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Nonlinear Dirichlet problem on a solid torus in the critical of supercritical case

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Abstract

We study a nonlinear elliptic problem on a solid torus $\overline{T} \subset \mathbb{R}^3$, when the data of the problem are invariants under the group $G = O(2) \times I \subset O(3)$. We find the best constants in the Sobolev inequalities which deal with the supercritical case (the critical of supercritical). We apply these results to solve the problem:

$$(\mathbf{P}) \qquad \Delta_q u + a(x)u^{q-1} = f(x)u^{p-1},\ u > 0 \ \text{ on } T,\ u|_{\partial T} = 0,$$

$$p = \frac{2q}{2-q}, \frac{3}{2} < q < 2.$$

1. Introduction

A lot of effort has been devoted to resolving nonlinear PDEs of the same type with the above equation. We refer for example to [1, 5, 6, 8, 9, 10, 13, 14, 17, 18, 19, 20, 21, 22] and the references therein. Best constants in Sobolev inequalities are fundamental in the study of non-linear PDEs on manifolds [1, 10, 11, 12, 18] and the references therein. It is also well known that Sobolev embeddings can be improved in the presence of symmetries [3, 7, 9, 11, 12, 16, 17, 18] and the references therein.

Given (M, g) a smooth, compact $n-$dimensional Riemannian manifold with boundary we define the Sobolev space $H_1^q(M)$ as the completion of $C^\infty(M)$ with respect to the norm $\|u\|_{H_1^q} = \|\nabla u\|_q + \|u\|_q$, $q \geq 1$ and $\overset{\circ}{H}{}_1^q(M)$ as the closure of $C_0^\infty(M)$ in $H_1^q(M)$.
As it is known [1, 15] by the Sobolev embedding theorem one has that for any $q \in [1, n)$ real, the embedding $H_1^q(M) \hookrightarrow L^p(M)$ is compact for $1 \leq p < nq/(n-q)$, while $H_1^q(M) \hookrightarrow L^{nq/(n-q)}(M)$ is only continuous.

Let G be a subgroup of the isometry group of (M, g) and k be the minimum orbit dimension of G. Denote by $H_{1,G}^q(M)$ the subspace of $H_1^q(M)$ of all $G-$invariant functions. We know by [16] that for any $q \in [1, n)$ real, the embedding $H_{1,G}^q(M) \hookrightarrow L_G^p(M)$ is compact for $1 \leq p < (n-k)q/(n-k-q)$, while $H_{1,G}^q(M) \hookrightarrow L_G^{(n-k)q/(n-k-q)}(M)$ is only continuous. In our case for any $q \in [1,2)$ real, the embedding $H^q_{1,G}(T) \hookrightarrow L^p_G(T)$ is compact for $1 \le p < 2q/(2-q)$, while $H^q_{1,G}(T) \hookrightarrow L^{2q/(2-q)}_{1,G}(T)$, is only continuous.

The equation $\Delta_p u + a(x)u^{p-1} = f(x)u^{p^*-1}$, with $\Delta_p u = -div\big(|\nabla u|^{p-2}\nabla u\big)$ on the sphere S_n is studied in [3], when the functions a and f are invariants under the group $O(m) \times O(k)$, with $m+k = n+1$, $k \ge m \ge 2$ and $p^* = pk/(k-p)$. Here the exponent p^* is supercritical: $p^* > pn/(n-p)$.

In the spirit of [1, 10] we determine:
The best constants of the Sobolev inequality

$$\|u\|^q_{L^p(T)} \le A\|\nabla u\|^q_{L^q(T)} + B\|u\|^q_{L^q(T)},$$

where $1/p = (1/q) - (1/2)$, $1 \le q < 2$, which concern the supercritical case (the critical of supercritical) $p = 2q/(2-q)$ (because $p > 3q/(3-q)$) and we use the above to solve the following problem:

$$(\mathbf{P}) \qquad \Delta_q u + a(x)u^{q-1} = f(x)u^{p-1},\ u > 0 \ \text{ on } T,\ u|_{\partial T} = 0.$$

2. Notations and preliminary results

We study the above nonlinear elliptic problem on a solid torus $\overline{T} \subset \mathbb{R}^3$, when the data of the problem are invariants under the group $G = O(2) \times I \subset O(3)$.

