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Sharp inequalities for Riesz, Bessel and Yukawa potential operators

Athanase Cotsiolis and Nikolaos K. Tavoularis

Abstract

We obtain best constants for inequalities with Riesz, Bessel and Yukawa potential operators. As an application, we give a theorem concerning conformally invariant higher-order differential operators on the sphere.

1. Introduction

Inequalities of Sobolev type have a wide range of applications and have been extensively studied (see for example [1], [2], [3], [4], [7], [10], [11], [12], [15], [16], [18], [20], [24]). Sometimes, it is also important to have precise estimates for the constants appearing in these inequalities. This has been the subject on many papers recently (cf. [2], [5], [6], [10], [13], [14], [19], [23], [25],[26] and the references therein).

More precisely given an integer $k \in \mathbb{N}$, the Sobolev space $H^k(\mathbb{R}^n)$ is defined as the space of those functions $f \in L^2$ satisfying $|\nabla^\ell f| \in L^2(\mathbb{R}^n)$, $1 \leq \ell \leq k$. The Sobolev imbedding theorem asserts that $H^k(\mathbb{R}^n) \subseteq L^q(\mathbb{R}^n)$ for $q = 2n/(n-2k)$ when $n > 2k$. For example, when $k = 1$, $n \geq 3$ and $q = \frac{2n}{n-2}$, we have the inequality

$$||f||^2_{\frac{2n}{n-2}} \leq C_n ||\nabla f||_2^2, \qquad f \in H^1(\mathbb{R}^n) \tag{1}$$

The best value for the constant C_n in the above inequality has been estimated to be (cf. [2], [23])

$$C_n = \pi^{-1} n^{-1} (n-2)^{-1} \left[\frac{\Gamma(n)}{\Gamma(n/2)} \right]^{2/n} \tag{2}$$

where $\Gamma(t)$ is the Gamma function.

Also using the formula $\frac{\Gamma(n)}{\Gamma(n/2)} = \frac{2^{n-1}}{\pi^{1/2}} \Gamma\left(\frac{n+1}{2}\right)$ we have

$$C_n = \frac{4}{n(n-2)} \omega_n^{-2/n} = 2^{-2/n} \pi^{-(n+1)/n} \frac{4}{n(n-2)} \left[\Gamma\left(\frac{n+1}{2}\right) \right]^{2/n}$$

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where $\omega_n = 2\pi^{(n+1)/2}/\Gamma\left(\frac{n+1}{2}\right)$ is the volume of $\mathbb{S}_n$ which is the sphere of radius 1 in $\mathbb{R}^{n+1}$.

We have equality in (1) if and only if

$$f(x) = A(\mu^2 + (x - x_0)^2)^{-(n-2)/2}, \qquad x \in \mathbb{R}^n$$

where $A \in \mathbb{R}$, $\mu > 0$ are fixed constants and $x_0 \in \mathbb{R}^n$.

Also a generalization of (1) is the following inequality (cf. [2], [23])

$$||f||_q \leq C_{n,p}||\nabla f||_p, \qquad f \in C_0^\infty(\mathbb{R}^n) \tag{3}$$

where let $1 \leq p < n$ and $1/q = 1/p - 1/n$ and the best value for the constant $C_{n,p}$ in the above inequality has been estimated to be (cf. [2])

$$C_{n,p} = \frac{1}{n}\left(\frac{n(p-1)}{n-p}\right)^{1-1/p}\left(\frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{p}\right)\Gamma\left(n+1-\frac{n}{p}\right)\omega_{n-1}}\right)^{1/n} \tag{4}$$

when $p > 1$, and ω_{n-1} is the volume of the standard unit sphere of $\mathbb{R}^n$.

We have equality in (3) if and only if

$$f(x) = A(\mu^2 + |x - x_0|^{p/(p-1)})^{(p-n)/p}, \qquad x \in \mathbb{R}^n$$

A, μ and x_0 as above.

For $x = (x_1, ..., x_n), k = (k_1, ..., k_n) \in \mathbb{R}^n$, we set $(k, x) = k_1x_1 + ... + x_nk_n$ and $|x| = (x, x)^{1/2}$. The gradient ∇f of a differentiable function f is $\nabla f = \left(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}\right)$.

If $f, g \in L^2(\mathbb{R}^n)$, then we set $< f, g >= \int_{\mathbb{R}^n} f(x)g(x)dx$.

Recall that the Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\widehat{f}(k) = \int_{\mathbb{R}^n} e^{-2\pi i(k,x)}f(x)dx$$

We set $(f)^\wedge(-x) = \widehat{f}(-x) = f^\vee(x)$. So $f = (\widehat{f})^\vee$.

Note that $\widehat{(-\Delta)f}(k) = (2\pi|k|)^2\widehat{f}(k)$ where Δ is the Laplacian in $\mathbb{R}^n$ $\Delta = (\partial^2/\partial^2 x_1) + ... + (\partial^2/\partial^2 x_n)$.

The operators $(-\Delta)^{s/2}$ were defined in ([21]) by

$$(-\Delta)^{s/2}f(x) = \left((2\pi|\xi|)^s\widehat{f}(\xi)\right)^\vee(x)$$

Also $H^{s,p}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm $\|f\|_{H^{s,p}(\mathbb{R}^n)} = ||f||_p + \|(-\Delta)^{s/2}f\|_p$.

Note that $\|\nabla f\|_2 = \|(-\Delta)^{1/2}f\|_2$ and that $\|\nabla f\|_p \neq \|(-\Delta)^{1/2}f\|_p$. By corollary 1 §7.1 in [18], for $\ell = 1, 2, ...,$ there exist positive number c and C that depend only on n, ℓ such that $c||(-\Delta)^{\ell/2}f||_p \leq ||\nabla^\ell f||_p \leq C||(-\Delta)^{\ell/2}f||_p$. (see remark in the end of the theorem 2.1).

The operators $(-\Delta)^{-s/2}$, $0 < s < n$, are called Riesz potential operators (c.f. [21]) and we have $(-\Delta)^{-s/2}(f) = I_s * f$, where I_s is the Riesz potential

$$I_s(x) = \frac{1}{c_s}|x|^{-n+s}, \quad \text{where} \quad c_s = \frac{\pi^{n/2}2^s\Gamma(s/2)}{\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)}$$

We have shown (c.f. [8] and [9]) the existence of a sharp constant for Sobolev inequalities with higher fractional derivatives. For $q = \frac{2n}{n-2s}$ any function $f \in H^{s,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$ satisfies

$$||f||_q^2 \leq S(n,s)||(-\Delta)^{s/2}f||_2^2 \tag{5}$$

where

$$S(n,s) = 2^{-2s}\pi^{-s}\frac{\Gamma\left(\frac{n-2s}{2}\right)}{\Gamma\left(\frac{n+2s}{2}\right)}\left[\frac{\Gamma(n)}{\Gamma(n/2)}\right]^{2s/n}. \tag{6}$$

There is equality in inequality (5) if and only if

$$f(x) = A\left(\mu^2 + (x-x_0)^2\right)^{-\frac{n-2s}{2}}$$

with $A \in \mathbb{R}$, $\mu > 0$ and $x_0 \in \mathbb{R}^n$.

