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Between closed maps and g-closed maps

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Abstract

In this paper we define g^*-closed maps and obtain new characterizations of normal spaces and regular spaces. It is shown that the continuous and g^*-closed surjection (resp. open continuous and g^*-closed surjection) images of normal (resp. regular) spaces are normal (resp. regular).

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1. Introduction

In 1970, N.Levine [10] introduced the notion of generalized closed sets (briefly g-closed) of a topological space. These sets were also considered by various authors (e.g. W. Dunham and N. Levine [8]. J. Dontchev [7], M. Caldas [5] and T. Noiri, H. Maki and J. Umehara [14]). S.R. Malghan [13] defined a clase of maps called g-closed maps by using g-closed sets and study some of their fundamental properties. In this direction Veera Kumar [17] defined the concept of g^*-closed sets in terms of g-closed sets. In the same paper he also introduced and studied g^*-continuous maps.

In the same spirit of S.R. Malghan [13], we introduce a new class of maps, namely g^*-closed maps as a natural dual to the class of g^*-continuous maps, which is properly placed between the class of closed maps and the class of g-closed maps. We show that this new class is a stronger form of the class of αg-closed maps, the class of gs-closed maps, the class of gp-closed maps, the class of rg-closed maps and in the class of gpr-closed maps and a weak form of the class of δ-closed maps. We show also that g^*-closedness is independent of β-closedness. As applications of this new class, we obtain new characterizations of normal spaces and regular spaces, show that under the continuous g^*-closed surjection the image of a normal space is normal and that regularity is preserved under open continuous g^*-closed surjection maps. Moreover we introduce and study the concepts of G^*O-compactness and G^*O- connectedness by involving g^*-open sets and we show that if a topological space (X,τ) is $T^*_{\frac{1}{2}}$, then the notions of connectedness and G^*O-connectedness are equivalent with each other.

Throughout the present paper spaces (X,τ), (Y,σ) and (Z,γ) (or simply X, Y and Z) always mean topological spaces on which no separation axiom is assumed unless explicitly stated. We denote the closure and interior of a subset A of a space (X,τ) by $Cl(A)$ and $Int(A)$ respectively.

2. Preliminaries

Since we shall require the following known definitions, notations, we recall them in this section.

Recall that $pCl(A)$ (resp. the $sCl(A)$, the $\alpha Cl(A)$) of a subset A of (X,τ) is the intersection of all preclosed sets (resp. semi-closed sets, α-closed sets) that contain A.

Definition 1. A subset A of a topological space (X,τ) is called:
(1) A semi preopen set [2] (=β-open set [1]) if $A \subset Cl(Int(Cl(A)))$.
(2) A δ-closed set [16] if $A = Cl_\delta(A)$ where $Cl_\delta(A) = \{x \in X : Int(Cl(U)) \cap A \neq \emptyset, U \in \tau \text{ and } x \in U\}$.
(3) A generalized closed set (briefly g-closed) [10] if $Cl(A) \subset U$ whenever $A \subset U$ and U is a subset open of X.
(4) A generalized open set (briefly g-open) [10] if $Int(A) \supset F$ whenever $A \supset F$ and F is a subset closed of X.
(5) A regular generalized closed set (briefly rg-closed) [15] if $Cl(A) \subset U$ whenever $A \subset U$ and U is a subset regular open of X.
(6) An α generalized closed set (briefly αg-closed) [11] if $\alpha Cl(A) \subset U$ whenever $A \subset U$ and U is a subset open of X.
(7) A generalized semi-closed set (briefly gs-closed) [3] if $sCl(A) \subset U$ whenever $A \subset U$ and U is a subset open of X.
(8) A generalized preclosed set (briefly gp-closed) [12] if $pCl(A) \subset U$ whenever $A \subset U$ and U is a subset open of X.
(9) A generalized preregular closed set (briefly gpr-closed) [9] if $pCl(A) \subset U$ whenever $A \subset U$ and U is a subset regular open of X.
(10) A g^*-closed set [17] if $Cl(A) \subset U$ whenever $A \subset U$ and U is a subset g-open of X.
(11) A g^*-open set [17] if its complement is a g^*-closed subset of X, equivalently, if $Int(A) \supset F$ whenever $A \supset F$ and F is a subset g-closed of X.

