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Invariant Einstein metrics on quaternionic Stiefel manifolds

Andreas Arvanitoyeorgos, V.V. Dzhepko and
Yu.G. Nikonorov

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Abstract

A Riemannian manifold (M, ρ) is called Einstein if the metric ρ satisfies the condition $\mathrm{Ric}(\rho) = c \cdot \rho$ for some constant c. We investigate G-invariant Einstein metrics with additional symmetries, on some homogeneous spaces G/H of classical groups. As a consequence, we obtain new invariant Einstein metrics on some quaternionic Stiefel manifolds $Sp(n)/Sp(l)$. Furthermore, we show that for any positive integer p there exists a quaternionic Stiefel manifold $Sp(n)/Sp(l)$ which admits at least p $Sp(n)$-invariant Einstein metrics.

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Introduction

A Riemannian manifold (M, ρ) is called Einstein if the metric ρ satisfies the condition $\mathrm{Ric}(\rho) = c \cdot \rho$ for some real constant c. We refer to [3] and [13] for a an exposition and various results on homogeneous Einstein manifolds. It is difficult to obtain general existence. Among the first important attempts are the works of G. Jensen [7] and M. Wang, W. Ziller [14]. Recently, a new existence approach was introduced by C. Böhm, M. Wang, and W. Ziller [4], [5].

The structure of the set of invariant Einstein metrics on a given homogeneous space is still not very well understood in general. The situation is only clear for few classes of homogeneous spaces, such as isotropy irreducible homogeneous spaces, low dimensional examples, certain flag manifolds, and some other special types of homogeneous spaces ([1], [3], [9], [10], [12]). For an arbitrary compact homogeneous space G/H it is not clear if the set of invariant Einstein metrics (up to isometry and up to scaling) is finite or not (cf. [15]). A finiteness conjecture states that this set is in fact finite if the isotropy representation of G/H consists of pairwise inequivalent irreducible components ([4, p. 683]).

Let G be a compact Lie group and H a closed subgroup so that G acts almost effectively on G/H. In this paper we use a technique developed in [2] to investigate G-invariant metrics on G/H with additional symmetries. More precisely, let K be a closed subgroup of G with $H \subset K \subset G$, and suppose that $K = L' \times H'$, where $\{e_{L'}\} \times H' = H$. It is clear that $K \subset N_G(H)$, the normalizer of H in G. If we denote $L = L' \times \{e_{H'}\}$, then the group $\widetilde{G} = G \times L$ acts on G/H by $(a, b) \cdot gH = agb^{-1}H$, and the isotropy subgroup at eH is $\widetilde{H} = \{(a, b) : ab^{-1} \in H\}$.

The set $\mathcal{M}^{\widetilde{G}}$ of $\widetilde{G}$-invariant metrics on $\widetilde{G}/\widetilde{H}$ is a subset of $\mathcal{M}^G$, the set of G-invariant metrics on G/H. Therefore, it would be simpler to search for invariant Einstein metrics on $\mathcal{M}^{\widetilde{G}}$. In this way we obtain existence results for Einstein metrics for certain quotients.

We apply this method for the quaternionic Stiefel manifolds $V_k(\mathbb{H}^n) = Sp(n)/Sp(n-k)$ of orthonormal k frames in the quaternionic inner product space $\mathbb{H}^n$. The case of the real Stiefel manifolds $V_k(\mathbb{R}^n)$ was treated in [2]. In [8] G. Jensen proved that $Sp(n)/Sp(n-k)$ admits at least two $Sp(n)$-invariant Einstein metrics. Note that, according to W. Ziller [16], the simplest case $S^{4n-1} = Sp(n)/Sp(n-1)$ admits exactly two invariant Einstein metrics. For $k \geq 2$ there is no obstruction for existence of more than two homogeneous invariant Einstein metrics on the quaternionic Stiefel manifolds $Sp(n)/Sp(n-k)$.

In particular we prove the following:

Theorem 1. *If $s > 1$ and $l \geq k \geq 1$ then the quaternionic Stiefel manifold $Sp(sk+l)/Sp(l)$ admits at least four $Sp(sk+l) \times (Sp(k))^s$-invariant Einstein metrics, two of which are Jensen's metrics.*

We also prove the following:

Theorem 2. *For any positive integer p there exists a quaternionic Stiefel manifold $Sp(n)/Sp(l)$ which admits at least p $Sp(n)$-invariant Einstein metrics.*

The paper is organized as follows: In Section 1 we review the basic construction for searching for invariant Einstein metrics with additional symmetries on G/H, as introduced in [2]. Metrics with this property for some homogeneous spaces of the group $Sp(n)$, are described in Section 2, Lemma 2.. In Section 3 we compute the scalar curvature for these metrics (Proposition 1.), and the variational approach to the Einstein metrics is given in Proposition 3.. In Section 4, as an application of our construction, we obtain Jensen's invariant Einstein metrics on the quaternionic Stiefel manifold $Sp(k_1+k_2)/Sp(k_2)$. In Section 5 we investigate $Sp(k_1+k_2+k_3) \times Sp(k_1) \times Sp(k_2)$–invariant Einstein metrics on $Sp(k_1+k_2+k_3)/Sp(k_3)$. Another construction for searching for invariant Einstein metrics on $Sp(sk+l)/Sp(l)$ is given in Section 6. Finally, in Section 7 the proofs of the main results are given.

