BULLETIN OF THE
GREEK MATHEMATICAL SOCIETY
Volume 53, 2007 (101–105)

On pseudo-Hilbertizable Q-normed algebras

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Received 23/11/2005 Accepted 19/04/2007

Abstract

We show that a semi-simple pseudo-Hilbertizable Q-normed algebra is finite dimensional if, and only if, the unit is a sum of mutually orthogonal indecomposable idempotents.

1991 *Mathematics subject classification*: 46C05 - 46C99
Keywords: Normed algebra, Q- algebra, pseudo-Hilbertizable algebra, idempotent element

1. Preliminaries and introduction

Let A be a complex algebra. An algebra norm on A is a linear norm $\|.\|$ satisfying $\|xy\| \leq \|x\| \|y\|$, for every $x, y \in A$. A normed unitary algebra will be called Q-normed algebra if the group $G(A)$ of its invertible elements is open. A pseudo-Hilbertizable Q-normed algebra is a Q-normed algebra $(A, \|.\|)$ on which is defined a scalar product $\langle , \rangle$ such that $\|x\|^2 = \langle x, x \rangle$, for every $x \in A$. An element $a \in A$ is called an idempotent if $a^2 = a$. An idempotent $a \neq 0$ is said to be decomposable if there are orthogonal idempotents e_1, e_2 different from zero, such that $a = e_1 + e_2$. A complex Banach algebra $(A, \|.\|)$ with identity e of norm 1 is said to be Hilbertizable in the sense of R. A. Hirschfeld [2] if $\|.\|$ is equivalent to a norm $\|.\|_H$ which turns A into a Hilbert space. All algebras considered here are complex ones. The spectral radius will be denoted by ρ_A that is, for every $a \in A$, $\rho_A(a) = \sup\{|z| : z \in Sp_A a\}$, where $Sp_A a$ denotes the spectrum of a.

In [2] R. A. Hirschfeld has proved that a semi-simple Hilbertizable algebra $(A, \|.\|)$ is isomorphic to the field of complex numbers if $\|e\|_H = 1$. In this paper, we extend and generalize this result to pseudo-Hilbertizable Q-normed algebras. We prove that if $(A, \|.\|)$ is a semi-simple pseudo-Hilbertizable Q-normed algebra with a unit e, then A is finite dimensional if, and only if, the unit e is the sum of mutually orthogonal indecomposable idempotents. A counter-example (see Remark 3.5) shows that semi-simplicity is not superfluous.

2. Examples

1) Let $A = \mathbb{C}^n$, $n \geq 2$, with coordinatewise multiplication and Euclidean norm $\|.\|$. Then $(A, \|.\|)$ is a complex Hilbertizable Banach algebra. Notice that this algebra admits an identity $e = (1, 1, ...1)$ with $\|e\| = \sqrt{n}$.

2) The classical H^*-algebras of W. Ambrose [3] are complex Hilbertizable Banach algebras. A special examples are the algebra of Hilbert-Schmidt operators in a fixed Hilbert space, as well as the convolution algebra $L^2(G)$ of a compact group G.

3) Let $\omega : \mathbb{R} \longrightarrow \mathbb{R}$ be a measurable non negative and locally integrable function such that

$$\omega^{-1} * \omega^{-1} \leq \omega^{-1}.$$

We consider the space $L^2_\omega(\mathbb{R})$ of all equivalence classes (under equality almost every where) f such that $|f|^2 \omega$ is a Lebesgue integrable function on $\mathbb{R}$, where the same symbol f is used to denote both a function and its equivalent class. The space $L^2_\omega(\mathbb{R})$ endowed with the norm

$$\|f\|_\omega = \left(\int_{\mathbb{R}} |f(t)|^2 \omega(t) dt \right)^{\frac{1}{2}}$$

becomes a Banach space. If f and g are complex functions in $\mathbb{R}$, their convolution $f * g$ is defined by

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y)\omega(y) dy$$

provided that the integral exists for all (or at least for almost all). Now using Cauchy-Schwarz inequality and the fact that $\omega^{-1} * \omega^{-1} \leq \omega^{-1}$, one has

$$\|f * g\|_\omega \leq \|f\|_\omega \|g\|_\omega, \quad f, g \in L^2_\omega(\mathbb{R}).$$