The class of all g^*-closed sets (resp. g-closed sets, closed sets) of (X,τ) is denoted of $G^*C(X,\tau)$ (resp. $GC(X,\tau)$, $C(X,\tau)$).

Definition 2. A map $f : (X,\tau) \to (Y,\sigma)$ is said to be:
(1) generalized semi-closed map (briefly as gs-closed map) [6] if for each closed set F of X, $f(F)$ is gs-closed in Y.

(2) generalized closed map (briefly as g-closed map) [13] if for each closed set F of X, $f(F)$ is g-closed in Y.
(3) generalized preclosed map (briefly as gp-closed map) [14] if for each closed set F of X, $f(F)$ is gp-closed in Y.
(4) δ-closed map if for each closed set F of X, $f(F)$ is δ-closed in Y.
(5) g^*-continuous [17] if $f^{-1}(G)$ is a g^*-closed subset of X for every closed subset G of Y.
(6) g^*-irresolute [17] if $f^{-1}(G)$ is a g^*-closed subset of X for every g^*-closed subset G of Y.
(7) gc-irresolute [4] if $f^{-1}(G)$ is a g-closed subset of X for every g-closed subset G of Y.
(8) β-closed map [1] if for each closed set F of X, $f(F)$ is β-closed in Y.

3. g^*-Closed maps and g^*-Open maps

Definition 3. A map $f : (X, \tau) \to (Y, \sigma)$ is called:
(1) g^*-closed if for each closed subset A of X, $f(A)$ is g^*-closed in Y.
(2) g^*-open if for each open subset A of X, $f(A)$ is g^*-open in Y.

Theorem 3.1 *Every closed map is g^*-closed and every g^*-closed map is g-closed, i.e., the class of the g^*-closed maps is properly placed between the class of closed maps and the class of g-closed maps.*

The converse of the above theorem is not true in general as it can be seen from the following example.

Example 3.2 Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{a, c\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = a$, $f(b) = c$ and $f(c) = b$. Then f is not a closed map since $\{a, c\}$ is a closed set of (X, τ) but $f(\{a, c\}) = \{a, b\}$ is not closed set of (Y, σ). However f is g^*-closed

Remark 3.3 (1) The Theorem 2.1 shows also that the class of all g^*-closed maps is properly contained in the class of αg-closed maps, the class of gs-closed maps, the class of gp-closed maps, the class of rg-closed maps and in the class of gpr-closed maps, but the converse is not true.
(2) Since every δ-closed set is a g^*-closed set [17], we have that the class of g^*-closed maps properly contains the class of δ-closed maps.

Example 3.4 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. Let $f : (X, \tau) \to (X, \tau)$ be defined by $f(a) = b$, $f(b) = a$ and $f(c) = c$. We have that $GC(X, \tau) = P(X, \tau)$ (where $P(X, \tau)$ is the power set of X) and $G^*C(X, \tau) = C(X, \tau)$. Then f is a g-closed map and hence αg-closed, gs-closed, gp-closed, rg-closed and gpr-closed. But f is not a g^*-closed map.

A similar result of Theorem 3.1 we can obtain for the class of g^*-open maps. Also the set $B = \{b\}$ in the Example 3.2 is an open set of (X, τ) but $f(b) = c$ is not an open set of (Y, σ). Thus f is not an open map. However f is g^*-open.

Remark 3.5 g^*-closedness and β-closedness are independent of each other as the next two example show.

