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On a nonlinear Volterra integrodifferential equation involving iterated integrals

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Abstract

The aim of this paper is to study the existence, uniqueness and other properties of solutions of a certain Volterra integrodifferential equation involving iterated integrals. The main tools employed in the analysis are based on the variation of constants formula, application of the Leray-Schauder alternative and a certain integral inequality which provides explicit bound on the unknown function.

Keywords: Volterra integrodifferential equation, iterated integrals, variation of constants formula, Leray-Schauder alternative, integral inequality, global existence, behavior of solutions.
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1. Introduction

Consider the nonlinear iterated Volterra integrodifferential system of the form

$$x'(t) = A(t)\,x(t) + f(t, x(t), K(t, x)),\; x(0) = x_0, \tag{1.1}$$

as a perturbation of the linear system

$$y'(t) = A(t)\,y(t),\; y(0) = x_0, \tag{1.2}$$

for $0 \le t < \infty$, where A is a $n \times n$ matrix,

$$K(t, x) = \int_0^t g(t, \sigma, x(\sigma), L(t, \sigma, x))\, d\sigma, \tag{1.3}$$

in which

$$L(t, \sigma, x) = \int_0^\sigma h(t, \sigma, \tau, x(\tau))d\tau, \tag{1.4}$$

and x, y, f, g, h are real n-dimensional vectors. Let R^n denotes the real n-dimensional Euclidean space of column vectors. The symbol $|.|$ will denote the norm in R^n or $n \times n$ matrix norm depending on whether is applied to vector or matrix. Let $J = [0, T]$ ($T > 0$ is a constant), $R_+ = [0, \infty)$ be the given subsets of R, the set of real numbers and $C(S_1, S_2)$ denotes the class of continuous functions from the set S_1 to the set S_2 . For $0 \leq \tau \leq \sigma \leq t < \infty$, we assume that $h \in C\left(R_+^3 \times R^n, R^n\right)$, $g \in C\left(R_+^2 \times R^n \times R^n, R^n\right)$, $f \in C\left(R_+ \times R^n \times R^n, R^n\right)$ and $A(t)$ is a continuous $n \times n$ matrix on R_+

Many papers have been devoted to the study of behavioral relationships between the solutions of the special versions of equation (1.1) and the solutions of equation (1.2) by using different techniques. The equation considered in (1.1) is in the general sprit of the investigations of Constantin [1], Éshmatov [3], Loginov [5], Pachpatte [7], Talpalaru [10] (see also [4]) and others. The purpose of this paper is to study the existence, uniqueness and other properties of the solutions of equation (1.1) treated as a perturbation of equation (1.2) under some suitable assumptions on the functions involved therein. The variation of constants formula, a simple and classical application of the Leray-Schauder alternative and a certain integral inequality with explicit estimate are used to establish the results.

2. Global existence

Before giving the main result concerning the global existence of solutions of equation (1.1), we note that, our approach and arguments led us to make use of the variation of constants formula, namely, any solution $x(t)$ of equation (1.1) considered as a perturbation of equation (1.2) can be represented by the equivalent integral equation

$$x(t) = Y(t) Y^{-1}(0) x_0 + \int_0^t Y(t) Y^{-1}(s) f(s, x(s), K(s, x)) ds, \tag{2.1}$$

where $Y(t)$ is the fundamental solution matrix of equation (1.2) such that $Y(0) = I$, the identity matrix.

We shall use the following form of the topological transversality theorem given by Granas in [2, p. 61] which is also known as Leray-Schauder alternative. For an excellent account on the applications of the topological transversality method, see the survay paper by Ntyouyas [6].

Lemma 1. *Let B be a convex subset of a normed linear space E and assume $0 \in B$. Let $S : B \to B$ be a completely continuous operator and let $U(S) = \{x \in B : x = \lambda Sx\}$ for some $0 < \lambda < 1$. Then either $U(S)$ is unbounded or S has a fixed point.*

We are now ready to state and prove the following theorem which deals with the global existence of solutions of equation (1.1).

