p_\alpha$, $\alpha \in I$.
For an $f \in D_\alpha(E, p_\alpha, b)$ we have $|f(x)| \leq p_\alpha(x)$, $x \in E$ and $f(b) = p_\alpha(b)$. So $\mathcal{N}_\alpha \subseteq \ker f$ from which implies that the linear form

$$F : E_\alpha \longrightarrow \mathbb{C} : x_\alpha \equiv x + \mathcal{N}_\alpha \longmapsto F(x_\alpha) := f(x) \quad (9)$$

is well defined $(: x + \mathcal{N}_\alpha = y + \mathcal{N}_\alpha \Leftrightarrow x - y \in \mathcal{N}_\alpha \Rightarrow f(x) = f(y))$.
Moreover, $F(b_\alpha) = p_\alpha(b) \equiv \|b_\alpha\|_\alpha$ and $|F(x_\alpha)| \leq \|x_\alpha\|_\alpha$, $x_\alpha \in E_\alpha$ so that $\|F\|_\alpha = 1$. Hence $F \in D(E_\alpha, \|\cdot\|_\alpha, b_\alpha)$.
On the other hand, for an $F \in D(E_\alpha, \|\cdot\|_\alpha, b_\alpha)$ there exists

$$f : E \longrightarrow \mathbb{C} : x \longmapsto f(x) := F(x_\alpha) \quad (10)$$

such that $|f(x/p_\alpha(x))| = \left|F(x_\alpha/\|x_\alpha\|_\alpha)\right| \leq \|F\|_\alpha \cdot \left\|x_\alpha/\|x_\alpha\|_\alpha\right\|_\alpha = 1$ and $|f(b)| = p_\alpha(b)$, so that $f \in D(E, p_\alpha, b)$.

Hence, the sets $D_\alpha(E, p_\alpha, b)$ and $D(E_\alpha, \|\cdot\|_\alpha, b_\alpha)$ are isomorphic for every $\alpha \in I$, so that, by (5) and (6),

$$V_b(E, a) = \bigcup_{\alpha\in I} V_{b_\alpha}(E_\alpha, a_\alpha) \quad (11)$$

for every $a \in E$.

On the other hand, if $\tilde{E}_\alpha$, $\alpha \in I$ are the Banach algebras, complesions of E_α, $\alpha \in I$, respectively, then

$$V_{b_\alpha}(\tilde{E}_\alpha, a_\alpha) = V_{b_\alpha}(E_\alpha, a_\alpha). \quad (12)$$

In fact, for $F \in D(\tilde{E}_\alpha, a_\alpha)$ we have $F_1 \equiv F|_{E_\alpha} \in D(E_\alpha, a_\alpha)$ and for $F_1 \in D(E_\alpha, a_\alpha)$ the Hahn-Banach theorem guaranties that there exists an $F \in D(\tilde{E}_\alpha, a_\alpha)$ with $F|_{E_\alpha} = F_1 \in D(E_\alpha, a_\alpha)$. So, by the relation (6) we take the relation (12).
Now, we have the following proposition, the proof of which is easy and therefore omitted.

Proposition 2.1 *Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha\in I})$ be an lmc algebra and $a, b, x \in E$. Then,*

(i) $V_x(E, a + b) \subseteq V_x(E, a) + V_x(F, b)$

(ii) $V_x(E, \lambda a) = \lambda \cdot V_x(E, a)$

(iii) $v_x(a + b) \leq v_x(a) + v_x(b)$

(iv) $v_x(\lambda a) \leq |\lambda| \cdot v_x(a)$

Proposition 2.2 *Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha \in I})$ be an lmc algebra and $a, b \in E$. Then, we have the following:*

(i) *For each $\alpha \in I$ the set $D_\alpha(E, p_\alpha, b)$ is compact.*

(ii) *The set $V_b(E, a)$ is bounded if and only if $\sup\limits_{\alpha \in I} p_\alpha(ab) < +\infty$. Moreover, if the family $\Gamma \equiv \{p_\alpha\}_{\alpha \in I}$ is finite, then $V_b(E, a)$ is a compact subset of $\mathbb{C}$.*

(iii) *If $p_\alpha(b) = 1$, $\alpha \in I$, then the set $V_b(E, a)$ is convex.*

Proof. (i) The set $D_\alpha(E, p_\alpha, b) \subseteq E'$ is closed, since the sets $(\mathcal{U}_\alpha(1))^0$ and $\hat{b}^{-1}(\{p_\alpha(b)\})$ are such. Moreover, the set $D_\alpha(E, p_\alpha, b)$ is equicontinuous since it is contained in the polar of a neighborhood of $o \in E$. So by the Alaoglu-Burbaki theorem it is relatively compact. Hence $D_\alpha(E, p_\alpha, b)$ $(\alpha \in I)$ is a compact subset of E'.