Example 3.6 Let X, Y, τ, σ and f be as in the Example 3.2, Then $C(X, \tau) = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$, and $G^*C(X, \tau) = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, Y\}$. f is not a β-closed map since $\{a, c\}$ is a closed set of (X, τ), but $f(\{a, c\}) = \{a, b\}$ is not a β-closed set of (Y, σ). However f is g^*-closed.

Example 3.7 Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{a, c\}, Y$ Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then f is not a g^*-closed map since $\{c\}$ is a closed set of (X, τ) but $f(\{c\}) = \{c\}$ is not g^*-closed set of (Y, σ). However f is β-closed.

The relationships between these new classes of maps and other corresponding types of maps are shown in the following diagram:

$$\begin{array}{ccccccc}
 & & \delta\text{-closed map} & & rg\text{-closed map} & \longrightarrow & gpr\text{-closed map} \\
 & \swarrow & \downarrow & \nearrow & \uparrow & & \uparrow \\
\text{closed map} & \longrightarrow & g^*\text{-closed map} & \longrightarrow & g\text{-closed map} & & gp\text{-closed map} \\
 & \searrow & \not\updownarrow & & \downarrow & \searrow & \uparrow \\
 & & \beta\text{-closed map} & & gs\text{-closed map} & \longleftarrow & \alpha g\text{-closed map}
\end{array}$$

It is well known that if $f : (X, \tau) \to (Y, \sigma)$ is a closed (resp. open) map, then for any subset A in (Y, σ) and any open (resp. closed) set O in (X, τ) containing $f^{-1}(A)$, there exists an open (resp. closed) $B \supset A$ such that $f^{-1}(B) \subset O$.

The following theorem is a version of it.

Theorem 3.8 *A map $f : (X, \tau) \to (Y, \sigma)$ is g^*-closed (resp. g^*-open) if and only if for each subset S of Y and for any open (resp. closed) set A in (X, τ) containing $f^{-1}(S)$, there exists a g^*-open (resp. g^*-closed) set B of Y such that $B \supset S$ and $f^{-1}(B) \subset A$.*

Proof. Necessity: Let S be a subset of Y and A be an open set of X such that $f^{-1}(S) \subset A$. Then $Y - f(X - A)$, say B is a g^*-open set containing S such that $f^{-1}(B) \subset A$.

Sufficiency: Let F be a closed set of X. Then $f^{-1}(Y - f(F)) \subset X - F$ and $X - F$ is open. By hypothesis there is a g^*-open set B of Y such that $Y - f(F) \subset B$ and $f^{-1}(B) \subset X - F$. Therefore we have $F \subset X - f^{-1}(B)$ and hence $Y - B \subset f(F) \subset f(X - f^{-1}(B)) \subset Y - B$, which implies $f(F) = Y - B$. Since $Y - B$ is g^*-closed, $f(F)$ is g^*-closed and thus f is a g^*-closed map.

The proof is similar for the case of an g^*-open map.

Theorem 3.9 *If $f : (X, \tau) \to (Y, \sigma)$ is gc-irresolute g^*-closed and A is g^*-closed in X, then $f(A)$ is g^*-closed in Y.*

Proof. If $f(A) \subset B$ where B is a g-open set of Y, then $A \subset f^{-1}(B)$ and hence $Cl(A) \subset f^{-1}(B)$ since A is g^*-closed. Thus $f(Cl(A)) \subset B$ and $f(Cl(A))$ is a g^*-closed set. Then $Cl(f(Cl(A))) \subset B$. It follows that $Cl(f(A)) \subset Cl(f(Cl(A))) \subset B$. Then $Cl(f(A)) \subset B$ and $f(A)$ is g^*-closed.

The fact that closed maps are g^*-closed maps gives that the Theorem 5.19 of Veera Kumar [17] is a consequence of Theorem 3.9.

Remark 3.10 The image of a g^*-open set under a *gc*-irresolute and g^*-closed map is not necessarily g^*-open as is shown by the following example.

