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Infinite Matrices and almost Convergence

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Abstract

The main purpose of this paper is to characterize the class of matrices $(c_o(p,s),\ \hat{c}_o(p))$, $(c_o(p,s),\hat{c})$,$(c_o(p,s),\ \hat{c}_o)$, $(\ell_\infty(\text{p,s}),\hat{c})$ and $(\ell_\infty(\text{p,s}),\hat{c}_o)$.

1. Introduction

Let X,Y be two nonempty subsets of the space S of all complex sequences and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}(n,k= 1,2,...). For every $x = (x_k) \in X$ and every integer n we write $A_n(x) = \sum_k a_{nk}x_k$ where the sum without limits is always taken from $k = 1$ to $k = \infty$. The sequence $Ax = (A_n(x))$, if it exists, is called the transformation of x by the matrix A. We say that $A \in (X, Y)$ if and only if $Ax \in Y$ whenever $x \in X$.

Let ℓ_∞ and c be the Banach spaces of bounded and convergent sequences $x = (x_k)$ with the usual norm $\|x\| = Sup_k\,|x_k|$ respectively. If $\text{x} = (\text{x}_k)_{k\geq 1}$ is a sequence, then define the shifting operator by $Dx = (x_k)_{k\geq 1}$, $\text{D}^2\text{x} = (\text{x}_k)_{k\geq 2}$and so on. It may be recalled that the Banach limit(see Banach [1]) L is a non-negative linear functional on ℓ_∞ such that L is invariant under shift operator(that is, $L(Dx) = L(x)\ \forall\ \ x \in \ell_\infty$) and $L(e) = 1$ where $e = (1, 1, \dots)$. A sequence $x \in \ell_\infty$ is almost convergent if all Banach limits of x coincide. Let $\hat{c}$ denotes the space of almost convergent sequences [5]. It is proved by Lorentz [5] that

$\hat{c}$ $=\{\text{x} = (\text{x}_n) : \lim_m \text{t}_{mn}(\text{x})$ exists uniformly in n $\}$

where$t_{mn}(x) = \frac{1}{m+1}\sum\limits_{i=0}^{m} D^i x_n = \frac{1}{m+1}\sum\limits_{i=0}^{m} x_{n+i}$.

If $p_k > 0$ is real and $p = (p_k)$ is such that $sup_k p_k < \infty$,and $s \geq 0$ then let us list the required sequence spaces as follows ([2], [6], [7], and [8]);

$c_o(p) = \{x = (x_k) : |x_k|^{p_k} \to 0, k \to \infty\}$,

$c_o(p, s) = \{x = (x_k) : k^{-s}\,|x_k|^{p_k} \to 0\text{as } k \to \infty,\}$,

$c_o(p, s) = \{x = (x_k) : k^{-s}\,|x_k|^{p_k} \to 0\text{as } k \to \infty,\}$, and

$\hat{c}_o(p) = \{x = (x_k) : |t_{m,n}(x)|^{p_m} \to 0\text{as}m \to \infty,\}$.

Necessary and sufficient conditions for the classes ($c_o(p,s)$,$\hat{c}$), (ℓ_∞, $\hat{c}$), ($c_o(p)$, $\hat{c}_o(p)$), ($c_o(p)$,$\hat{c}$),($c_o(p,s)$,$\hat{c}$), (ℓ_∞(p), $\hat{c}$) and (ℓ_∞(p), $\hat{c}_o$) have been characterized by King [4], Duran [3], and Nanda [7].

In the sequel, the object of this paper is to characterize the class of matrices ($c_o(p,s)$, $\hat{c}_o(p)$), ($c_o(p,s)$,$\hat{c}$),($c_o(p,s)$, $\hat{c}_o$), (ℓ_∞(p,s), $\hat{c}$) and (ℓ_∞(p,s), $\hat{c}_o$).

In Theorem 1 we determine the matrices in the class ($c_o(p,s)$, $\hat{c}_o(p)$). Theorem 2 characterizes the matrices in the class ($c_o(p,s)$,$\hat{c}$). In Theorem 3 we determine the matrices in the class (ℓ_∞(p,s), $\hat{c}$). The following notations are used throughout. For all integers $n, m \geq 1$

$t_{mn}(Ax) = \frac{1}{m+1}\sum_{i=0}^{m} A_{n+i}(x) = \sum_k a(n,k,m)x_k$ where $a(n,k,m) = \frac{1}{m+1}\sum_{i=0}^{m} a_{n+i,k}$.

2. Main results

We now characterize the matrices in the class ($c_o(p,s)$, $\hat{c}_o(p)$).