(ii) Let $(\tilde{E}_\alpha)_{\alpha \in I}$ be the family of Banach algebras in the Arens-Michael decomposition of E. For the Banach algebra $\tilde{E}_\alpha$ $(\alpha \in I)$ we have

$$o \leq v_{b_\alpha}(\tilde{E}_\alpha, a_\alpha) \leq p_\alpha(ab) = \|a_\alpha b_\alpha\|_\alpha$$

where $a_\alpha = a + \mathcal{N}_\alpha$, $b_\alpha = b + \mathcal{N}_\alpha$ (cf. [3; Lemma 2.4(ii)]). So, taking the suprema we have

$$o \leq \sup_{\alpha \in I} v_{b_\alpha}(\tilde{E}_\alpha, a_\alpha) \leq \sup_{\alpha \in I} p_\alpha(ab)$$

and by the relations (11) and (12) we get

$$o \leq v_b(E, a) \leq \sup_{\alpha \in I} p_\alpha(ab),$$

from which implies that $V_b(E, a)$ is bounded if and only if $\sup\limits_{\alpha \in I} p_\alpha(ab) < +\infty$. On the other hand, if the family of seminorms $\Gamma \equiv \{p_\alpha\}_{\alpha \in I}$ is finite, then the seminorm $r(x) = \max\limits_{\alpha \in I} p_\alpha(x)$, $x \in E$, defines on E the same topology as that the family Γ and $D(E, b) = D_r(E, p_r, b)$. Since by (i) $D_r(E, p_r, b)$ is compact, one has that $V_b(E, a)$ has the same property.

(iii) For $f, g \in D(E, b)$ there are $\alpha, \beta \in I$ such that $|f(x)| \leq p_\alpha(x)$ and $|g(x)| \leq p_\beta(x)$, $x \in E$.

Since the family of seminorms $\Gamma \equiv \{p_\alpha\}_{\alpha \in I}$ is upper directed, there is a $\gamma \in I$ such that $f, g \in D_\gamma(E, p_\gamma, b) \subseteq D(E, b)$. Now, if h is a convex combination of f and g, then by the convexity of $D_\gamma(E, p_\gamma, b)$ we have $h \in D_\gamma(E, p_\gamma, b) \subseteq D(E, b)$, i.e. $D(E, b) \subseteq E'$ is a convex subset and by (6), $V_b(E, a) \subseteq \mathbb{C}$ is also a convex subset. □

A. Gaur and T. Husain have proved (see [3; Lemma 2.4]) that, in case of Banach algebras the relative numerical range is always bounded. In the following Corollary 2.3, we prove that this is moreover closed. In point of fact we prove that the relative numerical range is compact. That is we have the following.

Corollary 2.3 *Let A be a Banach algebra and $b \in A$. Then,*

(i) *The set $D(A,b)$ is compact.*

(ii) *For each $a \in A$ the set $V_b(A,a)$ is compact, therefore this is also closed.*

(iii) *If $\|b\| = 1$ then $V_b(A,a)$ is a (compact and) convex subset of $\mathbb{C}$.*

Proposition 2.4 *Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha\in I})$ be an lmc algebra and $a, b \in E$. Moreover, let that $p_\alpha(b) = 1$, $\alpha \in I$.*

(i) *If $ab = b$ then $V_b(E,a) = \{1\}$.*

(ii) *If $V_b(E,a) = \{1\}$ then, either $ab = b$ or $0 \leq \inf_{\lambda\in\mathbb{C}} p_\alpha(b - \lambda ab) \leq 1$ for every $\alpha \in I$.*

Proof. (i) It follows from the definition.

(ii) Let $V_b(E,a) = \{1\}$. If $b = \lambda ab$ for some $\lambda \in \mathbb{C}$, then for any $f \in D(E,b)$ we have $f(b) = \lambda \cdot f(ab) = \lambda$. On the other hand, for $f \in D(E,b)$ there is $\alpha \in I$ with $f \in D_\alpha(E, p_\alpha, b)$ so that $f(b) = p_\alpha(b) = 1$. Hence $b = ab$.

Now, let that $b \notin \mathbb{C}ab$ and let $(E_\alpha)_{\alpha\in I}$ and $(\tilde{E}_\alpha)_{\alpha\in I}$ be the families of normed and Banach algebras respectively, in the Arens-Michael decomposition of E. Then, the set $D_\alpha(E, p_\alpha, b)$ is isomorphic to $D(E_\alpha, b_\alpha)$, $b_\alpha = b + \ker p_\alpha$, $\alpha \in I$, (see comments before of the Proposition 2.1).