Let a solid torus

$$\overline{T} = \left\{(x,y,z) \in \mathbb{R}^3 / \left(\sqrt{x^2+y^2} - l\right)^2 + z^2 \le r^2,\ l > r > 0\right\}$$

and

$$\mathcal{A} = \{(\Omega_i, \xi_i)\,/ i = 1,2\}$$

an atlas on T defined by

$$\Omega_1 = \left\{(x,y,z) \in T/(x,y,z) \notin H^+_{XZ}\right\}$$

$$\Omega_2 = \left\{(x,y,z) \in T/(x,y,z) \notin H^-_{XZ}\right\}$$

where

$$H^+_{XZ} = \left\{(x,y,z) \in \mathbb{R}^3 / x > 0\,,\ y = 0\right\}$$

$$H^-_{XZ} = \left\{(x,y,z) \in \mathbb{R}^3 / x < 0\,,\ y = 0\right\}$$

and

$$\xi_i : \Omega_i \to I_i \times D, \quad i = 1,2$$

with

$$I_1 = (0, 2\pi), I_2 = (-\pi, \pi), D = \left\{(t,s) \in \mathbb{R}^2 / t^2 + s^2 < 1\right\}$$

and

$$\xi_i(x,y,z) = (\omega_i, t, s), \quad i = 1,2$$

with

$$cos\omega_i = \frac{x}{\sqrt{x^2+y^2}}, \quad sin\omega_i = \frac{y}{\sqrt{x^2+y^2}}, \quad i = 1,2$$

where

$$\omega_1 = \begin{cases} \arctan\frac{y}{x} & , \quad x \neq 0 \\ \pi/2 & , \quad x = 0, y > 0 \\ 3\pi/2 & , \quad x = 0\,,\, y < 0 \end{cases} \quad , \quad \omega_2 = \begin{cases} \arctan\frac{y}{x} & , \quad x \neq 0 \\ \pi/2 & , \quad x = 0, y > 0 \\ -\pi/2 & , \quad x = 0\,,\, y < 0 \end{cases}$$

and

$$t = \frac{\sqrt{x^2+y^2} - l}{r} \quad , \qquad s = \frac{z}{r}.$$

The Euclidean metric g on $(\Omega, \xi) \in \mathcal{A}$ can be expressed as

$$\left(\sqrt{g} \circ \xi^{-1}\right)(\omega, t, s) = r^2(l + rt).$$

Consider the spaces of all $G-$invariant functions under the action of the group $G = O(2) \times I \subset O(3)$

$$C^{\infty}_{0,G} = \{u \in C^{\infty}_0(T)/u \circ \tau = u\,,\, \forall\, \tau \in G\}$$

and

$$H^q_{1,G} = \{u \in H^q_1(T)/u \circ \tau = u\,,\, \forall\, \tau \in G\}\,,$$

where $H^q_1(T)$ is the completion of $C^{\infty}(T)$ with respect the to norm $\|u\|_{H^q_1} = \|\nabla u\|_q + \|u\|_q$.

We denote $\overset{\circ}{H}{}^q_{1,G}$ the completion of $C^{\infty}_{0,G}$ with respect to the norm $\|u\|_{H^q_1}$ and for all $G-$invariants u we define the functions $\phi(t,s) = (u \circ \xi^{-1})(\omega, t, s)$. Then we have

$$||u||^p_{L^p(T)} = 2\pi r^2 \int_D |\phi(t,s)|^p (l + rt)\, dt\, ds \tag{2.1}$$

and

$$||\nabla u||^q_{L^q(T)} = 2\pi r^{2-q} \int_D |\nabla\phi(t,s)|^q (l + rt)\, dt\, ds \tag{2.2}$$