Let us recall that the operators $(I-\Delta)^{s/2}$ have been defined by (cf. [21])

$$\widehat{(I-\Delta)^{s/2}f}(\xi) = (1 + (2\pi|\xi|)^2)^{s/2}\widehat{f}(\xi).$$

The operators $(I-\Delta)^{-s/2}$, for $s > 0$, are called Bessel potential operators (cf. [11]) and they are given by convolution with the Bessel potential

$$G_s(x) = \frac{1}{\alpha(s)}\int_0^\infty e^{-|x|^2/4\delta}e^{-\delta}\delta^{(-n+s)/2}\frac{d\delta}{\delta},$$

where $\alpha(s) = \Gamma(s/2)(4\pi)^{n/2}$.

Also $H^s(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm

$$\|f\|_{H^s(\mathbb{R}^n)} = \|(I-\Delta)^{s/2}f\|_2.$$

It is well known that $H^s(\mathbb{R}^n)$ is continuously imbedded into $L^q(\mathbb{R}^n)$ if $0 \leq s < n/2$, $2 \leq q \leq 2n/(n-2s)$ or $n = 2s$, $2 \leq q < \infty$ or $n < 2s$, $2 \leq q \leq \infty$.

We next remind the Green's function for $-\Delta + \mu^2$ with $\mu > 0$. It is well known (c.f. [15]) that for each $n \geq 1$ and $\mu > 0$ there is a function G_y^μ that satisfies

$$(-\Delta + \mu^2)G_y^\mu = \delta_y \tag{7}$$

and is given by

$$G_y^\mu(x) = G^\mu(x-y), \tag{8}$$

$$G^\mu(x) = \int_0^\infty (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t} - \mu^2 t} dt. \tag{9}$$

The function G_y^μ is called the Yukawa potential, at least for $n = 3$, and played an important role in the theory of elementary particles (mesons), for which H. Yukawa won a Nobel prize.

The Fourier transform of G^μ is ([15])

$$\widehat{G^\mu}(k) = \left((2\pi|k|)^2 + \mu^2\right)^{-1}. \tag{10}$$

2. Main Results

Theorem 2.1 *Let $n > 2s > 0$ and $p = \frac{2n}{n+2s}$. Then*

$$||f||_2^2 \leq S(n,s)||(-\Delta)^{s/2} f||_p^2 \ , f \in H^{s,p}(\mathbb{R}^n) \tag{11}$$

where

$$S(n,s) = 2^{-2s}\pi^{-s} \frac{\Gamma\left(\frac{n-2s}{2}\right)}{\Gamma\left(\frac{n+2s}{2}\right)} \left[\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\right]^{\frac{2s}{n}}. \tag{12}$$

We have equality in (11) if and only if $f = I_s * F$ with $F(x) = A(\mu^2 + (x - x_0)^2)^{-(n+2s)/2}$ where $A \in \mathbb{R}$, $\mu > 0$ and $x_0 \in \mathbb{R}^n$ are fixed constants.

Proof. We have $(-\Delta)^{s/2} f \in L^p(\mathbb{R}^n)$ and therefore $\widehat{(-\Delta)^{s/2}} f$ exists by the Hausdorff - Young inequality (Theorem 5.7, [15]).

Also the function

$$g := c'_{n-s}|x|^{s-n} * (-\Delta)^{s/2} f$$

where $c'_\alpha := \pi^{-\alpha/2}\Gamma(\alpha/2)$, $\alpha > 0$, is a $L^p(\mathbb{R}^n)$ - function by the Hardy - Littlewood - Sobolev inequality (Theorem 4.3, [15]).

We have

$$\widehat{g}(k) = c'_s|k|^{-s}\widehat{(-\Delta)^{s/2}} f(k) = c'_s(2\pi)^s \widehat{f}(k)$$

Moreover (Corollary 5.10, [15]),

$$c'_{2s}\int_{\mathbb{R}^n} |k|^{-2s}\left|\widehat{(-\Delta)^{s/2}} f(k)\right|^2 dk =$$

$$c'_{n-2s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \overline{(-\Delta)^{s/2} f(x)}(-\Delta)^{s/2} f(y)|x-y|^{2s-n} dxdy$$

By the Hardy - Littlewood - Sobolev inequality we have

$$\begin{aligned} ||f||_2^2 \quad &\leq \quad \frac{c'_{n-2s}}{c'_{2s}(2\pi)^{2s}}\pi^{(n-2s)/2} \\ &\times \frac{\Gamma\left(\frac{n}{2} - \frac{n-2s}{2}\right)}{\Gamma\left(n - \frac{n-2s}{2}\right)}\left[\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)}\right]^{-1+\frac{n-2s}{n}} ||(-\Delta)^{s/2} f||_p^2 \end{aligned} \tag{13}$$

and therefore the inequality (11) follows.

Finally, let us observe that in order to have equality in (11) we must have equality in (13) and as it is well known by the Hardy-Littlewood-Sobolev inequality (Theorem 4.3, [15]), this happens if and only if $(-\Delta)^{s/2}f = A(\mu^2 + (x - x_0)^2)^{-(n+2s)/2}$ i.e. $f = I_s * F$ for fixed constants $A \in \mathbb{R}$, $\mu > 0$ and $x_0 \in \mathbb{R}^n$.

Remarks (i) We have shown in [9] that $||\nabla^\ell f||_2 = C||(-\Delta)^{\ell/2}f||_2$ ($\ell \in \mathbb{N}$), with

$$C = 2^{-\ell} \prod_{h=-\ell}^{\ell-1} (n + 2h)^{1/2} \left[\frac{\Gamma\left(\frac{n-2\ell}{2}\right)}{\Gamma\left(\frac{n+2\ell}{2}\right)} \right]^{1/2}.$$

(ii) For $p = \frac{2n}{n+2}$ and $\ell = 1$ we can give the constant C such that

$$||\nabla f||_p^2 = C||(-\Delta)^{1/2} f||_p^2$$

We have that $||f||_2^2 \leq C_{n,p}^2 ||\nabla f||_p^2$ for $p = \frac{2n}{n+2}$ and

$$C_{n,2n/(n+2)}^2 = (n-2)^{(n-2)/n} n^{(2-3n)/n} \left(\frac{\Gamma(n+1)}{\Gamma\left(\frac{n+2}{2}\right)\Gamma\left(\frac{n}{2}\right)\omega_{n-1}} \right)^{2/n}$$

is given by (4). Combining this inequality with the fact that $c||(-\Delta)^{1/2}f|||_p^2 \leq ||\nabla f||_p^2 \leq C||(-\Delta)^{1/2}f||_p^2$ (corollary 1 §7.1 in [18]), we have that

$$||f||_2^2 \leq C_{n,p}^2 ||\nabla f||_p^2 \leq CC_{n,p}^2 ||(-\Delta)^{1/2} f||_p^2$$

Also $||f||_2^2 \leq S(n,1)||(-\Delta)^{1/2}f||_p^2 \leq S(n,1)\frac{1}{c}||\nabla f||_p^2$. However, $S(n,1)$ and $C_{n,p}^2$ are the best constants for the inequalities $||f||_2^2 \leq S(n,1)||(-\Delta)^{1/2}f||_p^2$ and $||f||_2^2 \leq C_{n,p}^2||\nabla f||_p^2$ respectively. Therefore we have that

$$c = C = \frac{S(n,1)}{C_{n,p}^2} = 4^{(n-1)/n} \left(\frac{n}{n-2} \right)^{(2n-2)/n} \left(\frac{\omega_{n-1}}{\omega_n} \frac{\left(\Gamma\left(\frac{n}{2}\right)\right)^2}{\Gamma(n)} \right)^{2/n}.$$