Theorem 1. *Let $Y(t)$ be the fundamental solution matrix of equation (1.2) such that*

$$\left|Y(t)\,Y^{-1}(s)\right| \leq M, \tag{2.2}$$

for $0 \leq s \leq t \leq T$, where M is a positive constant and let $c = M\left|x_0\right|$. Suppose that the functions f, g, h in equation (1.1) satisfy the conditions

$$\left|f(t, x(t), K(t,x))\right| \leq p(t)\, w_1(|x(t)|) + |K(t,x)|, \tag{2.3}$$

$$\left|g(t,\sigma,x(\sigma), L(t,\sigma,x))\right| \leq q(t,\sigma)\, w_2(|x(\sigma)|) + |L(t,\sigma,x)|, \tag{2.4}$$

$$\left|h(t,\sigma,\tau,x(\tau))\right| \leq r(t,\sigma,\tau)\, w_3(|x(\tau)|), \tag{2.5}$$

where $p(t) \in C(R_+, R_+)$ and for $0 \leq \tau \leq \sigma \leq \infty$, $q(t,\sigma) \in C\left(R_+^2, R_+\right)$, $r(t,\sigma,\tau) \in C\left(R_+^3, R_+\right)$ and for $i = 1,2,3$, $w_i : R_+ \to (0,\infty)$ are continuous and nondecreasing functions. Let $w(u) = \max\{w_1(u), w_2(u), w_3(u)\}$. Then the equation (1.1) has a solution defined on J provided T satisfies

$$M\int_0^T D(s)\,ds < \int_0^\infty \frac{ds}{w(s)}, \tag{2.6}$$

where

$$D(t) = p(t) + \int_0^t \left\{ q(t,\sigma) + \int_0^\sigma r(t,\sigma,\tau)\,d\tau \right\} d\sigma. \tag{2.7}$$

Proof. First we establish the priori bounds for the solutions of the problem

$$x'(t) = A(t)\,x(t) + \lambda f(t, x(t), K(t,x)),\ x(0) = x_0, \tag{2.8}$$

for $t \in J$ and $\lambda \in (0,1)$. If $x(t)$ is a solution of equation (2.8), then it satisfies the equivalent integral equation

$$x(t) = y(t) + \lambda \int_0^t Y(t)\,Y^{-1}(s)\, f(s, x(s), K(s,x))\,ds, \tag{2.9}$$

where $y(t) = Y(t)\,Y^{-1}(0)\,x_0$ is a solution of equation (1.2). From (2.9) and using the hypotheses (2.2)-(2.5) we have

$$|x(t)| \leq |y(t)| + \int_0^t \left|Y(t)\,Y^{-1}(s)\right| \left|f(s, x(s), K(s,x))\right| ds$$

$$\leq c + M \int_0^t \left[p(s)\, w_1(|x(s)|) + \int_0^s \left\{ q(s,\sigma)\, w_2(|x(\sigma)|) + \int_0^\sigma r(s,\sigma,\tau)\, w_3(|x(\tau)|)\, d\tau \right\} d\sigma \right] ds. \tag{2.10}$$

Define a function $u(t)$ by the right hand side of (2.10), then $|x(t)| \leq u(t), u(0) = c$ and

$$u'(t) = M \left[p(t)\, w_1(|x(t)|) + \int_0^t \left\{ q(s,\sigma)\, w_2(|x(\sigma)|) + \int_0^\sigma r(s,\sigma,\tau)\, w_3(|x(\tau)|)\, d\tau \right\} d\sigma \right]$$

$$\leq M \left[p(t)\, w_1(u(t)) + \int_0^t \left\{ q(s,\sigma)\, w_2(u(\sigma)) + \int_0^\sigma r(s,\sigma,\tau)\, w_3(u(\tau))\, d\tau \right\} d\sigma \right]$$

$$\leq MD(t)\, w(u(t)),$$

i.e.

$$\frac{u'(t)}{w(u(t))} \leq MD(t). \tag{2.11}$$

Integration of (2.11) from 0 to $t \in J$ and the use of the change of variable and the condition (2.6) gives

$$\int_c^{u(t)} \frac{ds}{w(s)} \leq M \int_0^t D(s)\, ds \leq M \int_0^T D(s)\, ds < \int_c^\infty \frac{ds}{w(s)}. \tag{2.12}$$

From (2.12) we conclude that there is a constant Q independent of $\lambda \in (0,1)$ such that $u(t) \leq Q$ for $t \in J$ and consequently $\|x\| = \sup_{t \in J} |x(t)| \leq Q$.