Let $K(2,q)$ be the best constant [1] of the Sobolev inequality

$$\|\varphi\|_p \leq K(2,q)\, \|\nabla\varphi\|_q$$

for all $\varphi \in H^q_1(\mathbb{R}^2)$. Consider a point $P_j(x_j, y_j, z_j) \in \overline{T}$, and by O_{P_j} denote the orbit of P_j under the action of the group G. Let $l_j = \sqrt{x_j^2 + y_j^2}$ be the horizontal distance of the orbit O_{P_j} from the axis $z'z$. For $\varepsilon > 0$ given and $\delta_j = l_j\varepsilon$, consider a finite covering $(T_j)_{j=1,\ldots N}$ with

$$T_j = \left\{(x,y,z) \in \overline{T}/\left(\sqrt{x^2+y^2} - l_j\right)^2 + (z - z_j)^2 < \delta_j^2\right\}$$

an open small solid torus (a tubular neighborhood of the orbit O_{P_j}). Then the following lemma holds.

Lemma 2.1 *For all $\varepsilon > 0$ and $p, q \in \mathbb{R}$ with $1/p = (1/q) - (1/2)$, $1 \leq q < 2$ there exist $\delta_j = \varepsilon l_j$, $j = 1, 2, ...N$ such that for all $u \in C^\infty_{0,G}$ the following inequality holds*

$$\left(\int_{T_j} |u|^p \, dV \right)^{1/p} \leq \frac{(1+\varepsilon)^{1/p}}{(1-\varepsilon)^{1/q}} \frac{K(2,q)}{\sqrt{2\pi l_j}} \left(\int_{T_j} |\nabla u|^q \, dV \right)^{1/q}$$

Proof of Lemma 2.1. On every T_j we define the subsets Ω_{ij}, $i = 1, 2$ of T_j in the same way we defined the subsets Ω_i, $i = 1, 2$ of T. Also define the maps $\xi_{ij} : \Omega_{ij} \to I_i \times D, i = 1, 2$. Then $\mathcal{A}_j = \{(\Omega_{ij}, \xi_{ij}) / i = 1, 2\}$ is an atlas on T_j and the Euclidean metric g on $(\Omega_j, \xi_j) \in \mathcal{A}_j$ can be expressed as

$$\left(\sqrt{g} \circ \xi_j^{-1} \right) (\omega, t, s) = \delta_j^2 (l_j + \delta_j t)$$

Let $u \in C^\infty_{0,G}$ and $\phi_j = u \circ \xi_j^{-1}$. According to (2.1) we have:

$$\left(\int\limits_{T_j} |u|^p \, dV \right)^{1/p} \leqslant (2\pi l_j)^{1/p} \, \delta_j^{2/p} \, (1+\varepsilon)^{1/p} \left(\int\limits_{D} |\phi_j|^p dt ds \right)^{1/p} \tag{2.3}$$

Because $\varphi_j \in C^\infty_0(D)$ and since the space $C^\infty_0(D)$ is dense in $\overset{\circ}{H}{}^q_1(D)$ with respect to the norm $\| \cdot \|_{H^q_1}$ according to lemma 7 of [2] and lemma 3.1 of [14] we have $\|\phi_j\|_p \leqslant K(2,q) \, \|\nabla \phi_j\|_q$, with $q \in [1, 2)$ and $(1/p) = (1/q) - (1/2)$.
Finally we have

$$\left(\int\limits_{T_j} |u|^p \, dV \right)^{1/p} \leqslant \left(2\pi l_j \delta_j^2 (1+\varepsilon) \right)^{1/p} K(2,q) \left(\int\limits_{D} |\nabla \phi_j|^q dt ds \right)^{1/q} \tag{2.4}$$

Moreover from (2.2) we have

$$\int\limits_{T_j} |\nabla u|^p \, dV \geq (1-\varepsilon) \, 2\pi l_j \delta_j^{2-q} \int\limits_{D} |\nabla \phi_j(t,s)|^q dt ds$$