Theorem 2.2 *Let $n > s > 0$ and $p = \frac{2n}{n+s}$. Then*

$$||f||_q \leq K(n,s)||(-\Delta)^{s/2} f||_p \ , f \in H^{s,p}(\mathbb{R}^n) \tag{14}$$

where $q = p' = \frac{2n}{n-s}$ and

$$K(n,s) = (4\pi)^{-s/2} \frac{\Gamma\left(\frac{n-s}{2}\right)}{\Gamma\left(\frac{n+s}{2}\right)} \left[\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)} \right]^{s/n} \tag{15}$$

Proof. Let $(-\Delta)^{s/2}f = g$ i.e.

$$f = (-\Delta)^{-s/2}g \tag{16}$$

We shall prove that $||(-\Delta)^{-s/2}g||_q \le K(n,s)||g||_p$. Indeed, it follows from Hardy - Littlewood - Sobolev inequality (Theorem 4.3, [15])

$$\begin{aligned} ||(-\Delta)^{-s/2}g||_q &= ||\frac{1}{c_s}|x|^{s-n} * g||_q \\ &\le \frac{1}{c_s}\frac{1}{r'}\left(\frac{n}{\omega_{n-1}}\right)^{1/r} C(n, n/r, p)|||x|^{s-n}||_{r,w}||g||_p \end{aligned} \tag{17}$$

where $\frac{1}{p}+\frac{1}{r} = 1+\frac{1}{q}$, i.e $r = \frac{n}{n-s}$ and hence $r' = \frac{n}{s}$. Also q' is the dual index of q i.e. $q' = p = \frac{2n}{n+s}$. We have

$$|||x|^{s-n}||_{r,w} = \frac{n}{s}\left[\frac{\omega_{n-1}}{n}\right]^{(n-s)/n} \tag{18}$$

and combining (17) and (18) we have

$$||(-\Delta)^{-s/2}g||_q \le \frac{\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)}{\pi^{n/2}2^s\Gamma(s/2)}C(n, n-s, \frac{2n}{n+s})||g||_p$$

i.e.

$$||(-\Delta)^{-s/2}g||_q \le \frac{\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)}{\pi^{n/2}2^s\Gamma(s/2)}\pi^{(n-s)/2}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{n+s}{2}\right)}\left[\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)}\right]^{-s/n}||g||_p$$

and it follows from (16)

$$||f||_q \le K(n,s)||(-\Delta)^{s/2}f||_p.$$

Theorem 2.3 *For $1 < p < \infty$, let $n > ps > 0$ and $q = \frac{np}{n-ps}$. Then*

$$||f||_q \le S(n,p,s)||(-\Delta)^{s/2}f||_p \ , f \in H^{s,p}(\mathbb{R}^n) \tag{19}$$

where the sharp constant $S(n,p,s)$ satisfies

$$\begin{aligned} S(n,p,s) &\le \frac{\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)}{\pi^{n/2}2^s\Gamma(s/2)}\frac{n}{s}\left(\frac{\omega_{n-1}}{n}\right)^{(n-s)/n}\frac{n(p-1)+ps}{np^2} \\ &\times\left[\left(\frac{p(n-s)}{n-ps}\right)^{(n-s)/n} + \left(\frac{p(n-s)}{n(p-1)}\right)^{(n-s)/n}\right]. \end{aligned} \tag{20}$$

Proof. We have to prove that $||(-\Delta)^{-s/2}f||_q \leq V(n,p,s)||f||_p$. Indeed, it follows from Hardy - Littlewood - Sobolev inequality

$$\begin{aligned} ||(-\Delta)^{-s/2}f||_q &= ||\frac{1}{c_s}|x|^{s-n} * f||_q \\ &\leq \frac{1}{c_s}\frac{1}{r'}\left(\frac{n}{\omega_{n-1}}\right)^{1/r} C(n,n/r,p)|||x|^{s-n}||_{r,w}||f||_p \qquad (21) \end{aligned}$$

where $\frac{1}{p}+\frac{1}{r}=1+\frac{1}{q}$, i.e $r=\frac{n}{n-s}$ and hence $r'=\frac{n}{s}$. Also q' is the dual index of q i.e. $q'=\frac{np}{np-n+ps}$. It follows from (18) and (21) that

$$||(-\Delta)^{-s/2}f||_q \leq \frac{1}{c_s}C(n,n-s,p)||f||_p. \qquad (22)$$

We can compute a upper bound for $C(n,n-s,p)$. We have,

$$\begin{aligned} C(n,n-s,p) &\leq \frac{n}{s}\left(\frac{\omega_{n-1}}{n}\right)^{(n-s)/n}\frac{n(p-1)+ps}{np^2} \\ &\times\left[\left(\frac{p(n-s)}{n(p-1)}\right)^{(n-s)/n}+\left(\frac{p(n-s)}{n-ps}\right)^{(n-s)/n}\right]. \qquad (23) \end{aligned}$$

Combining (22) and (23) the theorem follows.

Remark The constant in the right side of (20) is not the best for the inequality (19) when $q=\frac{np}{n-ps}$ $\left(p=\frac{qn}{n+qs}\right)$, but it is a upper bound for the best constant $S(n,p,s)$. For the case $p=\frac{2n}{n+2s}$ we have

$$\begin{aligned} S(n,\frac{2n}{n+2s},s) &\leq \frac{\Gamma\left(\frac{n-s}{2}\right)}{\pi^{n/2}2^s\Gamma\left(\frac{s}{2}\right)}\left(\frac{\omega_{n-1}}{n}\right)^{(n-s)/n}\frac{n+2s}{4s} \\ &\times\left[\left(\frac{2(n-s)}{n}\right)^{(n-s)/n}+\left(\frac{2(n-s)}{n-2s}\right)^{(n-s)/n}\right]. \end{aligned}$$

However, the best constants are estimated for this case ($p=\frac{2n}{n+2s}$) by the previous theorem 2.1 and for the case $p=q'$ $\left(p=\frac{2n}{n+s}, q=\frac{2n}{n-s}\right)$ by the theorem 2.2.

Lemma 2.1 *Let $f\in L^p(\mathbb{R}^n)$ with $p=2n/(n+s)$, then $\widehat{f}$ exists (by Hausdorff-Young's inequality (Theorem 5.7, [15])). The function $g:=G_{2s}*f$ is an $L^2(\mathbb{R}^n)$-function and hence has a Fourier transform $\widehat{g}$*

$$\widehat{g}(k)=\widehat{G_{2s}}(k)\widehat{f}(k)=(1+(2\pi|k|)^2)^{-s}\widehat{f}(k)$$

Also, we have

$$||G_{2s}*f||_2^2 \leq B(n,s)||f||_p^2 \qquad (24)$$

where the best constant $B(n,s)$ is

$$B(n,s) = \left(\frac{1}{(\Gamma(s))^2(4\pi)^{s/2}}\right)\frac{2^s n^{-s/2}s^{s/2}(n-s)^{(n-s)/2}}{(n+s)^{(n+s)/2}} \times \int_0^\infty\int_0^\infty e^{-(\delta+\varepsilon)}(\delta\varepsilon)^{s-1}(\delta+\varepsilon)^{-s/2}d\delta d\varepsilon \tag{25}$$

and the integral above converges for $0<s<1/2$.

Equality can occur in the inequality (24) above if and only if $f(x)$ is a multiple of $Exp[-\frac{2n}{n-s}(x-x_0,J(x-x_0))]$ where $x_0\in\mathbb{R}^n$ and J is any real, symmetric, positive-definite matrix.