We define $B = C(J, R^n)$ be the Banach space of all continuous functions endowed with sup-norm as above and rewrite the initial value problem (1.1) as follows. If $z \in B$ and $x(t) = y(t) + z(t)$, then it is easy to see that $z(t)$ satisfies

$$z(0) = z_0 = 0,$$

$$z(t) = \int_0^t Y(t) Y^{-1}(s) f(s, y(s) + z(s), K(s, y + z))\, ds,$$

if and only if $x(t)$ satisfies equation (1.1). Define $S : B_0 \to B_0, B_0 = \{z \in B : z_0 = 0\}$ by

$$Sz(t) = \int_0^t Y(t) Y^{-1}(s) f(s, y(s) + z(s), K(s, y + z))\, ds, \tag{2.13}$$

for $t \in J$. Clearly S is continuous. Next, we prove that S is completely continuous. Let $\{a_m\}$ be a bounded sequence in B_0, i.e. $\|a_m\| \le d$ for all m, where d is a positive constant. From (2.13), using the hypotheses and letting $\bar{D} = \sup_{t \in J} \{D(t)\}$ we have

$$\begin{aligned}
|Sa_m(t)| &\le \int_0^t M \Bigg[p(s) w_1(c + |a_m(s)|) \\
&\quad + \int_0^s \Bigg\{ q(s, \sigma) w_2(c + |a_m(\sigma)|) \\
&\quad + \int_0^\sigma r(s, \sigma, \tau) w_3(c + |a_m(\tau)|)\, d\tau \Bigg\} d\sigma \Bigg] ds \\
&\le \int_0^t M \Bigg[p(s) w_1(c + d) + \int_0^s \Bigg\{ q(s, \sigma) w_2(c + d) \\
&\quad + \int_0^\sigma r(s, \sigma, \tau) w_3(c + d)\, d\tau \Bigg\} d\sigma \Bigg] ds \\
&\le Mw(c + d) \int_0^t D(s)\, ds \\
&\le Mw(c + d) \int_0^T D(s)\, ds \\
&\le Mw(c + d) \bar{D} T.
\end{aligned}$$

Consequently, we have

$$\|Sa_m\| \le Mw(c + d) \bar{D} T.$$

This means that $\{Sa_m\}$ is uniformly bounded.

Now, we shall show that $\{Sa_m\}$ is equicontinuous. Let $0 \leq t_1 \leq t_2 \leq T$. Then from (2.13), using the hypotheses and letting $\bar{D} = \sup_{t \in J} \{D(t)\}$ we have

$$|Sa_m(t_2) - Sa_m(t_1)|$$

$$= \left| \int_{t_1}^{t_2} Y(t_2) Y^{-1}(s) f(s, y(s) + a_m(s), K(s, y + a_m)) \, ds \right.$$

$$\left. + \int_0^{t_1} [Y(t_2) - Y(t_1)] Y^{-1}(s) f(s, y(s) + a_m(s), K(s, y + a_m)) \, ds \right|$$

$$\leq \int_{t_1}^{t_2} \left| Y(t_2) Y^{-1}(s) \right| |f(s, y(s) + a_m(s), K(s, y + a_m))| \, ds$$

$$+ \int_0^{t_1} |Y(t_2) - Y(t_1)| \left| Y^{-1}(s) \right| |f(s, y(s) + a_m(s), K(s, y + a_m))| \, ds$$

$$\leq \int_{t_1}^{t_2} M \left[\quad p(s) w_1(|y(s)| + |a_m(s)|) \right.$$

$$+ \int_0^s \left\{ \quad q(s, \sigma) w_2(|y(\sigma)| + |a_m(\sigma)|) \right.$$

$$\left. \left. + \int_0^\sigma r(s, \sigma, \tau) w_3(|y(\tau)| + |a_m(\tau)|) \, d\tau \right\} d\sigma \right] ds$$

$$+ \int_0^{t_1} |Y(t_2) - Y(t_1)| M \left[\quad p(s) w_1(|y(s)| + |a_m(s)|) \right.$$