Therefore

$$\left(\int\limits_{D} |\nabla \phi_j|^q \, dt ds \right)^{1/q} \leqslant \left[(1-\varepsilon) \, 2\pi \, l_j \delta_j^{2-q} \right]^{-1/q} \left(\int\limits_{T_j} |\nabla u|^q \, dv(g) \right)^{1/q} \tag{2.5}$$

From (2.4) and (2.5) because of $(1/p) = (1/q) - (1/2)$ we obtain

$$\left(\int_{T_j} |u|^p \, dV\right)^{1/p} \leqslant \frac{(1+\varepsilon)^{1/p}}{(1-\varepsilon)^{1/q}} \frac{K(2,q)}{\sqrt{2\pi\, l_j}} \left(\int_{T_j} |\nabla u|^q \, dV\right)^{1/q}$$

□

3. Results

3.1. Best constants on the solid Torus

Theorem 3.1 *Let $\overline{T}$ be the solid torus and p, q be two positive real numbers such that $1/p = (1/q) - (1/2)$ with $1 \le q < 2$. Then for all $\varepsilon > 0$, there exists a constant $B = B(\varepsilon, q)$ such that:*

1. *For all $u \in \overset{\circ}{H}{}^q_{1,G}$ the following inequality holds*

$$\|u\|_p^q \le \left[\left(\frac{K(2,q)}{\sqrt{2\pi(l-r)}}\right)^q + \varepsilon\right] \|\nabla u\|_q^q + B\,\|u\|_q^q \tag{3.1}$$

2. *For all $u \in H^q_{1,G}$ the following inequality holds*

$$\|u\|_p^q \le \left[\left(\frac{K(2,q)}{\sqrt{\pi(l-r)}}\right)^q + \varepsilon\right] \|\nabla u\|_q^q + B\,\|u\|_q^q \tag{3.2}$$

The constants $\frac{K(2,q)}{\sqrt{2\pi(l-r)}}$ and $\frac{K(2,q)}{\sqrt{\pi(l-r)}}$ are the best constants for which the inequalities 1. and 2. hold for all $u \in \overset{\circ}{H}{}^q_{1,G}$ and $u \in H^q_{1,G}$ respectively.

Because of the concentration phenomenon on the orbit of a sequence of solutions of nonlinear deferential equations, we establish ([16, 12]) inequalities without ε.

Theorem 3.2 *Let $\overline{T}$ be the solid torus and p, q be two positive real numbers such that $1/p = (1/q) - (1/2)$ with $1 < q < 2$. Then there exists $B = B(q) > 0$ such that:*

1. *For all $u \in \overset{\circ}{H}{}^q_{1,G}$*

$$\|u\|_p^q \le \left(\frac{K(2,q)}{\sqrt{2\pi(l-r)}}\right)^q \|\nabla u\|_q^q + B\,\|u\|_q^q \tag{3.3}$$

2. *For all $u \in H^q_{1,G}$*

$$\|u\|_p^q \le \left(\frac{K(2,q)}{\sqrt{\pi(l-r)}}\right)^q \|\nabla u\|_q^q + B\,\|u\|_q^q \tag{3.4}$$

3.2. Resolution of the problem

We give now an application resolving the problem (**P**).
Consider the functional

$$I(u) = \int_T \Big(|\nabla u|^q + a(x)|u|^q \Big) dV$$

and suppose that the operator

$$L_q(u) = \Delta_q u + a(x)u^{q-1}$$

is coercive. That is, there exists a real number $\lambda > 0$, such that, for all $u \in H^q_{1,G}$

$$I(u) \geq \lambda \int_T |u|^q dV$$

For

$$\frac{3+2}{3-2} + 1 = 6 < p = \frac{2q}{2-q}, \ \frac{3}{2} < q < 2$$

and for all $u \in \mathcal{H}_p$ set

$$\mu = inf I(u),$$

where

$$\mathcal{H}_p = \left\{ u \in H^q_{1,G} \ , \ u > 0 \ / \ \int_T f(x) u^p dV = 1 \right\}.$$