Proof. We can find a sequence $f^1,f^2,...$ of functions in $C_0^\infty(\mathbb{R}^n)$ such that $f^j\to f$ strongly in $L^p(\mathbb{R}^n)$. The function g is in $L^2(\mathbb{R}^n)$.Indeed,

$$\begin{aligned}
||g||_2^2 &= \int_{\mathbb{R}^n}|g(x)|^2dx \\
&= \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\frac{1}{\Gamma(s)(4\pi)^{n/2}}f(y)\int_0^\infty e^{-\frac{|x-y|^2}{4\delta}}e^{-\delta}\delta^{-n/2+s-1}d\delta dy\right|^2dx \\
&= \left(\frac{1}{\Gamma(s)(4\pi)^{n/2}}\right)^2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left[\int_0^\infty e^{-\frac{|x-y|^2}{4\delta}}e^{-\delta}\delta^{-n/2+s-1}d\delta f(y)dy\right] \\
&\quad\times\int_{\mathbb{R}^n}\left[\int_0^\infty e^{-\frac{|x-z|^2}{4\varepsilon}}e^{-\varepsilon}\varepsilon^{-n/2+s-1}d\varepsilon\overline{f(z)}dz\right]dx \\
&= \left(\frac{1}{\Gamma(s)(4\pi)^{n/2}}\right)^2\int_0^\infty\int_0^\infty\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{-\delta}\delta^{-n/2+s-1}e^{-\varepsilon}\varepsilon^{-n/2+s-1} \\
&\quad\times\left[\int_{\mathbb{R}^n}e^{-\frac{|x-y|^2}{4\delta}}e^{-\frac{|x-z|^2}{4\varepsilon}}dx\right]f(y)\overline{f(z)}dydzd\delta d\varepsilon
\end{aligned}\tag{26}$$

We have

$$\int_{\mathbb{R}^n}e^{-\pi\frac{|x-y|^2}{c}}e^{-\pi\frac{|x-z|^2}{d}}dx=(cd)^{n/2}(c+d)^{-n/2}e^{-\pi\frac{|z-y|^2}{c+d}} \tag{27}$$

Indeed,

$$\begin{aligned}
\left(e^{-\pi\frac{|x|^2}{c}}*e^{-\pi\frac{|x|^2}{d}}\right)^\wedge(k) &= \left(e^{-\pi\frac{|x|^2}{c}}\right)^\wedge(k)\left(e^{-\pi\frac{|x|^2}{d}}\right)^\wedge(k) \\
&= c^{n/2}e^{-\pi c|k|^2}d^{n/2}e^{-\pi d|k|^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(e^{-\pi\frac{|x|^2}{c}}*e^{-\pi\frac{|x|^2}{d}}\right)(y) &= (cd)^{n/2}\left(e^{-\pi(c+d)|k|^2}d^{n/2}\right)^\vee(y) \\
&= (cd)^{n/2}\left(e^{-\pi(c+d)|k|^2}\right)^\wedge(-y) \\
&= (cd)^{n/2}(c+d)^{-n/2}e^{-\frac{\pi|y|^2}{c+d}}
\end{aligned}$$

i.e.

$$\left(e^{-\pi\frac{|x|^2}{c}} * e^{-\pi\frac{|x|^2}{d}}\right)(Y) = \int_{\mathbb{R}^n} e^{-\pi\frac{|Z|^2}{c}} e^{-\pi\frac{|Y-Z|^2}{d}} dZ = (cd)^{n/2}(c+d)^{-n/2} e^{-\pi\frac{|Y|^2}{c+d}}$$

If we set $Y = z - y$ and $Z = x - y$ then we will have (27).

Combining (26) and (27) (for $c = 4\pi\delta$ and $d = 4\pi\varepsilon$) we have

$$||g||_2^2 = \left(\frac{1}{(\Gamma(s))^2(4\pi)^{n/2}}\right)\int_0^\infty\int_0^\infty e^{-(\delta+\varepsilon)}(\delta\varepsilon)^{s-1}(\delta+\varepsilon)^{-n/2} \times \left[\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-|z-y|^2/4(\delta+\varepsilon)} f(y)\overline{f(z)}dydz\right] d\delta d\varepsilon \tag{28}$$

Applying the Young inequality (Th. 4.2, [15]) we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\pi\frac{|z-y|^2}{4\pi(\delta+\varepsilon)}} f(y)\overline{f(z)}dydz \le C_{p,q,p;n}||f||_p^2 \left|\left|e^{-\pi\frac{|x|^2}{4\pi(\delta+\varepsilon)}}\right|\right|_q \tag{29}$$

for $p = r = \frac{2n}{n+s}$ and $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 2$ i.e. $q = \frac{n}{n-s}$ and also $q' = n/s$, $p' = \frac{2n}{n-s}$

We can calculate

$$\left|\left|e^{-\pi\frac{|x|^2}{4\pi(\delta+\varepsilon)}}\right|\right|_q = (4\pi)^{(n-s)/2}(\delta+\varepsilon)^{(n-s)/2}\left(\frac{n-s}{n}\right)^{(n-s)/2} \tag{30}$$

and also

$$C_{p,q,p;n} = \frac{2^s n^{n/2} s^{s/2}}{(n+s)^{(n+s)/2}} \tag{31}$$

Combining (30), (31) and (29)

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\pi\frac{|z-y|^2}{4\pi(\delta+\varepsilon)}} f(y)\overline{f(z)}dydz \le \frac{2^s n^{n/2} s^{s/2}}{(n+s)^{(n+s)/2}}(4\pi)^{(n-s)/2}(\delta+\varepsilon)^{(n-s)/2} \times \left(\frac{n-s}{n}\right)^{(n-s)/2} ||f||_p^2 \tag{32}$$

Combining (28) and (32)

$$||g||_2^2 \le \left(\frac{1}{(\Gamma(s))^2(4\pi)^{s/2}}\right)\frac{2^s n^{n/2} s^{s/2}}{(n+s)^{(n+s)/2}}\left(\frac{n-s}{n}\right)^{(n-s)/2} \times \int_0^\infty\int_0^\infty e^{-(\delta+\varepsilon)}(\delta\varepsilon)^{s-1}(\delta+\varepsilon)^{-s/2} d\delta d\varepsilon ||f||_p^2 \tag{33}$$

and therefore $g \in L^2(\mathbb{R}^n)$.

Let

$$g^j := G_{2s} * f^j$$

Since $f^j \to f$, we have $\widehat{f^j} \to \widehat{f}$ in $L^{p'}(\mathbb{R}^n)$ (by the Hausdorff-Young inequality). By the Young inequality we have $\widehat{g^j} \to \widehat{g}$ in $L^2(\mathbb{R}^n)$.

Indeed,

$$||g^j - g||_2^2 \leq B(n,s)||f^j - f||_p^2 \to 0.$$

We have $\widehat{g^j}(k) = \left(1 + (2\pi|k|)^2\right)^{-s}\widehat{f^j}(k)$. Our problem is to show that $\widehat{g}(k) = \left(1 + (2\pi|k|)^2\right)^{-s}\widehat{f}(k)$.