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1. The main construction

We review the method described in [2]. Let G be a compact Lie group and H a closed subgroup so that G acts almost effectively on G/H. Let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras of G and H, and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be a reductive decomposition of $\mathfrak{g}$ with respect to some $\mathrm{Ad}(G)$-invariant inner product of $\mathfrak{g}$. The orthogonal complement $\mathfrak{p}$ can be identified with the tangent space $T_{eH}G/H$. Any G-invariant metric ρ of G/H corresponds to an $\mathrm{Ad}(H)$-invariant inner product $(\cdot,\cdot)$ on $\mathfrak{p}$ and vice-versa. For G semisimple, the negative of the Killing form B of $\mathfrak{g}$ is an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$, therefore we can choose the above decomposition with respect to this form. We will use such a decomposition later on. Moreover, the restriction $\langle\cdot,\cdot\rangle = -B|_{\mathfrak{p}}$ is an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{p}$, which generates a G-invariant metric on G/H called *standard*.

The normalizer $N_G(H)$ of H in G acts on G/H by $(a, gH) \mapsto ga^{-1}H$. For a fixed a this action induces a G-equivariant diffeomorphism $\varphi_a : G/H \to G/H$. Note that if $a \in H$ this diffeomorphism is trivial, so the action of the gauge group $N_G(H)/H$ is well defined. However, it is simpler from technical point of view to use the action of $N_G(H)$. Let ρ be a G-invariant metric of G/H with corresponding inner product $(\cdot,\cdot)$. Then the diffeomorphism φ_a is an isometry of $(G/H,\rho)$ if and only if the operator $\mathrm{Ad}(a)|_{\mathfrak{p}}$ is orthogonal with respect to $(\cdot,\cdot)$.

Let K be a closed subgroup of G with $H \subset K \subset G$ such that $K = L' \times H'$, where $\{e_{L'}\} \times H' = H$, and consider $L = L' \times \{e_{H'}\}$. It is clear that $K \subset N_G(H)$. The group $\widetilde{G} = G \times L$ acts on G/H by $(a,b)\cdot gH = agb^{-1}H$, and the isotropy $\widetilde{H}$ at eH is isomorphic to K (see details in [3]).

The set $\mathcal{M}^G$ of G-invariant metrics on G/H is finite dimensional. We consider the subset $\mathcal{M}^{G,K}$ of $\mathcal{M}^G$ corresponding to $\mathrm{Ad}(K)$-invariant inner products on $\mathfrak{p}$ (and not only $\mathrm{Ad}(H)$-invariant).

Let $\rho \in \mathcal{M}^{G,K}$ and $a \in K$. The above diffeomorphism φ_a is an isometry of $(G/H,\rho)$. The action $\widetilde{G}$ on $(G/H,\rho)$ is isometric, so any metric form $\mathcal{M}^{G,K}$ can be identified a metric in $\mathcal{M}^{\widetilde{G}}$ and vice-versa. Therefore, we may think of $\mathcal{M}^{\widetilde{G}}$ as $\mathcal{M}^{G,K}$, which is a subset of $\mathcal{M}^G$.

Since metrics in $\mathcal{M}^{G,K}$ correspond to $\mathrm{Ad}(K)$-invariant inner products on $\mathfrak{p}$, we call these metrics $\mathrm{Ad}(K)$*-invariant metrics on* G/H.

The aim of the present work is to apply the above construction for $Sp(n)$, and prove existence of Einstein metrics in the set $\mathcal{M}^{G,K}$ for various choices of the subgroup $K = L' \times H'$.

Let $n \in \mathbb{N}$ and $k_1,\ k_2, \ldots, k_s, k_{s+1}, \ldots, k_{s+t}$ be natural numbers such that $k_1 + \cdots + k_s = l$, $k_{s+1} + \cdots + k_{s+t} = m$, $l + m = n$. Let $G = Sp(n)$ and $K = L' \times H'$, where $L' = Sp(k_1) \times \cdots \times Sp(k_s)$ and $H' = Sp(k_{s+1}) \times \cdots \times Sp(k_{t+s})$. The embedding of K in G is the standard one.

We note that for $s = 0$ we obtain a flag manifold of symplectic groups. Invariant Einstein metrics on the spaces $Sp(k_1+k_2+k_3)/Sp(k_1)\times Sp(k_2)\times Sp(k_3)$, were studied in [10], [11].

2. $\mathrm{Ad}(K)$-invariant metrics on the space G/H

Let $\mathfrak{p}_i$ be the subalgebra $sp(k_i)$ in $\mathfrak{g}$, $1 \le i \le s+t$. We note that for $1 \le i \le s$ the submodule $\mathfrak{p}_i$ of $\mathfrak{p}$ is $\mathrm{Ad}(K)$-invariant and $\mathrm{Ad}(K)$-irreducible submodule. For $1 \le i < j \le s+t$ we denote by $\mathfrak{p}_{(i,j)}$ the $\mathrm{Ad}(K)$-invariant and $\mathrm{Ad}(K)$-irreducible submodule of $\mathfrak{p}$ which is determined by the equality $sp(k_i+k_j) = sp(k_i)\oplus sp(k_j)\oplus \mathfrak{p}_{(i,j)}$, where $\mathfrak{p}_{(i,j)}$ is orthogonal to $sp(k_i) \oplus sp(k_j))$ with respect to the Killing form B.