$$+ \int_0^s \left\{ \quad q(s, \sigma) w_2(|y(\sigma)| + |a_m(\sigma)|) \right.$$

$$\left. \left. + \int_0^\sigma r(s, \sigma, \tau) w_3(|y(\tau)| + |a_m(\tau)|) \, d\tau \right\} d\sigma \right] ds$$

$$\leq \int_{t_1}^{t_2} M \left[\quad p(s) w_1(c + d) + \int_0^s \left\{ \quad q(s, \sigma) w_2(c + d) \right. \right.$$

$$+\int_0^{\sigma} r(s,\sigma,\tau)\, w_3(c+d)\, d\tau \Bigg\} d\sigma \Bigg] ds$$

$$+\int_0^T |Y(t_2)-Y(t_1)|\, M\Bigg[\quad p(s)\, w_1(c+d) + \int_0^s \Bigg\{ \quad q(s,\sigma)\, w_2(c+d)$$

$$+\int_0^{\sigma} r(s,\sigma,\tau)\, w_3(c+d)\, d\tau \Bigg\} d\sigma \Bigg] ds$$

$$\le \int_{t_1}^{t_2} Mw(c+d)\, D(s)\, ds + \int_0^T Mw(c+d)\, D(s)\, |Y(t_2)-Y(t_1)|\, ds$$

$$\le \int_{t_1}^{t_2} Mw(c+d)\, \bar{D} ds + \int_0^T Mw(c+d)\, \bar{D}\, |Y(t_2)-Y(t_1)|\, ds \tag{2.14}$$

From (2.14) and by virtue of the continuity of $Y(t), t \in J$, we conclude that $\{Sa_m\}$ is equicontinuous and hence by Arzela-Ascoli theorem the operator S is completely continuous.

Moreover, the set $U(S) = \{z \in B_0 : z = \lambda Sz; \lambda \in (0,1)\}$ is bounded, since for every z in $U(S)$ the function $x(t) = y(t) + z(t)$ is a solution of (2.9), for which we have proved that $\|x\| \le Q$ and hence $\|z\| \le Q + c$. Now an application of Lemma 1, the operator S has a fixed point in B_0 . This means that the problem (2.8) has a solution $x(t)$ on J . The proof is complete. □

Remark 1. If we take $MD(t) = 1$ in (2.6) and the right hand side in (2.6) is assumed to diverse, then the solution of equation (1.1) exists for every $T < \infty$, that is on the entire interval R_+.

3. Behavior of solutions

In this section we study the behavior of solutions of equation (1.1) under some suitable conditions on the functions involved in equation (1.1) and the fundamental solution matrix of equation (1.2). In our subsequent discussion we assume that the solutions to equation (1.1) exist on R_+.

We need the following inequality due to Bykov and Salpagarov (see [9, Theorem 1.4.2, p. 32]) in the analysis which follows. For detailed account on such inequalities, see [8,9].

Lemma 2. *Let* $u(t), p(t) \in C(R_+, R_+)$ *and for* $0 \le \tau \le \sigma \le t < \infty$, $q(t,\sigma) \in C(R_+^2, R_+)$, $r(t,\sigma,\tau) \in C(R_+^3, R_+)$. *If*

$$u(t) \le k + \int_0^t \left[p(s) u(s) + \int_0^s \left\{ q(s,\sigma) u(\sigma) + \int_0^\sigma r(s,\sigma,\tau) u(\tau) d\tau \right\} d\sigma \right] ds, \tag{3.1}$$

for $t \in R_+$, *where* $k \ge 0$ *is a constant, then*

$$u(t) \le k \exp\left(\int_0^t F(s) ds \right), \tag{3.2}$$

for $t \in R_+$, *where*

$$F(t) = p(t) + \int_0^t \left\{ q(t,\sigma) + \int_0^\sigma r(t,\sigma,\tau) d\tau \right\} d\sigma. \tag{3.3}$$

The following theorem deals with the uniqueness of solutions of equation (1.1).