To do this, we pass to a subsequence. Since $\widehat{g^j} \to \widehat{g}$ in $L^2(\mathbb{R}^n)$ and $\widehat{f^j} \to \widehat{f}$ in $L^p(\mathbb{R}^n)$ we have $\widehat{g^j}(k) \to \widehat{g}(k)$ and $\widehat{f^j}(k) \to \widehat{f}(k)$ pointwise a.e. (by the theorem of completeness of L^p-spaces). Thus,

$$\begin{aligned}\widehat{g}(k) &= \lim_{j\to\infty}\left(1 + (2\pi|k|)^2\right)^{-s}\widehat{f^j}(k) = \left(1 + (2\pi|k|)^2\right)^{-s}\lim_{j\to\infty}\widehat{f^j}(k)\\ &= \left(1 + (2\pi|k|)^2\right)^{-s}\widehat{f}(k)\end{aligned}$$

for almost every k.

By Plancherel's theorem (Theorem 5.3, [15]) we have

$$(F,G) = \int_{\mathbb{R}^n}\overline{F(x)}G(x)dx = \int_{\mathbb{R}^n}\overline{\widehat{F}(k)}\widehat{G}(k)dk.$$

For $F = G = G_{2s} * f$

$$||\widehat{g}||_2^2 = \int_{\mathbb{R}^n}\left(1 + (2\pi|k|)^2\right)^{-2s}|\widehat{f}(k)|^2dk = ||g||_2^2 \leq B(n,s)||f||_p^2$$

By the Young inequality (Theorem 4.2, [15]) we have that equality can occur in the inequality (29) and therefore in the inequality of the theorem if and only if $f(x)$ is a multiple of $Exp[-\frac{2n}{n-s}(x - x_0, J(x - x_0))]$ where $x_0 \in \mathbb{R}^n$ and J is any real, symmetric, positive-definite matrix.

Remark Using the Mathematica package we can compute the integral

$$I = \int_0^\infty\int_0^\infty e^{-(\delta+\varepsilon)}(\delta\varepsilon)^{s-1}(\delta+\varepsilon)^{-s/2}d\delta d\varepsilon$$

For $s = 1$, $I = \sqrt{\pi}/2$

Generally,

$$\begin{aligned}I &= \Gamma\left(\frac{s}{2}\right)\Gamma(s)_2F_1\left[\frac{s}{2}, s, 1-\frac{s}{2}, 1\right]\\ &\quad+\frac{1}{2}\frac{\cos(\pi s)\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(-\frac{s}{2}\right)\Gamma(s)}{\Gamma\left(1-\frac{3s}{2}\right)}{}_2F_1\left[s, \frac{3s}{2}, 1+\frac{s}{2}, 1\right]\end{aligned}$$

$$-\frac{1}{2}\frac{2^s\pi^{1/2}\cos(3\pi s/2)\Gamma\left(-\frac{s}{2}\right)\Gamma\left(\frac{3s}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)}{}_2F_1\left[s,\frac{3s}{2},1+\frac{s}{2},1\right]$$
$$-\frac{1}{2}\frac{\Gamma\left(1-\frac{s}{2}\right)\Gamma(s)\Gamma\left(\frac{3s}{2}\right)}{\Gamma\left(1+\frac{s}{2}\right)}{}_2F_1\left[s,\frac{3s}{2},1+\frac{s}{2},1\right]$$

where ${}_2F_1[\alpha,\beta,\gamma,z]$ is the hypergeometric function (Chapter II, [17])

$${}_2F_1[\alpha,\beta,\gamma,z]=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)}\frac{z^n}{n!}$$

The series converges in the unit circle $|z|<1$. If $|z|=1$ the series converges absolutely for $Re(\alpha+\beta-\gamma)<0$. In this case,

$${}_2F_1[\alpha,\beta,\gamma,1]=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)},$$

with $\gamma\neq 0,-1,-2,....$

Particularly,

$${}_2F_1\left[\frac{s}{2},s,1-\frac{s}{2},1\right]=\frac{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(1-2s\right)}{\Gamma\left(1-s\right)\Gamma\left(1-\frac{3s}{2}\right)}$$

and

$${}_2F_1\left[s,\frac{3s}{2},1+\frac{s}{2},1\right]=\frac{\Gamma\left(1+\frac{s}{2}\right)\Gamma\left(1-2s\right)}{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(1-s\right)}$$

So, the integral I converges absolutely for $0<s<1/2$ and we have

$$I=\frac{4\pi^2\cos(\pi s)\Gamma(1-2s)}{3s\sin(2\pi s)\sin(3\pi s)\left(\Gamma(1-s)\right)^2\Gamma\left(-\frac{3s}{2}\right)}.$$

Theorem 2.4 *Let $f\in H^{s,2}(\mathbb{R}^n)$ and $q=\frac{2n}{n-s}$, $n>s$. Then the following inequality holds:*

$$||f||_q^2\leq B(n,s)||(I-\Delta)^s f||_2^2 \tag{34}$$

where $B(n,s)$ is the best constant for the inequality (34) and

$$B(n,s) = \left(\frac{1}{(\Gamma(s))^2(4\pi)^{s/2}}\right)\frac{2^s n^{-s/2}s^{s/2}(n-s)^{(n-s)/2}}{(n+s)^{(n+s)/2}} \times\frac{4\pi^2\cos(\pi s)\Gamma(1-2s)}{3s\sin(2\pi s)\sin(3\pi s)\left(\Gamma(1-s)\right)^2\Gamma\left(-\frac{3s}{2}\right)}$$

if $0<s<1/2$. Equality in the inequality (34) attained if and only if $f(x)$ is a multiple of $Exp[-\frac{2n}{n+s}(x-x_0,J(x-x_0))]$ where $x_0\in\mathbb{R}^n$ and J is any real, symmetric, positive-definite matrix.

Proof. Let $f, g \in C_0^\infty(\mathbb{R}^n)$. Then we have

$$(f,g) \quad = \quad (\widehat{f},\widehat{g}) = \int_{\mathbb{R}^n} \left[\left(1+(2\pi|k|)^2\right)^s \overline{\widehat{f}(k)}\right]\left[\left(1+(2\pi|k|)^2\right)^{-s}\widehat{g}(k)\right] dk$$

$$\begin{aligned} |(f,g)|^2 &\leq \int_{\mathbb{R}^n} \left(1+(2\pi|k|)^2\right)^{2s} |\widehat{f}(k)|^2 dk \int_{\mathbb{R}^n} \left(1+(2\pi|k|)^2\right)^{-2s} |\widehat{g}(k)|^2 dk \\ &= ||(I-\Delta)f||_2^2 \int_{\mathbb{R}^n} \left(1+(2\pi|k|)^2\right)^{-2s} |\widehat{g}(k)|^2 dk \\ &\leq ||(I-\Delta)^s f||_2^2 B(n,s)||g||_p^2 \end{aligned} \tag{35}$$

by the Lemma 2.1, with $p = \frac{2n}{n+s}$.

Now setting $g = f^{q-1} \in L^p(\mathbb{R}^n)$ one obtains from (35) that

$$||f||_q^{2q} \leq B(n,s)||f^{q-1}||_p^2||(I-\Delta)^s f||_2^2$$

i.e. $||f||_q^2 \leq B(n,s)||(I-\Delta)^s f||_2^2$.

To have equality in (35) and hence in (34), it is necessary according to the Lemma 2.1 f^{q-1} be a multiple of $Exp[-\frac{2n}{n-s}(x-x_0, J(x-x_0))]$ where $x_0 \in \mathbb{R}^n$ and J is any real, symmetric, positive-definite matrix.