Example 3.11 Let $X = \{a\}$ and $Y = \{b, c\}$, $\tau = \{\emptyset, X\}$ and $\sigma = \{\emptyset, \{b\}, Y\}$. Let $f(a) = c$. Then f is a *gc*-irresolute g^*-closed map. However $\{a\}$ is a g^*-open subset of (X, τ) but $f(\{a\}) = \{c\}$ is not g^*-open subset of (Y, σ).

Theorem 3.12 *If A is a g^*-open set of (X, τ) and if $f : (X, \tau) \to (Y, \sigma)$ is bijective gc-irresolute and g^*-closed , then $f(A)$ is g^*-open in Y.*

Proof. If A is any g^*-open set of (X, τ) then $X - A$ is g^*-closed. By Theorem 3.9 also $f(X - A) = f(X) - f(A) = Y - f(A)$ is g^*-closed. Thus $f(A)$ is g^*-open in Y.

N.Levine in ([10], Theorem 6.3) showed that if $f : (X, \tau) \to (Y, \sigma)$ is continuous and closed and if A is a g-open (or a g-closed) subset of Y, then $f^{-1}(A)$ is g-open (or g-closed) in X. The following theorem is a version of this Theorem.

Theorem 3.13 *If $f : (X, \tau) \to (Y, \sigma)$ is g^*-continuous and pre g-closed (i.e., if for each g-closed set U of X, $f(U)$ is g-closed in Y) and if G is a g^*-open (or g^*-closed) subset of Y, then $f^{-1}(G)$ is g^*-open (or g^*-closed) in X.*

Proof. Let G be a g^*-open subset of Y. Let $F \subset f^{-1}(G)$ where F is g-closed in X. Therefore $f(F) \subset G$ holds. Since $f(F)$ is g-closed i and G is g^*-open in Y, $f(F) \subset Int(G)$ holds. Hence $F \subset f^{-1}(Int(G))$. Since f is g^*-continuous and $Int(G)$

is open in Y, $F \subset Int(f^{-1}(Int(G))) \subset Int(f^{-1}(G))$. Therefore $f^{-1}(G)$ is g^*-open in X.

By taking complements, we can show that if G is a g^*-closed in Y, then $f^{-1}(G)$ is g^*-closed in X.

Theorem 3.14 *If $f : (X, \tau) \to (Y, \sigma)$ is bijective g^*-continuous and pre g-open (i.e., if for each g-open set U of X, $f(U)$ is g-open in Y) and if B is a g^*-closed subset of Y, then $f^{-1}(B)$ is g^*-closed in X.*

Proof. Suppose B is is a g^*-closed subset of Y and that $f^{-1}(B) \subset A$ where A is g-open in X. We will shown that $Cl(f^{-1}(B)) \subset A$. Since f is surjective, $B \subset f(A)$ holds. Hence $Cl(B) \subset f(A)$ because $f(A)$ is g-open and B is g^*-closed in Y . Therefore $f^{-1}(Cl(B)) \subset A$. Since f is g^*-continuous and $Cl(B)$ is closed in Y, $Cl(f^{-1}(Cl(B))) \subset A$ and so $Cl(f^{-1}(B)) \subset A$. Therefore $f^{-1}(B)$ is g^*-closed in X.

Corollary 3.15 *If $f : (X, \tau) \to (Y, \sigma)$ is bijective g^*-continuous and pre g-open and if A is a g^*-open subset of Y, then $f^{-1}(A)$ is g^*-open in X.*

The composition map of two g^*-closed (g^*-open) maps is not always g^*-closed (g^*-open). But the following theorem holds, the proof of which can be carried out easily.

Theorem 3.16 *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \gamma)$ be two functions. Then*
(1) $g \circ f$ is g^*-open (g^* -closed) if f is open (closed) and g is g^*-open (g^*-closed).
(2) $g \circ f$ is g^*-closed If f is g^*-closed and g is gc-irresolute and g^*-closed.
(3) $g \circ f$ is g^*-open If f is g^*-open and g is bijective gc-irresolute and g^*-closed.
(4) Let (Y, σ) be T^*. Then $g \circ f$ is g^*-closed if f and g are g^*-closed.