Denote by d_i and $d_{(i,j)}$ the dimensions of the modules $\mathfrak{p}_i$ and $\mathfrak{p}_{(i,j)}$ respectively. It is easy to obtain that $d_i = 2k_i^2 + k_i$, $d_{(i,j)} = 4k_ik_j$.

We have a decomposition of $\mathfrak{p}$ into a sum of $\mathrm{Ad}(K)$-invariant and $\mathrm{Ad}(K)$-irreducible submodules:

$$\mathfrak{p} = \bigoplus_{i=1}^{s} \mathfrak{p}_i \bigoplus \bigoplus_{1\le i<j\le s+t} \mathfrak{p}_{(i,j)}\,. \tag{1}$$

Lemma 1. *([2]) There are no pairwise* $\mathrm{Ad}(K)$*-isomorphic submodules among the* $\mathfrak{p}_i$ *(*$i = 1, \ldots, s$*) and* $\mathfrak{p}_{(i,j)}$ *(*$1 \le i < j \le s+t$*).*

Proof. It is clear that $\mathfrak{p}_i$ and $\mathfrak{p}_j$ act nontrivially on $\mathfrak{p}_{(i,j)}$, and that each $\mathfrak{p}_i$ acts nontrivially on itself. Therefore, there are no pairwise $\mathrm{Ad}(K)$-isomorphic submodules. □

By the previous lemma we have a complete description of the $\mathrm{Ad}(K)$-invariant metrics on G/H. Let ρ be any $\mathrm{Ad}(K)$-invariant metric on G/H with corresponding $\mathrm{Ad}(K)$-invariant inner product $(\cdot,\cdot)$ on $\mathfrak{p}$.

Lemma 2. *Any* $\mathrm{Ad}(K)$*-invariant inner product on* $\mathfrak{p}$ *has the form*

$$(\cdot,\cdot) = \sum_{i=1}^{s} x_i \cdot \langle \cdot,\cdot\rangle|_{\mathfrak{p}_i} + \sum_{1\le i<j\le s+t} x_{(i,j)} \cdot \langle\cdot,\cdot\rangle|_{\mathfrak{p}_{(i,j)}} \tag{2}$$

for positive constants $x_i > 0$ and $x_{(i,j)} > 0$, where $\langle\cdot,\cdot\rangle = -B|_{\mathfrak{p}}$. Therefore, the set of $\mathrm{Ad}(K)$*-invariant metrics on G/H depends on $(s+t)(s+t-1)/2+s$ parameters.*

Note that in the case of pairwise $\mathrm{Ad}(K)$-isomorphic modules $\mathfrak{p}_\alpha$ and $\mathfrak{p}_\beta$ the set of $\mathrm{Ad}(K)$-invariant metrics have a more complicated structure ([14]).

3. The scalar curvature and the Einstein condition

Let $\{e_\alpha^j\}$ be an orthonormal basis of $\mathfrak{p}_\alpha$ with respect to $\langle\cdot,\cdot\rangle$, where $1 \le j \le d_\alpha$ (here α means any of the symbols of type i or (k,i)). We define the numbers (cf. [14]) $[\alpha\beta\gamma]$ by the relation

$$[\alpha\beta\gamma] = \sum_{i,j,k} \langle \left[e_\alpha^i, e_\beta^j\right], e_\gamma^k\rangle^2,$$

where i, j, and k vary from 1 to d_α, d_β, and d_γ respectively. The symbols $[\alpha\beta\gamma]$ are symmetric with respect to all three indices, as follows from the $\mathrm{Ad}(G)$-invariance of $\langle\cdot,\cdot\rangle$.

For any Lie algebra $\mathfrak{q}$ we shall use the symbol $B_{\mathfrak{q}}$ for the Killing form of $\mathfrak{q}$. If a simple algebra $\mathfrak{q}$ is a subalgebra of a Lie algebra $\mathfrak{r}$, then we denote by $\alpha_{\mathfrak{r}}^{\mathfrak{q}}$ a real number which satisfies the equality $B_{\mathfrak{q}} = \alpha_{\mathfrak{r}}^{\mathfrak{q}} \cdot B_{\mathfrak{r}}|_{\mathfrak{q}}$. We will need the following:

Lemma 3. *([2]) Let $\mathfrak{q} \subset \mathfrak{r}$ be arbitrary subalgebras in $\mathfrak{g}$ with $\mathfrak{q}$ simple. Consider in $\mathfrak{q}$ an orthonormal (with respect to $-B_{\mathfrak{r}}$) basis $\{f_j\}$ $(1 \le j \le \dim(\mathfrak{q}))$. Then*

$$\sum_{j,k=1}^{\dim(\mathfrak{q})} (-B_{\mathfrak{r}}([f_i, f_j], f_k))^2 = \alpha_{\mathfrak{r}}^{\mathfrak{q}}, \quad i = 1, \ldots, \dim(\mathfrak{q}),$$

$$\sum_{i,j,k=1}^{\dim(\mathfrak{q})} (-B_{\mathfrak{r}}([f_i, f_j], f_k))^2 = \alpha_{\mathfrak{r}}^{\mathfrak{q}} \cdot \dim(\mathfrak{q}),$$

where $\alpha_{\mathfrak{r}}^{\mathfrak{q}}$ is determined by the equation $B_{\mathfrak{q}} = \alpha_{\mathfrak{r}}^{\mathfrak{q}} \cdot B_{\mathfrak{r}}|_{\mathfrak{q}}$.