This implies that the space $\left(L^2_\omega(\mathbb{R}), \|.\|_\omega\right)$ endowed with convolution as the product is a complex Banach algebra. Moreover, if f, $g \in L^2_\omega(\mathbb{R})$, then $f\sqrt{\omega}$, $g\sqrt{\omega} \in L^2(\mathbb{R})$ and the expression

$$\langle f, g\rangle_\omega = \int_{\mathbb{R}} f(t)\overline{g(t)}\omega(t) dt, \quad f, g \in L^2_\omega(\mathbb{R})$$

is an inner product in $L^2_\omega(\mathbb{R})$ such that

$$\|f\|^2_\omega = \langle f, f\rangle_\omega, \text{ for every } f \in L^2_\omega(\mathbb{R}).$$

Whence $\left(L^2_\omega(\mathbb{R}), \|.\|_\omega\right)$ is a complex Hilbertizable Banach algebra. Notice that this example contains, in particular, a concrete example used in the theory of Sobolev spaces. Indeed, for $s > \frac{1}{2}$, put

$$\omega_s(t) = \left(1 + |t|^2\right)^s \text{ for every } t \in \mathbb{R}.$$

By a simple calculation, the reader can prove that

$$\omega_s^{-1} * \omega_s^{-1} \leq c_s \omega_s^{-1},$$

where

$$c_s = \left(2^{1+2s} \int_{\mathbb{R}} \omega_s^{-1}(t)dt\right) < +\infty$$

is a positive constant depending only on s. As above, we obtain

$$\|f * g\|_{\omega_s} \leq c_s \|f\|_{\omega_s} \|g\|_{\omega_s}, \quad f, g \in L^2_{\omega_s}(\mathbb{R}).$$

Therefore, without loss of generality, we may suppose that $\left(L^2_{\omega_s}(\mathbb{R}), \|.\|_{\omega_s}\right)$ is a complex Hilbertizable Banach algebra.

3. Hilbertizable Q-normed algebras

The main result of this note is the following.

Theorem 3.1 *Let $(A, \|.\|)$ be a complex semi-simple Hilbertizable Q-normed algebra with unit e. If there exists a net of mutually orthogonal indecomposable idempotents $\{e_\alpha : \alpha \in \Lambda\}$ in A such that $\|e_\alpha\| = 1$, for every $\alpha \in \Lambda$, and $e = \sum_{\alpha\in\Lambda} e_\alpha$, then*

1) *$e_\alpha A e_\alpha = \mathbb{C}e_\alpha$, for every $\alpha \in \Lambda$,*

2) *$\dim(e_\alpha A e_\beta) \leq 1$, for every $\alpha \neq \beta$,*

3) *A is finite dimensional.*

Proof. **1)** Observe first that $A_\alpha = e_\alpha A e_\alpha$ is a closed subalgebra of A, having e_α as a unit element, such that

$$Sp_{A_\alpha} a \cup \{0\} = Sp_A a, \text{ for every } a \in A_\alpha. \tag{1}$$

Let $x \in A_\alpha$ such that $\langle x, e_\alpha\rangle \neq 0$. Then $\left\langle \langle x, e_\alpha\rangle^{-1} x, e_\alpha \right\rangle \neq 1$. So, replacing x by $\langle x, e_\alpha\rangle^{-1} x$, we may suppose, without loss of generality, that $\langle x, e_\alpha\rangle = 1$. For $t = \|x\|^{-2}$, we have $\|e_\alpha - tx\|^2 = 1 - 2t + t^2 \|x\|^2 < 1$. Since A is a Q-normed algebra, we have $\rho_{A_\alpha}(e_\alpha - tx) < 1$ by (1) (see also [4: Theorem 4.1 and Corollary 4.1]). Hence tx is invertible in A_α and so x itself. Now put $N_\alpha = \{x \in A_\alpha : \langle x, e_\alpha\rangle = 0\}$ and let I be a maximal left ideal of A_α. Then $I \subset N_\alpha$. It follows that I is the unique maximal left ideal of A_α. Indeed let $J \neq I$ be another maximal left ideal of A_α, then $I + J \subset N_\alpha$, but $I + J = A_\alpha$, thus $e_\alpha \in I$; contradiction. Therefore $\{0\}$ is a maximal left ideal of A_α. Whence A_α is a normed division algebra. Therefore $A_\alpha = \{\lambda e_\alpha : \lambda \in \mathbb{C}\}$ by the theorem of Gelfand-Mazur.