Theorem 3.17 *Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \gamma)$ be two maps and let $g \circ f : (X, \tau) \to (Z, \gamma)$ a g^*-closed (g^*-open) map. Then:*
(1) If f is surjective and continuous, then g is g^*-closed (g^*-open).
(2) If g is bijective g^*-continuous and pre g-open, then f is g^*-closed (g^*-open).

Proof. (1) suppose B is an arbitrary closed set of Y. Then $f^{-1}(B)$ is closed in X because f is continuous. Since $g \circ f$ is g^*-closed and f is surjective, $g \circ f(f^{-1}(B)) = g(f(f^{-1}(B)) = g(B)$ is g^*-closed in (Z, γ). This implies that g is g^*-closed map.
(2) Suppose F is an arbitrary closed set in (X, τ). Then $(g \circ f)(F)$ is g^*-closed in (Z, γ) because $g \circ f$ is g^*-closed. Since g is injective, we have $g^{-1}[(g \circ f)(F)] = f(F)$. It follows immediately from Theorem 3.14, that $f(F)$ is g^*-closed in (Y, σ). This implies that f is g^*-closed.

The proof is similar for g^*-open.

Corollary 3.18 *If $f : (X, \tau) \to (Y, \sigma)$ is a g^*-continuous and pre g-closed map and $g : (Y, \sigma) \to (Z, \gamma)$ is g^*-continuous, then $g \circ f : (X, \tau) \to (Z, \gamma)$ is g^*-continuous.*

Theorem 3.19 *(1) If $f : (X, \tau) \to (Y, \sigma)$ is a g^*-open (resp. g^*-closed) and $A = f^{-1}(B)$ for some open (resp. closed) set B in Y, then the restriction $f_A : A \to Y$ is g^*-open (resp. g^*-closed).*
(2) If $f : (X, \tau) \to (Y, \sigma)$ is bijective g^*-irresolute and g^*-closed and A is a g^*-open subset of X, then the restriction $f_A : A \to Y$ is g^*-open.
(3) If $f : (X, \tau) \to (Y, \sigma)$ is a g^*-open (resp. g^*-closed) and A is an open (resp. closed) subset of X, then the restriction $f_A : A \to Y$ is g^*-open (resp. g^*-closed).

Proof. (1) Let H be an open set of A. Then there is an open set G of X such that $H = G \cap A$. Therefore $f_A(H) = f(H) = f(G \cap A) = f(G \cap f^{-1}(B)) = f(G) \cap B$. Since f is g^*-open and G is open in X, $f(G)$ is g^*-open in Y, so $f(G) \cap B$ is g^*-open in Y. Thus f_A is g^*-open.
The proof for g^*-closed subset is same as above.
(2) Follows from Theorem 3.13.
(3) Obvious.

Recall that, a space X is said to be normal if for any pair of disjoint closed sets A, B of X, there exist disjoint open sets U, V such that $A \subset U$ and $B \subset V$.

By using g^*-open sets, we obtain some new characterizations of normal spaces.

Theorem 3.20 *The followings properties are equivalent for a space X.*
(1) X is normal.
(2) For any pair of disjoint closed sets A, B of X, there exist disjoint g^*-open sets U, V such that $A \subset U$ and $B \subset V$.
(3) For any pair of disjoint closed sets A, B of X, there exist disjoint g-open sets U, V such that $A \subset U$ and $B \subset V$.
(4) For any closed sets A and any open set V containing A, there exists g-open sets U such that $A \subset U \subset Cl(U) \subset V$.