Lemma 2.2 *Let $f \in L^p(\mathbb{R}^n)$ with $p = \frac{2n}{n+1}$, then $\widehat{f}$ exists (by Hausdorff-Young's inequality). The function $g := G^\mu * f$, where*

$$G^\mu(x) = \int_0^\infty (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t} - \mu^2 t} dt$$

is the Yukawa potential (see Introduction), is an $L^2(\mathbb{R}^n)$-function and hence has a Fourier transform $\widehat{g}$

$$\widehat{g}(k) = \widehat{G^\mu}(k)\widehat{f}(k) = \left((2\pi|k|)^2 + \mu^2\right)^{-1}\widehat{f}(k).$$

We have

$$||G^\mu * f||_2^2 \leq U(n,\mu)||f||_p^2 \tag{36}$$

where $U(n,\mu) = \frac{n^{1/2}(n-1)^{(n-1)/2}}{(n+1)^{(n+1)/2}} \frac{1}{2\mu^3}$ is the best constant for the inequality (36).

Equality can occur in the inequality (36) above if and only if $f(x)$ is a multiple of $Exp[-\frac{2n}{n-1}(x-x_0, J(x-x_0))]$ where $x_0 \in \mathbb{R}^n$ and J is any real, symmetric, positive-definite matrix.

Proof. The proof is similar to the proof of Lemma 2.1.

We can find a sequence $f^1, f^2, \ldots$ of functions in $C_0^\infty(\mathbb{R}^n)$ such that $f^j \to f$ strongly in $L^p(\mathbb{R}^n)$. The function g is in $L^2(\mathbb{R}^n)$. Indeed, we have similarly to the proof of

Lemma 2.1

$$||g||_2^2 = (4\pi)^{-n/2}\int_0^\infty\int_0^\infty e^{-\mu^2 t}e^{-\mu^2\varepsilon}(t+\varepsilon)^{-n/2} \times\left[\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\pi\frac{|z-y|^2}{2\pi(t+\varepsilon)}} f(y)\overline{f(z)}dydz\right]dtd\varepsilon \tag{37}$$

Applying the Young inequality (Theorem 4.2, [15]) we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-\pi\frac{|z-y|^2}{4\pi(t+\varepsilon)}} f(y)\overline{f(z)}dydz \le C_{p,q,p;n}||f||_p^2\left|\left|e^{-\pi\frac{|x|^2}{4\pi(t+\varepsilon)}}\right|\right|_q \tag{38}$$

for $p=r=\frac{2n}{n+1}$ and $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=2$ i.e. $q=\frac{n}{n-1}$ and also $q'=n$, $p'=\frac{2n}{n-1}$

We can calculate

$$\left|\left|e^{-\pi\frac{|x|^2}{4\pi(t+\varepsilon)}}\right|\right|_q = 2^{n-1}\pi^{(n-1)/2}(t+\varepsilon)^{(n-1)/2}\left(\frac{n-1}{n}\right)^{(n-1)/2} \tag{39}$$

and also

$$C_{p,q,q;n} = 2n^{n/2}(n+1)^{-(n+1)/2} \tag{40}$$

Combining (37), (39), (40) and (38)

$$||g||_2^2 \le \pi^{-1/2}(n-1)^{(n-1)/2}(n+1)^{-(n+1)/2}n^{1/2} \times\int_0^\infty\int_0^\infty e^{-\mu^2 t}e^{-\mu^2\varepsilon}(t+\varepsilon)^{-1/2}dtd\varepsilon||f||_p^2 \tag{41}$$

But,

$$\int_0^\infty\int_0^\infty e^{-\mu^2 t}e^{-\mu^2\varepsilon}(t+\varepsilon)^{-1/2}dtd\varepsilon = \frac{\pi^{1/2}}{2\mu^3}$$

So,

$$\begin{aligned}||g||_2^2 &= \int_{\mathbb{R}^n}\left((2\pi|k|)^2+\mu^2\right)^{-2}|\widehat{f}(k)|^2dk\\ &\le \frac{n^{1/2}(n-1)^{(n-1)/2}}{2\mu^3(n+1)^{(n+1)/2}}||f||_p^2\end{aligned}$$

and therefore $g\in L^2(\mathbb{R}^n)$.

Using a simple approximation argument as in the proof of the Lemma 2.1 we have $\widehat{g}(k)=\left((2\pi|k|)^2+\mu^2\right)^{-1}\widehat{f}(k)$.

By Plancherel's theorem we have

$$(F,G)=\int_{\mathbb{R}^n}\overline{F(x)}G(x)dx=\int_{\mathbb{R}^n}\overline{\widehat{F}(k)}\widehat{G}(k)dk$$

For $F = G = G^{\mu} * f$

$$||\widehat{g}||_2^2 = \int_{\mathbb{R}^n} \left((2\pi|k|)^2 + \mu^2\right)^{-2} |\widehat{f}(k)|^2 dk = ||g||_2^2 \leq U(n,\mu)||f||_p^2$$

By the Young inequality we have that equality can occur in the inequality (38) and therefore in the inequality of the theorem if and only if $f(x)$ is a multiple of $Exp[-\frac{2n}{n-1}(x-x_0, J(x-x_0))]$ where $x_0 \in \mathbb{R}^n$ and J is any real, symmetric, positive-definite matrix.

Theorem 2.5 *Let a function f such that $||(-\Delta+\mu^2)f||_2 < \infty$ and $q = \frac{2n}{n-1}$, $n > 1$. Then the following inequality holds:*

$$||f||_q^2 \leq U(n,\mu)||(-\Delta+\mu^2)f||_2^2 \tag{42}$$

where

$$U(n,\mu) = \frac{n^{1/2}(n-1)^{(n-1)/2}}{(n+1)^{(n+1)/2}} \frac{1}{2\mu^3}$$

is the best constant for the inequality (42).

There is equality in the inequality above if and only if $f(x)$ is a multiple of $Exp[-\frac{2n}{n+1}(x-x_0, J(x-x_0))]$ where $x_0 \in \mathbb{R}^n$ and J is any real, symmetric, positive-definite matrix.

Proof. Let $f, g \in C_0^{\infty}(\mathbb{R}^n)$. Then we have

$$(f,g) = (\widehat{f},\widehat{g}) = \int_{\mathbb{R}^n} \left[\left((2\pi|k|)^2+\mu^2\right)\overline{\widehat{f}(k)}\right]\left[\left((2\pi|k|)^2+\mu^2\right)^{-1}\widehat{g}(k)\right] dk$$

$$\begin{aligned} |(f,g)|^2 &\leq \int_{\mathbb{R}^n} \left((2\pi|k|)^2+\mu^2\right)^2 |\widehat{f}(k)|^2 dk \int_{\mathbb{R}^n} \left((2\pi|k|)^2+\mu^2\right)^{-2} |\widehat{g}(k)|^2 dk \\ &= ||(-\Delta+\mu^2)f||_2^2 \int_{\mathbb{R}^n} \left((2\pi|k|)^2+\mu^2\right)^{-2} |\widehat{g}(k)|^2 dk \\ &\leq ||(-\Delta+\mu^2)f||_2^2 U(n,\mu)||g||_p^2 \end{aligned} \tag{43}$$

by the Lemma 2.2, with $p = \frac{2n}{n+1}$.

Now setting $g = f^{q-1} \in L^p(\mathbb{R}^n)$ one obtains from (43) that

$$||f||_q^{2q} \leq U(n,\mu)||f^{q-1}||_p^2||(-\Delta+\mu^2)f||_2^2$$

i.e. $||f||_q^2 \leq U(n,\mu)||(-\Delta+\mu^2)f||_2^2$.