Using this lemma we obtain an explicit expression for $[\alpha\beta\gamma]$. It is clear that the only non-zero symbols (up to permutation of indices) are

$$[aaa], \quad [a(a,b)(a,b)], \quad [b(a,b)(a,b)],$$

where $1 \le a < b \le s+t$, and $[(a,b)(b,c)(a,b)]$, with $1 \le a < b < c \le s+t$.

Lemma 4. *The following relations hold:*

$$[aaa] = \frac{k_a(k_a+1)(2k_a+1)}{n+1}, \quad [a(a,b)(a,b)] = \frac{k_a k_b(2k_a+1)}{n+1},$$

$$[b(a,b)(a,b)] = \frac{k_a k_b(2k_b+1)}{n+1}, \quad [(a,b)(b,c)(a,c)] = \frac{2k_a k_b k_c}{n+1}.$$

Proof. For the standard embedding $sp(k) \subset sp(n)$ we have $\alpha_{sp(n)}^{sp(k)} = \frac{k+1}{n+1}$ (see e.g. [6]).

The first equality $[aaa] = \frac{k_a(k_a+1)(2k_a+1)}{n+1}$ follows from Lemma 3.. In fact, $d_a = \dim(sp(k_a)) = 2k_a^2 + k_a$ and $\alpha_{sp(n)}^{sp(k_a)} = \frac{k_a+1}{n+1}$.

To prove the second equality we consider the subalgebra $sp(k_a + k_b) \subset sp(n)$. It is clear that $[\mathfrak{p}_a, \mathfrak{p}_b] = 0$, $[\mathfrak{p}_a, \mathfrak{p}_{(a,b)}] \subset \mathfrak{p}_{(a,b)}$. According to Lemma 3. we have that

$$[aaa] + [a(a,b)(a,b)] = \dim(\mathfrak{p}_a) \cdot \alpha_{sp(n)}^{sp(k_a+k_b)} = \frac{k_a(2k_a+1)(k_a+k_b+1)}{n+1},$$

which proves the second equality. The third equality can be obtained analogously.

To prove the forth equality we consider the subalgebra $sp(k_a + k_b + k_c) \subset sp(n)$. It is clear that

$$\dim(\mathfrak{p}_{(a,b)}) \cdot \alpha_{sp(n)}^{sp(k_a+k_b+k_c)} = 2\left([(a,b)a(a,b)] + [(a,b)b(a,b)] + [(a,b)(b,c)(a,c)]\right),$$

from which we obtain the last equality.

□

According to [14], the scalar curvature S of $(\cdot,\cdot)$ is given by

$$S((\cdot,\cdot)) = \frac{1}{2}\sum_{\alpha} \frac{d_\alpha}{x_\alpha} - \frac{1}{4}\sum_{\alpha,\beta,\gamma} [\alpha\beta\gamma]\frac{x_\gamma}{x_\alpha x_\beta},$$

where α, β and γ are arbitrary symbols of the type i $(1 \le i \le s)$ or of the type (i,j) $(1 \le i < j \le s+t)$.

For the metric (2) this formula takes the following form.

Proposition 1. *The scalar curvature S of an $\mathrm{Ad}(K)$-invariant metric (2) has the form*

$$S = \sum_{a=1}^{s} \frac{k_a(k_a+1)(2k_a+1)}{4(n+1)} \cdot \frac{1}{x_a} + 2\sum_{1\le a<b\le s+t} \frac{k_a k_b}{x_{(a,b)}}$$

$$-\frac{1}{4(n+1)} \sum_{1\le a\le s,\, a+1\le b\le s+t} k_a k_b(2k_a+1)\frac{x_a}{x_{(a,b)}^2} - \frac{1}{4(n+1)}\sum_{1\le a<b\le s} k_a k_b(2k_b+1)\frac{x_b}{x_{(a,b)}^2}$$

$$-\frac{1}{n+1}\sum_{1\le a<b<c\le s+t} k_a k_b k_c\left(\frac{x_{(a,b)}}{x_{(a,c)}x_{(b,c)}} + \frac{x_{(a,c)}}{x_{(a,b)}x_{(b,c)}} + \frac{x_{(b,c)}}{x_{(a,b)}x_{(a,c)}}\right). \qquad (3)$$

Denote by $\mathcal{M}_1^G$ the set of all G-invariant metrics with a fixed volume element on the space G/H. The following variational principle for invariant Einstein metrics is well known.

Proposition 2. **([3])** *Let G/H be a homogeneous space, where G and H are compact. Then the G-invariant Einstein metrics on the homogeneous space G/H are precisely the critical points of the scalar curvature functional S restricted to $\mathcal{M}_1^G$.*

For the general construction as described in Section 1, the above variational principle implies the following:

Proposition 3. **([2])** *Let $\mathcal{M}_1^{G,K}$ be the subset of $\mathcal{M}^{G,K}$ with fixed volume element. Then a metric in $\mathcal{M}_1^{G,K}$ is Einstein if and only if it is a critical point of the scalar curvature functional S restricted to $\mathcal{M}_1^{G,K}$.*

Proof. The set $\mathcal{M}_1^{G,K}$ is precisely the set of $\widetilde{G}$-invariant metrics with fixed volume element on $\widetilde{G}/\widetilde{H}$. □

The volume condition for the metric (2) takes the form

$$\prod_{i=1}^{s} x_i^{d_i} \cdot \prod_{1\leq i<j\leq s+t} x_{(i,j)}^{d_{(i,j)}} = \text{constant}\,. \qquad (4)$$

By using Proposition 3. the problem of searching for $\mathrm{Ad}(K)$-invariant Einstein metrics on G/H reduces to a Lagrange-type problem for the scalar curvature functional S under the restriction (4).