Proof. (1) $\to$ (2) $\to$ (3): The proof are obvious since every open set is g^*-open and every g^*-open set is g-open.
(3) $\to$ (4): Let A be any closed set and V an open set containing A. Since A and $X - V$ are disjoint closed sets of X, there exist g-open sets U, W of X such that $A \subset U$ and $X - V \subset W$ and $U \cap W = \emptyset$. By definition we have $X - V \subset Int(W)$. Since $U \cap Int(W) = \emptyset$, we have $Cl(U) \cap Int(W) = \emptyset$ and hence $Cl(U) \subset X - Int(W) \subset V$, therefore, we obtain $A \subset U \subset Cl(U) \subset V$.
(4) $\to$ (1): Let A and B be any disjoint closed sets of X. Since $X - B$ is an open set containing A, there exists a g-open set G such that $A \subset G \subset Cl(G) \subset X - B$. By definition, we have $A \subset Int(G)$. Put $U = Int(G)$ and $V = X - Cl(G)$. Then U and

V are disjoint open sets, such that $A \subset U$ and $B \subset V$. Therefore X is normal.

Recall that, a space X is said to be regular if for each closed set F of X and each point $x \in X - F$, there exist disjoint open sets U and V such that $F \subset U$ and $x \in V$.

Theorem 3.21 *The followings properties are equivalent for a space X.*
(1) X is regular.
(2) For each closed set F of X and each point $x \in X - F$, there exists an open set U and a g-open set V such that $x \in U$, $F \subset V$ and $U \cap V = \emptyset$.
(3) For each subset A of X and each closed set F such that $A \cap F = \emptyset$, there exist an open set U of X and a g-open set V such that $A \cap U \neq \emptyset$, $F \subset V$ and $U \cap V = \emptyset$.

Proof. (1) $\rightarrow$ (2) : The proof is obvious since every open set is g-open.
(2) $\rightarrow$ (3) : Let A be a subset of X and F a closed set of X such that $A \cap F = \emptyset$. For a point $x \in A$, $x \in X - F$ and hence there exist an open set U and a g-open set V such that $x \in U$, $F \subset V$ and $U \cap V = \emptyset$.
(3) $\rightarrow$ (1) : Let F be any closed set of X and $x \in X - F$. Then $\{x\} \cap F = \emptyset$ and there exists an open set U of X and a g-open set W such that $x \in U$, $F \subset W$ and $U \cap W = \emptyset$. Put $V = Int(W)$, then by definition we have $F \subset V$, where V is a subset open of X and $U \cap V = \emptyset$. Therefore X is regular.

It is known that normality (resp. regularity) is preserved under a continuous and g-closed surjection ([13], Theorem 1.12) (resp. an open continuous and g-closed surjection ([13], Theorem 1.13)) The following two corollary are a consequence of it.

Corollary 3.22 *If $f : (X,\tau) \rightarrow (Y,\sigma)$ is a continuous, g^*-closed surjection from a normal space (X,τ) to a space (Y,σ), then (Y,σ) is normal.*

Corollary 3.23 *If $f : (X,\tau) \rightarrow (Y,\sigma)$ is an open continuous, g^*-closed surjection from a regular space (X,τ) to a space (Y,σ), then (Y,σ) is regular.*

4. Applications of maps and g^*-closed sets.

Recall [17], that a space (X,τ) is called a:
(i) $T^*_{\frac{1}{2}}$ space if every g^*-closed set of (X,τ) is a closed set, equivalently, every singleton of (X,τ) is either g-closed or open.
(ii) ${}^*T_{\frac{1}{2}}$ space if every g-closed set of (X,τ) is a g^*-closed set.
(iii) $T_{\frac{1}{2}}$ space if and only if it is ${}^*T_{\frac{1}{2}}$ and $T^*_{\frac{1}{2}}$.