Theorem 2. *Let* $Y(t)$ *be the fundamental solution matrix of equation (1.2) such that*

$$\left| Y(t) Y^{-1}(s) \right| \le N, \tag{3.4}$$

for $0 \le s \le t < \infty$, *where* N *is a positive constant. Suppose that the functions* f, g, h *in equation (1.1) satisfy the conditions*

$$|f(t, x(t), u_1) - f(t, y(t), u_2)| \le p(t) |x(t) - y(t)| + |u_1 - u_2|, \tag{3.5}$$

$$|g(t, \sigma, x(\sigma), v_1) - g(t, \sigma, y(\sigma), v_2)| \le q(t,\sigma) |x(\sigma) - y(\sigma)| + |v_1 - v_2|, \tag{3.6}$$

$$|h(t, \sigma, \tau, x(\tau)) - h(t, \sigma, \tau, y(\tau))| \le r(t,\sigma,\tau) |x(\tau) - y(\tau)|, \tag{3.7}$$

where p, q, r *are as in Lemma 2 and*

$$\int_0^\infty F(s) ds < \infty, \tag{3.8}$$

in which $F(t)$ *is given by (3.3). Then the equation (1.1) has at most one solution on* R_+.

Proof. Let $x_1(t)$ and $x_2(t)$ for $t \in R_+$ be two solutions of equation (1.1). Then we have

$$x_1(t) - x_2(t) = \int_0^t Y(t) Y^{-1}(s) \{ f(s, x_1(s), K(s, x_1)) - f(s, x_2(s), K(s, x_2)) \} ds. \tag{3.9}$$

From (3.9) and using the hypotheses we have

$$|x_1(t) - x_2(t)| \le \int_0^t N \Bigg[p(s) |x_1(s) - x_2(s)| + \int_0^s \Bigg\{ q(s, \sigma) |x_1(\sigma) - x_2(\sigma)| + \int_0^\sigma r(s, \sigma, \tau) |x_1(\tau) - x_2(\tau)| d\tau \Bigg\} d\sigma \Bigg] ds. \tag{3.10}$$

Now a suitable application of Lemma 2 (when $k = 0$) to (3.10) yields $x_1(t) = x_2(t)$, that is, the equation (1.1) has at most one solution on R_+. □

The next theorem deals with the boundedness of solutions of equation (1.1).

Theorem 3. *Let $Y(t)$ be the fundamental solution matrix of equation (1.2) satisfying the condition (3.4). Suppose that the functions f, g, h in equation (1.1) satisfy the conditions*

$$|f(t, x(t), u)| \le p(t) |x(t)| + |u|, \tag{3.11}$$

$$|g(t, \sigma, x(\sigma), v)| \le q(t, \sigma) |x(\sigma)| + |v|, \tag{3.12}$$

$$|h(t, \sigma, \tau, x(\tau))| \le r(t, \sigma, \tau) |x(\tau)|, \tag{3.13}$$

where p, q, r are as in Lemma 2 and the condition (3.8) holds. Then all solutions of equation (1.1) are bounded on R_+.

Proof. Any solution $x(t)$, $t \in R_+$ of equation (1.1) can be represented by the equivalent integral equation (2.1). From (2.1) and using the hypotheses we have

$$|x(t)| \le N |x_0| + \int_0^t N \Bigg[p(s) |x(s)| + \int_0^s \Bigg\{ q(s, \sigma) |x(\sigma)| + \int_0^\sigma r(s, \sigma, \tau) |x(\tau)| d\tau \Bigg\} d\sigma \Bigg] ds. \tag{3.14}$$

Now an application of Lemma 2 to (3.14) yields

$$|x(t)| \le N|x_0| \exp\left(\int_0^t F(s)\,ds\right).$$

The above estimation, in view of the assumption (3.8) implies the boundedness of all solutions of equation (1.1) on R_+. □

Our Theorem 4 below demonstrates that all solutions of equation (1.1) approach zero as $t \to \infty$.

Theorem 4. *Let $Y(t)$ be the fundamental solution matrix of equation (1.2) such that*

$$\left|Y(t)\,Y^{-1}(s)\right| \le He^{-\alpha(t-s)}, \tag{3.15}$$

for $0 \le s \le t < \infty$, where H and α are positive constants. Suppose that the functions f, g, h in equation (1.1) satisfy the conditions (3.11)-(3.13) and

$$\int_0^\infty \left[p(s) + \int_0^s \left\{q(s,\sigma)\,e^{\alpha(s-\sigma)} + \int_0^\sigma r(s,\sigma,\tau)\,e^{\alpha(s-\tau)}d\tau\right\}d\sigma\right]ds < \infty, \tag{3.16}$$

holds. Then all solutions of equation (1.1) approach zero as $t \to \infty$.