To have equality in (43) and hence in (42), it is necessary according to the Lemma 2.2, f^{q-1} be a multiple of $Exp[-\frac{2n}{n-1}(x-x_0, J(x-x_0))]$ where $x_0 \in \mathbb{R}^n$ and J is any real, symmetric, positive-definite matrix.

Remark We can see that $U(n,1) = B(n,1)$.

Theorem 2.6 *Let $||(-\Delta+\mu^2)f||_2<\infty$. For $n=1$ the inequality*

$$||f||_q^2 \le V(q,\mu)||(-\Delta+\mu^2)f||_2^2 \tag{44}$$

holds for all $2<q<\infty$ with a constant $V(q,\mu)$ that satisfies

$$V(q,\mu) < (q-1)^{-1+1/q} q^{1-2/q} \left[\frac{1}{2\sqrt{\pi}} \mu^{\frac{3q+2}{2-q}} \frac{\Gamma\left(\frac{2+3q}{2q-4}\right)}{\Gamma\left(\frac{2q}{q-2}\right)} \right]^{(q-2)/q}$$

Proof. We have $\int_{\mathbb{R}} \left((2\pi|k|)^2\mu^2\right)^2 |\widehat{f}(k)|^2 dk < \infty$. Let $1<p<2$ be the dual index of $q>2$, i.e. $\frac{1}{p}+\frac{1}{q}=1 \Rightarrow p=\frac{q}{q-1}$ and also $\frac{p}{2-p}=\frac{q}{q-2}$.

We have that

$$\begin{aligned} ||\widehat{f}||_p^p &= \int_{\mathbb{R}} |\widehat{f}(k)|^p dk \\ &= \int_{\mathbb{R}} \left(|\widehat{f}(k)|^2 \left((2\pi|k|)^2+\mu^2\right)^2\right)^{p/2} \left((2\pi|k|)^2+\mu^2\right)^{-p} dk \end{aligned} \tag{45}$$

The function $F(k)=\left(|\widehat{f}(k)|^2\left((2\pi|k|)^2+\mu^2\right)^2\right)^{p/2}$ is in $L^{2/p}(\mathbb{R})$ since

$$||F||_{2/p} = ||(-\Delta+\mu^2)f||_2^p < \infty \tag{46}$$

The function $G(k)=\left((2\pi|k|)^2+\mu^2\right)^{-p}$ is in $L^{2/(2-p)}(\mathbb{R})$ since

$$\begin{aligned} ||G||_{2/(2-p)}^{2/(2-p)} &= \int_{\mathbb{R}} \left((2\pi|k|)^2+\mu^2\right)^{-\frac{2p}{2-p}} dk \\ &= \frac{1}{2\mu\sqrt{\pi}} (\mu^2)^{\frac{-2+3p}{p-2}} \frac{\Gamma\left(\frac{2-5p}{2p-4}\right)}{\Gamma\left(-\frac{2p}{p-2}\right)} \end{aligned} \tag{47}$$

Since $\frac{1}{2/(2-p)}+\frac{1}{2/p}=1$ by the Hölder inequality, (45) gives

$$||\widehat{f}||_p^p \le ||F||_{2/p}||G||_{2/(2-p)} \tag{48}$$

Combining (46), (47) and (48)

$$||\widehat{f}||_p \le ||(-\Delta+\mu^2)f||_2 \left[\frac{1}{2\mu\sqrt{\pi}} (\mu^2)^{\frac{-2+3p}{p-2}} \frac{\Gamma\left(\frac{2-5p}{2p-4}\right)}{\Gamma\left(-\frac{2p}{p-2}\right)} \right]^{(2-p)/2p}$$

i.e.

$$||\widehat{f}||_p \le \left[\frac{1}{2\sqrt{\pi}} \mu^{\frac{3q+2}{2-q}} \frac{\Gamma\left(\frac{2+3q}{2q-4}\right)}{\Gamma\left(\frac{2q}{q-2}\right)} \right]^{\frac{q-2}{2q}} ||(-\Delta+\mu^2)f||_2 \tag{49}$$

and therefore $\widehat{f} \in L^p(\mathbb{R})$.

Applying the sharp Hausdorff-Young inequality (Theorem 5.7, [15]) we have that

$$||\widehat{\widehat{f}}||_q \leq C_p||\widehat{f}||_p$$

with $1/p + 1/q = 1$ and $C_p = [p^{1/p}q^{-1/q}]^{1/2}$.

Also $f \in L^2(\mathbb{R})$ and by the inversion formula it is $f = (\widehat{f})^\vee$ where $f^\vee(x) = \widehat{f}(-x)$, i.e. $f(x) = \widehat{\widehat{f}}(-x)$ and also $||\widehat{\widehat{f}}||_q = ||f||_q$. Therefore

$$||f||_q^2 \leq (q-1)^{-1+1/q}q^{1-2/q}||\widehat{f}||_p^2 \tag{50}$$

Combining (49) and (50) we have that

$$||f||_q^2 \leq (q-1)^{-1+1/q}q^{1-2/q}\left[\frac{1}{2\sqrt{\pi}}\mu^{\frac{3q+2}{2-q}}\frac{\Gamma\left(\frac{2+3q}{2q-4}\right)}{\Gamma\left(\frac{2q}{q-2}\right)}\right]^{(q-2)/q} ||(-\Delta+\mu^2)f||_2^2$$

which proves the theorem.

Theorem 2.7 *We consider the operator semigroups* $e^{-t(I-\Delta)^s}$, $t > 0$ *defined by*

$$\left(e^{-t(I-\Delta)^s}f\right)^\wedge(k) = e^{-t\left(1+(2\pi|k|)^2\right)^{2s}}\widehat{f}(k).$$

We have $||(I-\Delta)^s f||_2 < \infty$ *if and only if*

$$A_t(f) = \frac{1}{t}\left[(f,f) - \left(f, e^{-t(I-\Delta)^s}f\right)\right]$$

is uniformly bounded and we have in which case

$$\sup_{t>0} A_t(f) = \lim_{t\to 0} A_t(f) = ||(I-\Delta)^s f||_2^2.$$

Proof. We have that $||(I-\Delta)^s f||_2^2 = \int_{\mathbb{R}^n}\left(1+(2\pi|k|)^2\right)^{2s}|\widehat{f}(k)|^2dk$ and also

$$\begin{aligned}(f,f) - \left(f, e^{-t(I-\Delta)^s}f\right) &= \int_{\mathbb{R}^n}|\widehat{f}(k)|^2dk - \int_{\mathbb{R}^n}\overline{\widehat{f}(k)}e^{-t\left(1+(2\pi|k|)^2\right)^{2s}}\widehat{f}(k)\\ &= \int_{\mathbb{R}^n}|\widehat{f}(k)|^2\left(1-e^{-t\left(1+(2\pi|k|)^2\right)^{2s}}\right)dk\end{aligned}$$

We have that $\lim_{t\to 0}\frac{1-e^{-t\left(1+(2\pi|k|)^2\right)^{2s}}}{t} = \left(1+(2\pi|k|)^2\right)^{2s}$ since $\frac{1-e^{-y}}{y}$ is decreasing function of $y > 0$.