Theorem 4.1 *Let $f : (X,\tau) \rightarrow (Y,\sigma)$ be a pre-g-closed and open bijection. If (X,τ) is a $T^*_{\frac{1}{2}}$-space, then (Y,σ) is also $T^*_{\frac{1}{2}}$-space.*

Proof. Let $y \in (Y, \sigma)$. Since (X, τ) is a $T^*_{\frac{1}{2}}$-space and since f is bijective then for some $x \in X$ with $f(x) = y$, we have that $\{x\}$ is open or g-closed. If $\{x\}$ is open then $\{y\} = f(\{x\})$ is open since f is open. If $\{x\}$ is g-closed then $\{y\}$ is g-closed, since f is pre-g-closed.

Theorem 4.2 *Let $f : (X, \tau) \to (Y, \sigma)$ be a gc-irresolute and g^*-closed bijection. If X is a $^*T_{\frac{1}{2}}$ -space, then (Y, σ) is also $^*T_{\frac{1}{2}}$-space.*

Proof. Assume that B is a g-closed subset of (Y, σ). Then $f^{-1}(B)$ is g-closed in X since f is gc-irresolute. But every g-closed set is g^*-closed in X. Therefore $f^{-1}(B)$ is g^*-closed in X . Hence by Theorem 3.9 B is g^*-closed. Therefore (Y, σ) is a $^*T_{\frac{1}{2}}$-space.

Theorem 4.3 *Let $f : (X, \tau) \to (Y, \sigma)$ be a g^*-closed. Then f is closed, if (Y, σ) is $T^*_{\frac{1}{2}}$.*

Definition 4. A For any bijection $f : (X, \tau) \to (Y, \sigma)$ is called a g^*-homeomorphism if f is both g^*-continuous and g^*-open map.

Every homeomorhhism is g^*-homeomorphism and the converse is not true as seen from the following example.

Example 4.4 Let f as in Example 3.2. Then f is a g^*-homeomorphism but it is not a homeomorphism.

We shall characterize g^*-homeomorphisms and g^*-open maps. The proofs are obvious and the proofs are omitted.

Theorem 4.5 *Let $f : (X, \tau) \to (Y, \sigma)$ be a bijective map. Then following statements are equivalent.*
(i) $f^{-1} : (Y, \sigma) \to (X, \tau)$ is g^*-continuous.
(ii)f is g^*-open.
(iii) f is g^*-closed.

Theorem 4.6 *Let $f : (X, \tau) \to (Y, \sigma)$ be a bijective and g^*–continuous map. Then the following statements are equivalent.*
(i) f is a g^*-open map.
(ii) f is a g^*-homeomorphism.
(iii) f is a g^*-closed map.

Now we introduce a class of maps which are included in the class of g^*-homeomorphisms. Moreover, the class of maps is closed under the composition of maps.

Definition 5. A bijection $f : (X, \tau) \to (Y, \sigma)$ is said to be g^*I-homeomorphism if both f and f^{-1} are g^*-irresolute. We say that spaces (X, τ) and (Y, σ) are g^*I-homeomorphic if there exists a g^*I-homeomorphism from (X, τ) in (Y, σ).

The family of all g^*I-homeomorphism (resp. g^*-homeomorphism) from (X, τ) onto itself is denoted by $g^*Ih(X, \tau)$ (resp. $g^*h(X, \tau)$).

Theorem 4.7 *The $g^*Ih(X, \tau)$ is a group and $g^*Ih(X, \tau) \subset g^*h(X, \tau)$.*

Proof. A binary operation $\mu : g^*Ih(X, \tau) \times g^*Ih(X, \tau) \to g^*Ih(X, \tau)$ is well defined by $\mu(f, h) = h \circ f$ (the composition of maps) for f and $h \in g^*Ih(X, \tau)$, because the composition of two g^*-irresolute maps is g^*-irresolute. It follows from definitions that $g^*Ih(X, \tau)$ is a group with the binary operation μ. Since every closed set is g^*-closed, $g^*Ih(X, \tau)$ is a subset of $g^*h(X, \tau)$.