Proof. Any solution $x(t)$, $t \in R_+$ of equation (1.1) can be represented by the equivalent integral equation (2.1). From (2.1) and using the hypotheses (3.15), (3.11)-(3.13) we have

$$\begin{aligned}|x(t)| &\le \left|Y(t)\,Y^{-1}(0)\right||x_0| + \int_0^t \left|Y(t)\,Y^{-1}(s)\right| |f(s,x(s),K(s,x))|\,ds \\ &\le He^{-\alpha t}|x_0| + \int_0^t He^{-\alpha(t-s)}\left[p(s)|x(s)| + |K(s,x)|\right]ds \\ &\le He^{-\alpha t}|x_0| + \int_0^t He^{-\alpha(t-s)}\left[p(s)|x(s)| + \int_0^s \left\{q(s,\sigma)|x(\sigma)|\right.\right. \\ &\qquad \left.\left. + \int_0^\sigma r(s,\sigma,\tau)|x(\tau)|\,d\tau\right\}d\sigma\right]ds.\end{aligned}$$

The above inequality can be written as

$$|x(t)|\,e^{\alpha t} \le H|x_0| + \int_0^t H\left[p(s)|x(s)|\,e^{\alpha s} + \int_0^s \left\{q(s,\sigma)\,e^{\alpha(s-\sigma)}|x(\sigma)|\,e^{\alpha\sigma}\right.\right.$$

$$+\int_0^{\sigma} r(s,\sigma,\tau)\, e^{\alpha(s-\tau)}\, |x(\tau)|\, e^{\alpha\tau} d\tau \Bigg\}\, d\sigma \Bigg]\, ds. \tag{3.17}$$

Now applying Lemma 2 with $u(t) = |x(t)|\, e^{\alpha t}$ to (3.17) and rewriting we obtain

$$|x(t)| \leq H\, |x_0|\, e^{-\alpha t} \exp\left(\int_0^t \left[\quad p(s) + \int_0^s \left\{ \quad q(s,\sigma)\, e^{\alpha(s-\sigma)} \right.\right.\right.$$

$$\left.\left.\left. + \int_0^{\sigma} r(s,\sigma,\tau)\, e^{\alpha(s-\tau)} d\tau \right\} d\sigma \right] ds \right). \tag{3.18}$$

The estimation (3.18) in view of the hypotheses (3.16) yields the desired result. The proof is complete. □

A continuous function $z(t)$, $t \in R_+$ will be called slowly growing if and only if for every $\varepsilon > 0$ there exists a constant N, which may depend on ε such that $|z(t)| \leq N e^{\varepsilon t}$, $t \in R_+$.

Finally, we shall give the following theorem which demonstrates that all the solutions of equation (1.1) grow more slowely than any positive exponential.

Theorem 5. *Let $Y(t)$ be the fundamental solution matrix of equation (1.2) such that*

$$\left|Y(t)\, Y^{-1}(s)\right| \leq H e^{\alpha(t-s)}, \tag{3.19}$$

for $0 \leq s \leq t < \infty$, where H and α are positive constants. Suppose that the functions f, g, h in equation (1.1) satisfy the conditions (3.11)-(3.13) and

$$\int_0^{\infty} \left[p(s) + \int_0^s \left\{ q(s,\sigma)\, e^{-\alpha(s-\sigma)} + \int_0^{\sigma} r(s,\sigma,\tau)\, e^{-\alpha(s-\tau)} d\tau \right\} d\sigma \right] ds < \infty, \tag{3.20}$$

holds. Then all solutions of equation (1.1) are slowely growing.

The proof of this theorem follows by the similar arguments as in the proof of Theorem 4 with suitable modifications. Here we omit the details.

Remark 2. We note that, one can easily extend the ideas of this paper to the equations of the form (1.1) when the functions f, g, h involved in equation (1.1) depends on the retarded arguments, under appropriate changes. For the study of such equations, see [6] and [9, p.185].

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