Theorem 1. *$A \in$ ($c_o(p,s)$, $\hat{c}_o(p)$) if and only if*

(i) *there exist an integer $B > 1$such that*

$C_n = \sup_m \left\{\sum_k |a(n,k,m)|\, k^{s/p_k} B^{-1/p_k}\right\}^{p_m} < \infty\, (\forall n),$

(ii) $\lim_m |a(n,k,m)|^{p_m} = 0$ *uniformly in n for each k.*

Proof. Necessity. Let A $\in$ ($c_o(p,s)$, $\hat{c}_o(p)$). Define $e_k = (\delta_{nk})_n$ where $\delta_{nk} = 0(n \neq k) = 1(n = k)$. Since $e_k \in c_o(p,s)$, (ii) must hold. Using the same kind of argument to that in [7], the necessity of the condition is obtained in a similar manner as done in Theorem 2 (i), by choosing a sequence $x\in c_o(p,s)$;

$$x_k^r = \left\{\begin{array}{cc} \delta^{M/p_k} k^{s/p_k} \operatorname{sgn} a(n,k,m), & 0 \leq k \leq r \\ 0, & \text{otherwise} \end{array}\right\},$$

for each r where $M = max(1, supp_k)$ and $0 < \delta < 1$.

Sufficiency. Let conditions (i) - (ii) hold and x $\in c_o(p,s)$. Fix $n\in$ Z$^+$. Given $\varepsilon > 0$ $\exists$ k$_0$ such that for k and m both larger than k$_0$, $B^{1/p_k}\left|k^{-s/p_k}x_k\right| < (\varepsilon/C_n)^{1/p_m}$. We have

$$|t_{mn}(Ax)|^{p_m} \leq L\{\left|\sum_{k=1}^{k_0} a(n,k,m)x_k\right|^{p_m} + \left|\sum_{k>k_0} a(n,k,m)x_k\right|^{p_m} = L(S_1 + S_2),$$

where $L = max(1, 2^{H-1})$ and $H = \sup p_m$. Since (ii) holds $\exists$ m$_0$ such that for $m > m_0$, $|a(n,k,m)| < \varepsilon^{1/p_m}$. Therefore for such m,

$$S_1 \leq (\sum_{k=1}^{k_0} |a(n,k,m)x_k|)^{p_m} < \varepsilon(\sum_{k=1}^{k_0} |x_k|)^{p_m}$$

$$< \varepsilon \max\{1, (\sum_{k=1}^{k_0} |x_k|)^M\}\ ,\ M = max(1, supp_k) \qquad (1)$$

For the sum S$_2$, we have,

$$S_2^{1/p_m} \geq \left| \sum_{k>k_0} a(n,k,m)x_k \right| = \sum_{k>k_0} |a(n,k,m)| k^{s/p_k} B^{-1/p_k} \left| k^{-s/p_k} x_k \right| B^{1/p_k}$$
$$< \sum_{k>k_0} |a(n,k,m)| k^{s/p_k} B^{-1/p_k} (\varepsilon/C_n)^{1/p_m}$$
$$< C_n^{1/p_m} (\varepsilon/C_n)^{1/p_m} = \varepsilon^{1/p_m} \qquad (2)$$

and consequently $S_2 < \varepsilon$, $(\forall\ m > m_0)$

Hence the sufficiency follows from (1) and (2)

Now we prove;

Theorem 2. *$A \in$ ($c_o(p,s)$,$\hat{c}$) if and only if*

(i) *there exist an integer $B > 1$ such that*

$D_n = \sup_m \sum_k |a(n,k,m)|\, k^{s/p_k} B^{-1/p_k} < \infty\, (\forall n),$

(ii) *$\exists \alpha_k \in C$ such that $\lim_m a(n,k,m) = \alpha_k$ uniformly in n for each k.*

Proof. Necessity. Let A $\in$ ($c_o(p,s)$,$\hat{c}$). Define $e_k = (\delta_{nk})_n$ where $\delta_{nk} = 0(n \neq k) = 1(n = k)$. Since $e_k \in c_o(p,s)$, (ii) must hold. Fix $n \in Z^+$. Put $f_{mn}(x) = t_{mn}(Ax).\{f_{mn}\}_m$ is a sequence of continuous linear functionals on$c_o(p,s)$, such that $lim_m f_{mn}(x)$exists uniformly in n. Therefore as in the necessity part of Theorem 1 the result follows from uniform boundedness principle.