2) For each $\alpha \in \Lambda$, it is clear that $e_\alpha A$ is a minimal right ideal, say, X_α. Thus X_α is an irreducible right A-module. Moreover the subset $\mathcal{D}$ of $L(X_\alpha)$, given by

$$\mathcal{D} = \{T \in L(X_\alpha) : aT(x) = T(ax), \ a \in A, x \in X_\alpha\}$$

is a normed division algebra over $\mathbb{C}$, and therefore, $\mathcal{D} = \mathbb{C}$, by the theorem of Gelfand-Mazur. Suppose now that $\alpha \neq \beta$ and there exist x_1,$x_2 \in Ae_\beta$ with $e_\alpha x_1, e_\alpha x_2$ linearly independent. Then, by theorem 24.10 of [1], there exists $a \in A$ with $ae_\alpha x_1 = e_\alpha x_1$ and $ae_\alpha x_2 = 0$. Therefore $e_\alpha ae_\alpha x_1 = e_\alpha^2 x_1 = e_\alpha x_1 \neq 0$, $e_\alpha ae_\alpha \neq 0$, $Ae_\alpha ae_\alpha = Ae_\alpha$. But then $e_\alpha x_2 \in Ae_\alpha x_2 = Ae_\alpha ae_\alpha x_2 = \{0\}$, which contradicts the linear independence of $e_\alpha x_1$, $e_\alpha x_2$. This implies that $\dim (e_\alpha Ae_\beta) \leq 1$.

3) Since $e = \sum_{\alpha\in\Lambda} e_\alpha$ and $\{e_\alpha : \alpha \in \Lambda\}$ are mutually orthogonal idempotents in A, we get $\|e\|^2 = \sum_{\alpha\in\Lambda} \|e_\alpha\|^2$. It follows that Λ is finite. Whence A is finite dimensional. □

As a consequence of **1.** of Theorem **3.1**, one has the following.

Corollary 3.2 *Let $(A, \|.\|)$ be a complex semi-simple Hilbertizable Q-normed algebra with unit e. If $\|e\| = 1$, then A is isomorphic to the field of complex numbers.*

In the commutative case, we have $e_\alpha Ae_\beta = \{0\}$, for every $\alpha \neq \beta$. It follows that $a = \sum_{\alpha\in\Lambda} e_\alpha ae_\alpha$, for every $a \in A$. So, by **1)** of theorem **3.1**, there exist $(\lambda_\alpha)_{\alpha\in\Lambda}$ such that $a = \sum_{\alpha\in\Lambda} \lambda_\alpha e_\alpha$ and $\|a\|^2 = \sum_{\alpha\in\Lambda} |\lambda_\alpha|^2$. This implies that $a \longmapsto (\lambda_\alpha)_{\alpha\in\Lambda}$ is a bijective mapping of A onto $\mathbb{C}^n$, where $n = card\,(\Lambda)$ and we have the following.

Corollary 3.3 *Let $(A, \|.\|)$ be a commutative complex semi-simple Hilbertizable Q-normed algebra with identity e. If there exists a net of mutually orthogonal indecomposable idempotents $\{e_\alpha : \alpha \in \Lambda\}$ in A such that $\|e_\alpha\| = 1$, for every $\alpha \in \Lambda$, and $e = \sum_{\alpha\in\Lambda} e_\alpha$, then A is isomorphic to $\mathbb{C}^n$, where $n = card\,(\Lambda)$.*

Remark 3.4 It is not always true that $e_\alpha Ae_\beta = \{0\}$, for every $\alpha \neq \beta$, as the following example shows. Let $A = \mathcal{M}_2(\mathbb{C})$ be the algebra of 2×2-matrices endowed with the scalar product

$$\langle M, N\rangle = tr\,(MN^*)\,, \quad \text{for every } M, N \in A.$$

It is clear that $e = e_1 + e_2$, $e_1e_2 = e_2e_1 = 0$, where

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

But $e_1Ae_2 \neq \{0\}$.

Remark 3.5 The assumption that A is semi-simple is not superfluousas it is pointed out in [2]. Indeed, just consider $A = \mathbb{C}$ with the product zero and consider $A^1 = A\oplus\mathbb{C}$ the unitization of A endowed with Hilbertizable structure product. It is clear that the identity e, of A^1, has norm 1, but $A^1 \neq \mathbb{C}$.

Acknowledgment. I wish to express my sincere thanks to Professor Anastasios Mallios for interesting discussion while he was visiting the E.N.S. of Rabat in March 2007. I am also indebted to the referee for careful reading of the manuscript and for valuable suggestions.

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