4. Jensen's metrics

As a first simple illustration of Proposition 3. we will show that Jensen's metrics ([8]) on the quaternionic Stiefel manifold $Sp(k_1+k_2)/Sp(k_2)$ can be obtained. We apply Proposition 1., formula (3) for $s=1$ and $t=1$. Then the scalar curvature reduces to

$$S = \frac{k_1(k_1+1)(2k_1+1)}{4(n+1)}\frac{1}{x_1} + 2\frac{k_1k_2}{x_{12}} - \frac{1}{4(n+1)}k_1k_2(2k_1+1)\frac{x_1}{{x_{12}}^2}.$$

The volume condition (4) is $V = x_1^{d_1}x_{12}^{d_{12}} = $ constant. By use of Lagrange method we obtain the equation

$$2(k_1+1)x_{12}^2 - 4(k_1+k_2+1)x_1x_{12} + (2k_1+2k_2+1)x_1^2 = 0,$$

which has two solutions

$$x_{12} = \frac{2(k_1+k_2+1) \pm \sqrt{2}\sqrt{1+k_1+2k_2+2k_1k_2+2k_2^2}}{2(k_1+1)}x_1.$$

These solutions are $Sp(k_1+k_2)\times Sp(k_1)$-invariant Einstein metrics on $Sp(k_1+k_2)/Sp(k_2)$, which were found in [8]. Note that if $k_1=1$ the above solutions simplify to $x_{12}=\frac{1}{2}x_1$ and $x_{12}=\frac{2k_2+3}{2}x_1$, which are Einstein metrics on the sphere $S^{4k_2+3} = V_1(\mathbb{H}^{k_2+1})$ (cf. [8, Example p. 612-613]). Also, the Einstein metric for the case $x_{12}=\frac{1}{2}x_1$ is the metric of constant curvature on S^{4k_2+3}.

5. New examples of Einstein metrics

In this section we will investigate $Sp(k_1+k_2+k_3)\times Sp(k_1)\times Sp(k_2)$–invariant Einstein metrics on the space $Sp(k_1+k_2+k_3)/Sp(k_3)$. Here $L' = Sp(k_1)\times Sp(k_2)$. By Lemma 2. these metrics depend on 5 parameters.

We apply Proposition 1. for $s=2$ and $t=1$ for the space $Sp(k_1+k_2+k_3)/Sp(k_3)$. By the Lagrange method we obtain the following system of equations:

$$
\begin{aligned}
&x_2x_{23}^2\Big((k_1+1)x_{12}^2x_{13}^2+k_2x_1^2x_{13}^2+k_3x_1^2x_{12}^2\Big)=x_1x_{13}^2\Big((k_2+1)x_{12}^2x_{23}^2+k_1x_2^2x_{23}^2+k_3x_2^2x_{12}^2\Big),\\
&2x_{13}\Big((k_2+1)x_{12}^2x_{23}^2+k_1x_2^2x_{23}^2+k_3x_2^2x_{12}^2\Big)=x_2x_{23}\Big(4(k_1+k_2+k_3+1)x_{12}x_{13}x_{23}\\
&-(2k_1+1)x_1x_{13}x_{23}-(2k_2+1)x_2x_{13}x_{23}+2k_3x_{12}^3-2k_3x_{12}x_{13}^2-2k_3x_{12}x_{23}^2\Big),\\
&x_{13}\Big(4(k_1+k_2+k_3+1)x_{12}x_{13}x_{23}-(2k_1+1)x_1x_{13}x_{23}\\
&-(2k_2+1)x_2x_{13}x_{23}+2k_3x_{12}^3-2k_3x_{12}x_{13}^2-2k_3x_{12}x_{23}^2\Big)=\\
&x_{12}\Big(4(k_1+k_2+k_3+1)x_{12}x_{13}x_{23}-(2k_1+1)x_1x_{12}x_{23}+2k_2x_{13}^3-2k_2x_{13}x_{12}^2-2k_2x_{13}x_{23}^2\Big),\\
&x_{23}\Big(4(k_1+k_2+k_3+1)x_{12}x_{13}x_{23}-(2k_1+1)x_1x_{12}x_{23}+2k_2x_{13}^3-2k_2x_{13}x_{12}^2-2k_2x_{13}x_{23}^2\Big)=\\
&x_{13}\Big(4(k_1+k_2+k_3+1)x_{12}x_{13}x_{23}-(2k_2+1)x_2x_{12}x_{13}+2k_1x_{23}^3-2k_1x_{23}x_{12}^2-2k_1x_{23}x_{13}^2\Big).
\end{aligned}
\tag{5}
$$