Sufficiency. Let conditions (i) - (ii) hold and $x \in c_o(p,s)$. Hence for $0 < \varepsilon < 1$, there exists r ;

$\forall k > r \left| k^{-s/p_k} x_k v_k \right|^{p_k/M} \leq \frac{\varepsilon}{B(2D_n+1)} < 1$

and therefore $k > r$

$B^{1/p_k} \left| k^{-s/p_k} x_k \right| < B^{M/p_k} \left| k^{-s/p_k} x_k \right| \quad < \left(\frac{\varepsilon}{2D_n+1} \right)^{M/p_k} \quad < \quad \frac{\varepsilon}{2D_n+1}$

where M = max (1, sup p_k). By (i) and (ii) we have

$$\sum_k |(a(n,k,m) - \alpha_k)x_k| \leq \sum_k |(a(n,k,m) - \alpha_k)| k^{s/p_k} B^{-1/p_k}$$
$$\leq \sum_k |a(n,k,m)| k^{s/p_k} B^{-1/p_k} + \sum_k |\alpha_k| k^{s/p_k} B^{-1/p_k}$$
$$< 2D_n \quad < \infty\, (\forall \text{n})$$

Hence

$\sum_{k>r} |(a(n,k,m) - \alpha_k)x_k| < \varepsilon.$

Also,$lim_m \sum_{k\leq r} |(a(n,k,m) - \alpha_k)x_k| = 0$ uniformly in n.

Therefore, we have $\lim_m \sum_k a(n,k,m)x_k = \sum_k \alpha_k x_k$ uniformly in n.

Let $\hat{c}_o$ denote the space of sequences almost convergent to zero.

Corollary. *$A \in$ ($c_o(p,s)$, $\hat{c}_o$) if and only if*

(i) *there exist an integer $B > 1$ such that*

$\sup_m \sum_k |a(n,k,m)|\, k^{s/p_k} B^{-1/p_k} < \infty\, (\forall \text{n}),$

(ii) $\lim_m a(n,k,m) = 0$ *uniformly in n for each k.*

Proof. Once we observe that $\alpha_k = 0$, in this case, the proof is immediate.

We now consider the class $(\ell_\infty(p,s),\hat{c}\ )$

Theorem 3. *$A \in (\ell_\infty(p,s),\hat{c}\ )$ if and only if*

(i) $\exists \alpha_k \in C$ *such that* $\lim_m a(n,k,m) = \alpha_k$ *uniformly in n for each k,*

(ii) $\sup_m \sum_k |a(n,k,m)| < \infty\ (\forall n)$

(iii) *for all integers* $N > 1 \lim_m \sum_k |a(n,k,m) - \alpha_k| k^{s/p_k} N^{1/p_k} = 0$ *uniformly in n.*

Proof. Necessity. Let A $\in (\ell_\infty(p,s)\ ,\ \hat{c}\ )$. Since $e_k \in \ell_\infty(p,s)$, (i) must hold. Fix n $\in$ Z$^+$. Put $f_{mn}(x) = t_{mn}(Ax).\{f_{mn}\}_m$is a sequence of bounded linear operators on $\ell_\infty(p,s)$ such that $\sup_m |t_{mn}(Ax)| < \infty$. Since $(\ell_\infty(p,s)\ ,\ \hat{c}\ ) \subseteq (\ell_\infty(p)\ ,\ \hat{c}\ )$ (ii) holds(see Nanda [7]). Let

$C = (C_{nk}) = \left(a_{nk} k^{s/p_k} N^{1/p_k}\right) \notin (\ell_\infty, c)$ be an infinite matrix. If (iii) is false, then the matrix $C \notin (\ell_\infty, \hat{c})$. So there exists $x \in l_\infty$with $\|x\|$=1 such that $Cx \notin \hat{c}$. Now $y = (y_k) = \left(k^{s/p_k} N^{1/p_k} x_k\right) \in \ell_\infty(p,s)$, but $Ay = Cx \notin \hat{c}$ and this contradiction completes the proof.

Sufficiency. Suppose that the conditions (i) - (iii) hold. Take an integer
$N > max(1, Sup_k k^{-s}|x_k|^p k)$. We have
$\sum_k |(a(n,k,m) - \alpha_k)x_k| \leq \sum_k |(a(n,k,m) - \alpha_k)| k^{s/p_k} N^{1/p_k}$
By (i) and (iii) it follows that
$\lim_m \sum_k a(n,k,m)x_k = \sum_k \alpha_k x_k$uniformly in n. This completes the proof.

Corollary. *$A \in (\ell_\infty(p,s),\hat{c}_o)$ if and only if*

(i) $\sup_m \sum_k |a(n,k,m)| < \infty$ ($\forall$n)

(ii) *for all integers* $N > 1 \lim_m \sum_k |a(n,k,m)| k^{s/p_k} N^{1/p_k} = 0$ *uniformly in n.*

Proof. Follows from Theorem 3 taking α_k= 0 for each k.

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