BULLETIN OF THE
GREEK MATHEMATICAL SOCIETY
Volume 53, 2007 (147–149)

Formulae for the kth Prime Number

Panayiotis G. Tsangaris

Received 28/03/2006 Accepted 10/06/2006

Introduction

In this note we first prove an explicit formula for the kth prime (Theorem 1), which improves a previous result of S. Regimbal [2]. Then we determine the smallest prime which is greater than a given natural number (Theorem 2), using a method which extends a previous idea of R. Ernvall [1].

Theorem 1. *Let k be a natural number. Then*

$$p_k = \sum_{m=2}^{k^2} \left[\frac{1}{1 + \left| k - \left[\frac{1}{\sum_{d=1}^{[\sqrt{m}]} \left[\frac{[m/d]}{m/d} \right]} \right] \sum_{n=2}^{m} \left[\frac{1}{\sum_{d=1}^{[\sqrt{n}]} \left[\frac{[n/d]}{n/d} \right]} \right] \right|} \right] m,$$

where [ ] *is the integral part function.*

Proof. The proof of Theorem has its origin to Regimbal's proof. We have:

$$\left[\frac{[n/d]}{n/d} \right] = \begin{cases} 1 & \text{if} \quad d \mid n \\ 0 & \text{if} \quad d \nmid n. \end{cases}$$

Let

$$f(n) = \sum_{d=1}^{[\sqrt{n}]} \left[\frac{[n/d]}{n/d} \right].$$

Mathematics subject classification: 11A51

A natural number $n > 1$ is composite if and only if it has a divisor d such that $1 < d \leqq [\sqrt{n}]$. Therefore

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is a prime} \\ 0 & \text{if } n \text{ is composite.} \end{cases}$$

We also define the function h by $h(0) = h(1) = 0$ and $h(n) = [1/f(n)]$ for $n > 1$. Then h is the characteristic function for the set of all prime numbers.

Let $\pi(m)$ be the number of primes $\leqq m$. Then

$$\pi(m) = \sum_{n=2}^{m} h(n).$$

Also

$$h(m)\pi(m) = \begin{cases} \pi(m), & \text{if } n \text{ is a prime} \\ 0, & \text{if } m \text{ is composite.} \end{cases}$$

If k is a natural number, then

$$\left[\frac{1}{1+\mid k - h(m)\pi(m) \mid}\right] = \begin{cases} 1, & \text{if } k = h(m)\pi(m) \\ 0, & \text{if } k \neq h(m)\pi(m). \end{cases}$$

Therefore

$$\left[\frac{1}{1+\mid k - h(m)\pi(m) \mid}\right] m = \begin{cases} m, & \text{if } m \text{ is the } k\text{th prime} \\ 0, & \text{otherwise.} \end{cases}$$

J. Rosser and L. Schoenfeld [2] have proved that $p_k < k(\log + \log\log k)$ for $k > 5$. Hence, $p_k < k^2$ for $k \geqq 3$. Consequently

$$p_k = \sum_{m=2}^{k^2} \left[\frac{1}{1+\mid k - h(m)\pi(m) \mid}\right] m,$$

which proves the Theorem. □

Theorem 2. (The Smallest Prime Greater than a Given Positive Integer) *Let m be a natural number with $m \geqq 2$. Let $p_1, p_2, \ldots, p_k$ be the primes not exceeding m. Let $c = \binom{2m}{m} / p_1^{a_1} \cdots p_k^{a_k}$, where $a_i = \sum_{s=1}^{\infty} ([2m/p_i^s] - 2[m/p_i^s])$ for each $i = 1, 2, \ldots, k$, (or let $c = (2m)!/p_1^{b_1} \cdots p_k^{b_k}$, where $b_i = \sum_{1 \leqq s \leqq [(\log 2m)/(\log p_i)]} [2m/p_i^s]$ for each $i = 1, 2, \ldots, k$). Let $t = c^c/(c^c, c!)$. Let r be the height of c in t, i.e. $c^r | t$ but $c^{r+1} \nmid t$. Then, the smallest prime greater than m is equal to the natural number*

$$c/(t/c^r, c).$$

Consequently, if $p_1, p_2, \ldots, p_k, \ldots$ *is the sequence of all primes, then*

$$p_{k+1} = c/(t/c^r, c),$$

where c, t, r *etc are defined as above, by putting* $m = p_k$.

Proof. Let $q_1, \ldots, q_n$ be all possible prime numbers in increasing order such that $m < q_i < 2m$ for any $i = 1, 2, \ldots, n$. It is easily seen that $c = q_1 q_2 \cdots q_n$. Let a_i be the height of q_i in $c!$. Then

$$a_i = \sum_{s=1}^{\infty} [c/q_i^s] < \sum_{s=1}^{\infty} c/q_i^s < c. \tag{1}$$

Also, the following hold true:

$$[c/q_{i+1}] = c/q_{i+1} < c/q_i = [c/q_i] \quad \text{for} \quad 1 \leqq i \leqq n-1 \tag{2}$$

$$[c/q_{i+1}^s] \leqq [c/q_i^s] \quad \text{for} \quad s = 2, 3, \ldots. \tag{3}$$

From (1), (2) and (3) we obtain:

$$c > a_1 > a_2 > \cdots > a_n > 0.$$

Hence $(c^c, c!) = q_1^{a_1} \cdots q_n^{a_n}$. Thus $t = q_1^{e_1} \cdots q_n^{e_n}$, where $e_i = c - a_i$ and consequently: $0 < e_1 < \cdots < e_n$. Moreover, $c^r \| t$ implies $r = e_1$. Therefore $t/c^r = q_2^{e_2 - e_1} \cdots q_n^{e_n - e_1}$ and $(t/c^r, c) = q_2 \ldots q_n$, where $c/(t/c^r, c) = q_1$. Finally,

$$p_{k+1} = c/(t/c^r, c),$$

which ends the proof. □

References

[1] R. Ernvall, *A formula for the least Prime greater than a given integer*, Elem., Math. 30(1975), pp. 13–14.

[2] S. Regimbal, *An Explicit Formula for the kth Prime Number*, Math. Mag. 48(1975), pp. 230–232.

[3] J.B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. 6(1972), pp. 64–94.

◇ Panayiotis G. Tsangaris
Department of Mathematics,
Athens University,
15784 Panepistimiopolis,
Athens, GREECE
ptsangaris@math.uoa.gr