So if $||(I-\Delta)^s f||_2 < \infty$ implies $\int_{\mathbb{R}^n}\left(1+(2\pi|k|)^2\right)^{2s}|\widehat{f}(k)|^2dk < \infty$ and therefore $A_t(f)$ is uniformly bounded and $\sup_{t>0} A_t(f) = \lim_{t\to 0} A_t(f) = ||(I-\Delta)^{2s}f||_2^2$. Conversely if $A_t(f)$ is uniformly bounded, the theorem of monotone convergence implies that

$$\sup_{t>0} A_t(f) = \lim_{t\to 0} A_t(f) = \int_{\mathbb{R}^n}\left(1+(2\pi|k|)^2\right)^{2s}|\widehat{f}(k)|^2dk.$$

So $||(I-\Delta)^s f||_2 < \infty$.

Remark We have announced a similar theorem to Theorem 2.7 in [8] for the operator semigroups $e^{-t(-\Delta)^s}$, $t>0$. Obviously, the theorem is valid for the operator semigroups $e^{-t(-\Delta+\mu^2)}$, $t>0$ defined by

$$\left(e^{-t(-\Delta+\mu^2)}f\right)^{\wedge}(k) = e^{-t\left((2\pi|k|)^2+\mu^2\right)^2}\widehat{f}(k).$$

3. An application on the sphere

Let $\mathbb{S}_n = \{x\in\mathbb{R}^{n+1} : |x|=1\}$ and let $g_{\mathbb{S}_n}$ the restriction of the Euclidean metric to $\mathbb{S}_n$. Let also $d\sigma$ be the area on $\mathbb{S}_n$ and let us denote by ω_n the volume of $\mathbb{S}_n$. Note that we have

$$\omega_n = \frac{2\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}$$

Intertwining operators on the sphere have been studied by Branson (c.f. [6]) who showed in particular that the A_s are the unique intertwiners on $\mathbb{S}_n$ and that they can be written in terms of the spherical Laplacian $\Delta_{\mathbb{S}_n}$ (see also [5] and [19]) as

$$A_s = \frac{\Gamma(B+(1+s)/2)}{\Gamma(B+(1-s)/2)}, \quad B=\sqrt{\Delta_{\mathbb{S}_n}+\left(\frac{n-1}{2}\right)^2}$$

where s be a positive real number.

Let $\pi:\mathbb{R}^n\to\mathbb{S}_n-\{0,...,0,-1\}$ denote stereographic projection and let J_π be the Jacobian of π.

The *intertwining* operators A_s on the sphere $(\mathbb{S}_n, g_{\mathbb{S}_n})$ are essentially the powers $(-\Delta)^{s/2}$ lifted to the sphere via the stereographic projection $\pi:\mathbb{R}^n\to\mathbb{S}_n-\{0,...,0,-1\}$, using the formula (c.f. [19])

$$(A_sF)\circ\pi = |J_\pi|^{-(n+s)/(2n)}(-\Delta)^{s/2}\left(|J_\pi|^{(n-s)/(2n)}(F\circ\pi)\right), F\in L^2(\mathbb{S}_n) \tag{51}$$

Beckner (c.f. [5]) showed that the following inequality

$$||F||^2_{L^q(\mathbb{S}_n)} \le \frac{\Gamma\left(\frac{n-2s}{2}\right)}{\Gamma\left(\frac{n+2s}{2}\right)}\int_{\mathbb{S}_n} FA_{2s}Fd\xi \tag{52}$$

is valid on $\mathbb{S}_n$ for $s=n/2-n/q$, $q>2$. The measure on sphere is the normalized surface measure which is

$$d\xi(x) = \pi^{-n/2}[\Gamma(n)/\Gamma(n/2)]\left(1+|x|^2\right)^{-n}dx$$

Alternatively, the proof of the inequality (52) can be given using the inequality (5) and the stereographic projection. So, we have shown (c.f. [9])

$$||F||^2_{L^q(\mathbb{S}_n)} \le (4\pi)^{-s}\frac{\Gamma\left(\frac{n-2s}{2}\right)}{\Gamma\left(\frac{n+2s}{2}\right)}\left[\frac{\Gamma(n)}{\Gamma(n/2)}\right]^{2s/n}\int_{\mathbb{S}_n} FA_{2s}Fd\sigma$$

where $n > 2s$.

A function $F : \mathbb{S}_n \to \mathbb{R}$ with $F \in L^2(\mathbb{S}_n)$ is said to be in $H^{s,p}(\mathbb{S}_n)$ if and only if

$$\int_{\mathbb{S}_n} |A_s F|^p d\sigma < \infty$$

Theorem 3.1 *Let $n > s > 0$ and $p = \frac{2n}{n+s}$. Then*

$$||F||_q \leq K(n,s)||A_s F||_p \ , F \in H^{s,p}(\mathbb{S}_n) \tag{53}$$

where $q = \frac{2n}{n-s}$ and

$$K(n,s) = (4\pi)^{-s/2} \frac{\Gamma\left(\frac{n-s}{2}\right)}{\Gamma\left(\frac{n+s}{2}\right)} \left[\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\right]^{\frac{s}{n}} \tag{54}$$

Proof. The conformal equivalence between $\mathbb{R}^n$ and $\mathbb{S}_n$ is effected by stereographic projection π from $\mathbb{R}^n$ to $\mathbb{S}_n - \{0, ..., 0, -1\}$ with the "north pole" corresponding to $(0, ..., 0, 1)$. Define $\sigma = (\sigma_1, \sigma_2, ..., \sigma_{n+1})$ by

$$\sigma_i = \frac{2x_i}{1+|x|^2} \quad \text{for} \quad i = 1, .., n \quad \text{and} \quad \sigma_{n+1} = \frac{1-|x|^2}{1+|x|^2}$$

The inverse map of π is

$$x_i = \frac{\sigma_i}{1+\sigma_{n+1}}, \qquad i = 1, ..., n$$

and the change of measure is given by

$$d\sigma = |J_\pi| dx = \left(\frac{2}{1+|x|^2}\right)^n dx$$

where $|J_\pi|$ is the determinant of the Jacobian of π and therefore

$$|J_\pi| = \left(\frac{2}{1+|x|^2}\right)^n \quad \textit{and} \quad |J_{\pi^{-1}}| = (1+\sigma_{n+1})^n.$$

We can lift functions in $L^q(\mathbb{R}^n)$ to the sphere $\mathbb{S}_n$. Simply define

$$F(\sigma) = |J_{\pi^{-1}}|^{1/q} f(\pi^{-1}(\sigma)) \tag{55}$$

or

$$f(x) = |J_\pi|^{1/q} F(\pi(x)) \tag{56}$$

and then

$$||F||_{L^q(\mathbb{S}_n)} = ||f||_{L^q(\mathbb{R}^n)} \tag{57}$$

Combining (51) and (56), we have that

$$\begin{aligned} A_s F &= |J_\pi|^{-(n+s)/(2n)} \left((-\Delta)^{s/2} f\right) \left(\pi^{-1}(x)\right) \\ &= |J_{\pi^{-1}}|^{(n+s)/(2n)} \left((-\Delta)^{s/2} f\right) \left(\pi^{-1}(x)\right) \end{aligned}$$

and therefore

$$||A_s F||_{L^p(\mathbb{S}_n)} = ||(-\Delta)^{s/2} f||_{L^p(\mathbb{R}^n)} \tag{58}$$

Applying theorem 2.2 and taking into account (57) and (58) the theorem follows.

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◇ Athanase Cotsiolis
Department of Mathematics, University of Patras
Patras 26110, Greece
`cotsioli@math.upatras.gr`

◇ Nikolaos K. Tavoularis

`niktav@master.math.upatras.gr`