On the other hand, for $f \in D(E,b)$ we have $f_\alpha \in D(E_\alpha, b_\alpha)$, $\alpha \in I$ (: $f_\alpha : E_\alpha \to \mathbb{C} : x_\alpha \mapsto f_\alpha(x_\alpha) := f(x)$), so that $f_\alpha(a_\alpha b_\alpha) = f_\alpha((ab)_\alpha) = f(ab) = 1$. Hence (see relation (12)) $V_{b_\alpha}(\tilde{E}_\alpha, a_\alpha) = V_{b_\alpha}(E_\alpha, a_\alpha) = \{1\}$ for every $\alpha \in I$. So, by [3; Theorem 2.5 (ii)] we take what we wanted to prove. □

3. Pseudoisometries and relative numerical ranges

Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha\in I})$, $(F, \Gamma' \equiv \{q_k\}_{k\in J})$ be two lmc algebras, b an element of E and $\phi : E \to F$ an algebraic homomorphism.

We say that ϕ is a *pseudoisometry at* b if for any $k \in J$ there exists $\alpha \in I$ with $q_k(\phi(x)) \leq p_\alpha(x)$, $x \in E$ and $q_k(\phi(b)) = p_\alpha(b)$ and for any $\beta \in I$ there exists $l \in J$ with $p_\beta(x) \leq q_l(\phi(x))$, $x \in E$ and $p_\beta(b) = q_l(\phi(b))$.

In this respect we have the following.

Theorem 3.1 *Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha\in I})$, $(F, \Gamma' \equiv \{q_k\}_{k\in J})$ be two lmc algebras and $b \in E$. Moreover, let $\phi : E \to F$ be a pseudoisometric algebraic homomorphism at b. Then,*

$$V_{\phi(b)}(F, \phi(a)) = V_b(E, a) \tag{13}$$

Proof. Let $\lambda \in V_{\phi(b)}(F, \phi(a))$. Then, there is $k \in J$ and $g \in (\mathcal{U}_k(1))^0 \subseteq F'$ with $g(\phi(b)) = q_k(\phi(b))$, such that $\lambda = g(\phi(a)\phi(b))$. Define f on E by $f(x) = g(\phi(x))$, $x \in E$. Clearly f is linear and

$$|f(x)| = |g(\phi(x)) \leq q_k(\phi(x))| \leq p_\alpha(x), \quad x \in E$$

for some $\alpha \in I$ such that $p_\alpha(b) = q_k(\phi(b))$. Moreover $f(b) = p_\alpha(b)$ so that $f \in D_\alpha(E, p_\alpha, b) \subseteq D(E, b)$. Hence $\lambda = f(ab) \in V_b(E, a)$.

Conversely, let $\lambda \in V_b(E, a)$. Then, there is $\beta \in I$ and $f \in (\mathcal{U}_\beta(1))^0 \subseteq E'$ with $f(b) = p_\beta(b)$, such that $\lambda = f(ab)$. Define h on $\phi(E) \leq F$ by

$$h(\phi(x)) := f(x), \; x \in E.$$

Then, h is well defined. In fact, let $\phi(x) = \phi(y)$, $x, y \in E$. By hypothesis for ϕ there exists $l \in J$ with $q_l(\phi(b)) = p_\beta(b)$ such, that

$$|f(x)| \leq p_\beta(x) \leq q_l(\phi(x)), \; x \in E.$$

So, $|f(x) - f(y)| \leq q_l(\phi(x - y)) = 0$ from which implies that $f(x) = f(y)$. On the other hand, since $|h(\phi(x)) \leq q_l(\phi(x))$, $x \in E$, Hahn-Banach theorem guaranties that, there is a linear form g on E with $g|_{\phi(E)} = h$ and $|g(y)| \leq q_l(y)$, $y \in F$. Hence $g \in (\mathcal{U}_l(1))^0 \subseteq F'$ and $g(\phi(b)) = q_l(\phi(b))$, so that $\lambda = g(\phi(ab)) \in V_{\phi(b)}(F, \phi(a))$. □

Corollary 3.2 *Let $(E, \Gamma \equiv \{p_\alpha\}_{\alpha\in I})$ be an lmc algebra, F a topological subalgebra of E and $b \in F$. Then, for every $a \in F$ we have*

$$V_b(F, a) = V_b(E, a). \; \blacksquare$$

Corollary 3.3 *Let E, F be two normed algebras and $\phi : E \to F$ an isometric algebraic homomorphism. Then, for every $a, b \in E$ we have*

$$V_{\phi(b)}(F, \phi(a)) = V_b(E, a). \; \blacksquare$$

The above Corollary 3.3 strengthen the Theorem 2.9 of [3].

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