If $x_{13}=x_{23}=z$ then the above system (5) reduces to the following:

$$
\begin{aligned}
&x_1=\tfrac{2(k_1-k_2)x_{12}+(2k_2+1)x_2}{2k_1+1},\\
&x_2=\tfrac{(k_2+1)z^2x_{12}}{(2k_2+k_1+1)z^2+k_3x_{12}^2},\\
&z^4x_{12}^3(k_2+1)(2k_2+1)(k_2-k_1)\Big((2k_2+2k_1+1)z^2+2k_3x_{12}^2\Big)\Big((k_2+k_1)z^2+k_3x_{12}^2\Big)^2=0,\\
&2(k_3^2+k_1k_3)x_{12}^4-4k_3(k_3+k_2+k_1+1)zx_{12}^3+(2k_1^2+2k_2^2+4k_1k_2\\
&+4k_1k_3+10k_2k_3+2k_1+3k_2+6k_3+1)z^2x_{12}^2-(8k_2k_3+4k_1k_3+12k_2k_1+4+4k_3\\
&+8k_1+12k_2+8k_2^2+4k_1^2)z^3x_{12}+(2k_1^2+10k_2k_1+6k_1+8k_2+8k_2^2+2)z^4=0.
\end{aligned}
\tag{6}
$$

From the third equation of (6) we get that a solution exists only when $k_1=k_2$. Let $k_1=k_2=k$, $k_3=l$, $x_1=x_2=x$, $x_{12}=1$, $x_{13}=x_{23}=z$. Then the original system (5) reduces to:

$$
\begin{aligned}
&2z(x-1)\Big(((3k+1)x-k-1)z^2+lx\Big)=0,\\
&z\Big((4kx+2x-8k-4)z^2+(8k+4l+4)z-(2k+1)x-2k-2l\Big)=0.
\end{aligned}
\tag{7}
$$

If $x=1$ we obtain Jensen's solutions, so assume that $x\neq 1$. Then from the first equation of the system (7) we get $x=\frac{(k+1)z^2}{(3k+1)z^2+l}$. In particular, $0<x<1$ for any positive z. Substituting this expression to the second equation of (7) we obtain the equation $F_{Sp}(z) = 0$, where

$$\begin{aligned} F_{Sp}(z) &= 2(10k^2 + 7k + 1)z^4 - 4(6k^2 + 3kl + 5k + l + 1)z^3 \\ &+ (8k^2 + 14kl + 5k + 6l + 1)z^2 - 4l(2k + l + 1)z + 2l(k + l). \end{aligned} \tag{8}$$

If the equation $F_{Sp}(z) = 0$ has a solution $z > 0$, then we get a new $Sp(2k + l) \times Sp(k) \times Sp(k)$-invariant Einstein metric. Some results of calculation are presented in Table 1. However, we can show that there exists an infinite series of new homogeneous Einstein manifolds, as the next proposition shows

Proposition 4. *If $l \geq k \geq 1$ then the quaternionic Stiefel manifold $Sp(2k+l)/Sp(l)$ admits at least four $Sp(2k + l) \times Sp(k) \times Sp(k)$-invariant Einstein metrics.*

Proof. We consider the polynomial (8). Then $F_{Sp}(0) = 2l(k + l) > 0$, $F_{Sp}(z) \to \infty$ as $z \to \infty$, and $F_{Sp}(1) = 4k^2 - 4kl - k - 2l - 1 - 2l^2 < 0$ for $l \geq k$, so $F_{Sp}(z) = 0$ has two positive solutions. From the above discussion these solutions are Einstein metrics, which are different from Jensen's Einstein metrics. Thus the result follows. □

Note that Proposition 4. is a special case of Theorem 1. stated in the Introduction.

k / l	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	2	2	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	2	2	2	2	2	2	0	0	0	0	0	0	0	0	0	0	0	0	0
3	2	2	2	2	2	2	2	2	2	0	0	0	0	0	0	0	0	0	0	0
4	2	2	2	2	2	2	2	2	2	2	2	2	0	0	0	0	0	0	0	0
5	2	2	2	2	2	2	2	2	2	2	2	2	2	2	0	0	0	0	0	0
6	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	0	0	0
7	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	0
8	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
9	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
10	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
11	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
12	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
13	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
14	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
15	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
16	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
17	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
18	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
19	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
20	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2

Table 1. The number of positive solutions of the equation $F_{Sp}(z) = 0$ for various (k, l). (New $Sp(2k + l) \times Sp(k) \times Sp(k)$-invariant Einstein metrics on $Sp(2k + l)/Sp(l)$).

6. Another construction for searching for Einstein metrics

As shown on Section 5 we can find by our method new invariant Einstein metrics on $Sp(k_1 + k_2 + k_3)/Sp(k_3)$, only when $k_1 = k_2$. It would be reasonable to extend this idea to the case $t=1$ and $k_1=k_2=\cdots=k_s=k$ $(s\geq 2)$, $k_{s+1}=l$. Then $n=sk+l$. If we choose $L'=(Sp(k))^s)$, then by Lemma 2. the set of $Sp(n)\times(Sp(k))^s$-invariant metrics depends on $(s^2+3s)/2$ parameters, which makes the problem difficult for big values s.