Theorem 4.8 *If $f : (X, \tau) \to (Y, \sigma)$ is a g^*I-homeomorphism, then f induces an isomorphism from the group $g^*Ih(X, \tau)$ onto $g^*Ih(X, \tau)$.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a g^*I-homeomorphism. Then the homeomorphism $f_* : g^*Ih(X, \tau) \to g^*Ih(X, \tau)$ is induced by $f_*(h) = f \circ h \circ f^{-1}$ for every $h \in g^*Ih(X, \tau)$. Then, by usual arguments it is proved that f_* is an isomorphism.

A subset B of a topological space (X, τ) is said to be G^*O-compact relative to X if for every cover $\{A_i : i \in \Omega\}$ of B by g^*-open subsets of (X, τ), i.e., $B \subset \bigcup\{A_i : i \in \Omega\}$ where A_i $(i \in \Omega)$ are g^*-open subsets of (X, τ), there exists a finite Ω_0 of Ω such that $B \subset \bigcup\{A_i : i \in \Omega_0\}$

Theorem 4.9 *Let $f : (X, \tau) \to (Y, \sigma)$ be a map and let B be a G^*O-compact set relative to (X, τ). Then,*
(i) If f is g^*-continuous, then $f(B)$ is compact in (Y, σ).
(ii) If f is g^*-irresolute, then $f(B)$ is G^*O-compact in (Y, σ).

Proof. (i) Let $\{U_i : i \in \Omega\}$ be any collection of open subsets of (Y, σ) such that $f(B) \subset \bigcup\{U_i : i \in \Omega\}$. Then $B \subset \bigcup\{f^{-1}(U_i) : i \in \Omega\}$ holds and there exist a finite subset Ω_0 of Ω such that $B \subset \bigcup\{f^{-1}(U_i) : i \in \Omega_0\}$ which shows that $f(B)$ is G^*O-compact in (Y, σ).
(ii) Analogous to (i).

A topological space (X, τ) is said to be G^*O-connected if X cannot be written as a disjoint union of two non-empty g^*-open sets. A subset of X is G^*O-connected if it is G^*O-connected as a subspace.

Theorem 4.10 *For a topological space (X, τ), the following are equivalent:*
(i) (X, τ) is G^*O-connected.

(ii) The only subsets of (X,τ) which are both g^*-open and g^*-closed are the empty set $\emptyset$ and X.

Proof. $(i) \to (ii)$: Let U be a g^*-open and g^*-closed subset of X. Then $X - U$ is both g^*-open and g^*-closed. Since X is the disjoint union of the g^*-open sets U and $X - U$, one of these must be empty, that is $U = \emptyset$ or $U = X$.
$(ii) \to (i)$: suppose that $X = A \cup B$ where A and B are disjoint non-empty g^*-open subsets of X. Then A is both g^*-open and g^*-closed. By assumption, $A = \emptyset$ or X. Therefore X is G^*O-connected.
It is obvious that every G^*O-connected space is connected. The following example shows that the converse is not true.

Example 4.11 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, X\}$. Then the topological space (X,τ) is connected. However, since $\{c\}$ is both g^*-open and g^*-closed, X is not G^*O-connected by Theorem 3.10.

Theorem 4.12 *Suppose that (X,τ) is a $T^*_{\frac{1}{2}}$ topological space. Then, X is connected if and only if X is G^*O-connected.*

Proof. It is obvious from definitions.

Theorem 4.13 *(i) If $f : (X,\tau) \to (Y,\sigma)$ is g^*-continuous surjection and X is G^*O-connected then Y is connected.*
(ii) If $f : (X,\tau) \to (Y,\sigma)$ is g^*-irresolute surjection and X is G^*O-connected then Y is G^*O-connected.

Proof. (i) Suppose that Y is not connected. Let $Y = A \cup B$ where A and B are disjoint non-empty open sets in Y. Since f is g^*-continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty and g^*-open in X. This contradics the fact that X is g^*-connected. Hence Y is connected.
(ii) It follows from the definitions.

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