However, if we choose $L'=N_{Sp(sk)}(Sp(k))^s$, the normalizer of $(Sp(k))^s$ in $Sp(sk)$, (this is an extension of $(Sp(k))^s$ by a discrete subgroup), then the number of parameters of corresponding $Sp(n)\times L$-invariant metrics reduces to three. More precisely, the following lemma holds:

Lemma 5. *If L' is chosen as above, and $K=L'\times H'$, where $H'=Sp(l)$, then we have a decomposition of $\mathfrak{p}$ into a sum of* $\mathrm{Ad}(K)$*-invariant and* $\mathrm{Ad}(K)$*-irreducible submodules*

$$\mathfrak{p}=\widetilde{\mathfrak{p}}_1\oplus\widetilde{\mathfrak{p}}_2\oplus\widetilde{\mathfrak{p}}_3, \tag{9}$$

where $\widetilde{\mathfrak{p}}_1=\oplus_{i=1}^s\mathfrak{p}_i$, $\widetilde{\mathfrak{p}}_2=\oplus_{1\leq i<j\leq s}\mathfrak{p}_{(i,j)}$, and $\widetilde{\mathfrak{p}}_3=\oplus_{i=1}^s\mathfrak{p}_{(i,s+1)}$ (cf. (1)). The submodules $\widetilde{\mathfrak{p}}_1,\widetilde{\mathfrak{p}}_2$ and $\widetilde{\mathfrak{p}}_3$ are pairwise inequivalent, therefore any $\mathrm{Ad}(K)$*-invariant inner product of $\mathfrak{p}$ is given by*

$$(\cdot,\cdot)=x\cdot\langle\cdot,\cdot\rangle|_{\widetilde{\mathfrak{p}}_1}+y\cdot\langle\cdot,\cdot\rangle|_{\widetilde{\mathfrak{p}}_2}+z\cdot\langle\cdot,\cdot\rangle|_{\widetilde{\mathfrak{p}}_3}. \tag{10}$$

Proof. For any $1\leq i<j\leq s$ any two of the submodules $\mathfrak{p}_i$ and $\mathfrak{p}_j$ are interchanged by $\mathrm{Ad}(a)$, for some $a\in L$. Similarly, any two of $\mathfrak{p}_{(i,s+1)}$ and $\mathfrak{p}_{(j,s+1)}$ $(1\leq i,j\leq s)$ are interchanged, and any two of $\mathfrak{p}_{(i,j)}$ and $\mathfrak{p}_{(i',j')}$ $(1\leq i<j\leq s,\ 1\leq i'<j'\leq s)$. Therefore decomposition (9) follows. The other statements are obvious. □

Next, we compute the scalar curvature for metric (10).

Proposition 5. *The scalar curvature S of an* $\mathrm{Ad}(K)$*-invariant metric (10) has the form*

$$\frac{4(n+1)}{sk}\cdot S=(k+1)(2k+1)\cdot\frac{1}{x}+2(s-1)k((s+2)k+2)\cdot\frac{1}{y}+8(ks+l+1)l\cdot\frac{1}{z}$$
$$-\left((s-1)k(2k+1)\cdot\frac{x}{y^2}+(2k+1)l\cdot\frac{x}{z^2}+2(s-1)kl\cdot\frac{y}{z^2}\right) \tag{11}$$

with volume condition $x^{s(2k+1)k}y^{2s(s-1)k^2}z^{4skl}=$ constant.

Proof. Metric (10) is a special case of metric (2) for which the scalar curvature was obtained in Proposition 1.. We apply these expressions for $t=1$, $k_1=\cdots=k_s=k$, $k_{s+1}=l$, and $x_a=x$ $(1\leq a\leq s)$, $x_{a,b}=y$ $(1\leq a<b\leq s)$, $x_{a,s+1}=z$ $(1\leq a\leq s)$ to obtain

$$\sum_{a=1}^s\frac{k_a(k_a+1)(2k_a+1)}{4(n+1)}\cdot\frac{1}{x_a}=\frac{sk(k+1)(2k+1)}{4(n+1)}\cdot\frac{1}{x};$$

$$\sum_{1\le a<b\le s+t}\frac{k_ak_b}{x_{(a,b)}}=\sum_{1\le a<b\le s}\frac{k_ak_b}{x_{(a,b)}}+\sum_{1\le a\le s,b=s+1}\frac{k_ak_b}{x_{(a,b)}}=\frac{s(s-1)k^2}{2}\cdot\frac{1}{y}+skl\cdot\frac{1}{z};$$

$$\sum_{1\le a\le s,\, a+1\le b\le s+t}k_ak_b(2k_a+1)\frac{x_a}{x^2_{(a,b)}}=\sum_{1\le a<b\le s}k_ak_b(2k_a+1)\frac{x_a}{x^2_{(a,b)}}$$

$$+\sum_{1\le a\le s,b=s+1}k_ak_b(2k_a+1)\frac{x_a}{x^2_{(a,b)}}=\frac{s(s-1)k^2(2k+1)}{2}\cdot\frac{x}{y^2}+sk(2k+1)l\cdot\frac{x}{z^2};$$

$$\sum_{1\le a<b\le s}k_ak_b(2k_b+1)\frac{x_b}{x^2_{(a,b)}}=\frac{s(s-1)k^2(2k+1)}{2}\cdot\frac{x}{y^2};$$

$$\sum_{1\le a<b<c\le s+t}k_ak_bk_c\left(\frac{x_{(a,b)}}{x_{(a,c)}x_{(b,c)}}+\frac{x_{(a,c)}}{x_{(a,b)}x_{(b,c)}}+\frac{x_{(b,c)}}{x_{(a,b)}x_{(a,c)}}\right)=$$

$$\sum_{1\le a<b<c\le s}k_ak_bk_c\left(\frac{x_{(a,b)}}{x_{(a,c)}x_{(b,c)}}+\frac{x_{(a,c)}}{x_{(a,b)}x_{(b,c)}}+\frac{x_{(b,c)}}{x_{(a,b)}x_{(a,c)}}\right)$$

$$+\sum_{1\le a<b\le s,c=s+1}k_ak_bk_c\left(\frac{x_{(a,b)}}{x_{(a,c)}x_{(b,c)}}+\frac{x_{(a,c)}}{x_{(a,b)}x_{(b,c)}}+\frac{x_{(b,c)}}{x_{(a,b)}x_{(a,c)}}\right)=$$

$$\frac{s(s-1)(s-2)k^3}{2}\cdot\frac{1}{y}+\frac{s(s-1)k^2l}{2}\cdot\left(\frac{y}{z^2}+\frac{2}{y}\right),$$

so equation (11) is easily obtained. The dimensions of $\widetilde{\mathfrak{p}}_1,\widetilde{\mathfrak{p}}_2$ and $\widetilde{\mathfrak{p}}_3$ are $sk(2k+1), 2s(s-1)k^2$ and $4skl$ respectively. Thus the volume conditions are obtained, and the proof is completed. □

In order to find the critical points of the scalar curvature S for the above two cases, note that the expression $\frac{4(n+1)}{sk}\cdot S$ and the volume are of the form

$$F(x,y,z)=\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-d\frac{x}{y^2}-e\frac{x}{z^2}-f\frac{y}{z^2},\quad G(x,y,z)=x^py^qz^r \tag{12}$$

respectively, where the constants a, b, c, d, e, f, p, q and r are positive, and

$$d=\frac{pb-qa}{q+2p},\quad f=\frac{qe}{p}. \tag{13}$$

We need to consider the following problem: Find all the critical points (with positive coordinates) of $F(x,y,z)$ under the constraint $G(x,y,z)=$ constant. This is a Lagrange-type problem, and it is easy to obtain the following lemma (see details in [2]):

Lemma 6. *The critical points of the function $F(x, y, z)$ with positive x, y, z under the restriction $G(x, y, z) = \text{constant}$ satisfy the following equations:*

1) If $x = y$, then

$$r(a + d)z^2 - pcxz + e(2p + 2q + r)x^2 = 0;$$

2) If $x \neq y$, then

$$x = \frac{aqyz^2}{pfy^2 + d(q + 2p)z^2},$$

$$(2d(q + 2p) + bq)drz^4 - (q + 2p)cdqyz^3 + (2d(r + q)(q + 2p) + (r + 2p)aq)fy^2z^2 - cfpqy^3z + (r + 2q)f^2py^4 = 0.$$

If in addition $d(q + 2p) > aq$, then $y > x$.

Now let

$$P(u) = (2d(q + 2p) + bq)dru^4 - (q + 2p)cdqu^3 + (2d(r + q)(q + 2p) + (r + 2p)aq)fu^2 - cfpqu + (r + 2q)f^2p. \quad (14)$$

It is clear that the equation $P(z/y) = 0$ is equivalent to the last one of Lemma 6.. The following simple lemma will be used for the proof of the main theorems

Lemma 7. *If $P(1) < 0$ then equation $P(u) = 0$ has at least two positive solutions.*

Proof. It is evident from the facts that $P(0) = (r + 2q)f^2p > 0$ and $P(u) \to +\infty$ when $u \to +\infty$. □

7. Proof of the main results

Proof of Theorem 1. We use Lemma 6. for values of a, b, c, d, e, f, p, q and r taken from the symplectic case of Proposition 5.. If $x = y$ then we obtain the two Jensen's solutions as derived in Section 4. If $x \neq y$ then for the polynomial (14) we have that

$$P(1) = 8s^2k^4l(2k + 1)(s - 1)^2(2sk^2 - 2skl - k - 2l - 1 - 2l^2).$$

It it easy to check that $2sk^2 - 2skl - k - 2l - 1 - 2l^2 < 0$ for $l \geq k$, $s \geq 2$, thus $P(1) < 0$. By Lemma 7. the equation $P(u) = 0$ has at least two positive solutions, so obtain at least two new invariant Einstein metrics. Since in this case it is also $d(q + 2p) > aq$, then $y > x$ for these new metrics. □

Proof of Theorem 2. Fix a positive integer p and choose positive integers n, l such that $n - l$ has at least p different prime factors $a_1, a_2, \ldots, a_p$ with $a_i < l$ $(i = 1, \ldots, p)$. Take k any of the a_i's, and positive integer s so that $n - l = sk$. For this choice of

k, l, s we use Theorem 1., and obtain that the homogenous space $Sp(n)/Sp(l)$ admits at least two $Sp(n) \times (Sp(k))^s$–invariant Einstein metrics which are not invariant under the group $Sp(n) \times Sp(n-l)$, that is they are not Jensen's metrics. It is easy to see that for different choices of $k = a_i$ we obtain pairwise different metrics (because they have different full motion groups). Therefore, we obtain at least $2p$ pairwise different $Sp(n)$-invariant Einstein metrics on the quaternionic Stiefel manifold $Sp(n)/Sp(l)$. □

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◇ Andreas Arvanitoyeorgos
University of Patras,
Department of Mathematics,
GR-26500 Rion, Greece
arvanito@math.upatras.gr

◇ V.V. Dzhepko
Rubtsovsk industrial institute,
ul. Traktornaya, 2/6,
Rubtsovsk, 658207, Russia
J_Valera_V@mail.ru

◇ Yu.G. Nikonorov
Rubtsovsk industrial institute,
ul. Traktornaya, 2/6,
Rubtsovsk, 658207, Russia
nik@inst.rubtsovsk.ru