Consequently, for the problem

(**P**) $\quad \Delta_q u + a(x)u^{q-1} = f(x)u^{p-1}, \ u > 0 \ \text{ on } T, \ u|_{\partial T} = 0,$

$\quad p = \frac{2q}{2-q}, \ \frac{3}{2} < q < 2,$

we have the theorem:

Theorem 3.3 *Let $\overline{T}$ be a solid torus , α and f be two smooth functions, $G-$invariant and p, q be two real numbers defined as in* (**P**)*. Suppose that $sup_{x \in T} f(x) > 0$ and the operator $L_q u = \Delta_q u + \alpha u^{q-1}$ is coercive. The problem* (**P**) *accepts a positive solution, that belongs to $C^{1,\alpha}$ for some $\alpha \in (0,1)$, if $\mu < \left(\frac{K(2,q)}{\sqrt{2\pi(l-r)}} \right)^{-q} (supf)^{-q/p}$.*

Corollary 3.1 *([7]) Let $\overline{T}$ be a solid torus , and α, f be two smooth functions, $G-$invariant. Then the problem*

$$\Delta u + a(x)u = f(x)u^{p-1}, \ u > 0 \ \text{ on } T, \ u|_{\partial T} = 0, \ \ p > 1$$

accepts a positive solution that belongs to $H^2_{1,G}$.

Throughout the rest of the paper we will denote $K = K(2,q)$ and $L = 2\pi(l-r)$.

4. Proofs of the theorems concerning the best constants

Proof of Theorem 3.1. **1**. Let $\varepsilon > 0$ given. Consider a point $P_j(x_j, y_j, z_j)$, $j \in J$. We denote by O_{P_j} the orbit of P_j under the action of the subgroup $G = O(2) \times Id$ of the group $O(3)$ of the type $(x, y, z) \to (A(x, y), z)$, $A \in O(2)$, $(x, y, z) \in \mathbb{R}^3$. Let $l_j = \sqrt{x_j^2 + y_j^2}$ be the horizontal distance of the orbit O_{P_j} from the axis $z'z$. Then we can choose an ε_0 depending on ε and P_j such that $T_j = \left\{Q \in \overline{T}/d(Q, O_{P_j}) < \delta_j\right\}$, with $\delta_j = \varepsilon_0 l_j$ having the following properties: $\overline{T}_j$ is a submanifold of $\overline{T}$ with boundary, $d^2(\cdot, O_{P_j})$ (where $d(\cdot, O_{P_j})$ is the distance to the orbit O_{P_j}) is a C^∞ function on $\overline{T}_j$, and $\overline{T}$ is covered by $(T_j)_{j\in J}$. Once more denote by $(T_j)_{j=1,\dots,N}$ a finite covering. According to lemma 2.1 and because of $infl_j = l - r$, for all $\varepsilon_0 > 0$, $j = 1, \dots, N$ and for all $u \in C^\infty_{0,G}(T_j)$ the following holds:

$$\left(\int_{T_j} |u|^p \, dV\right)^{q/p} \leqslant \frac{(1+\varepsilon_0)^{q/p}}{1-\varepsilon_0} \left(\frac{K}{\sqrt{L}}\right)^q \int_{T_j} |\nabla u|^q dV$$

From the last inequality according to lemma 1 of [11] we have:

$$\left(\int_T |u|^p dV\right)^{q/p} \leqslant \left[f(\varepsilon_0)\left(\frac{K}{\sqrt{L}}\right)^q + \varepsilon\right] \int_T |\nabla u|^q dV + B \int_T |u|^q dV, \qquad (4.1)$$

where $f(\varepsilon_0) = (1+\varepsilon_0)^{q/p}/(1-\varepsilon_0)$.
Now it's sufficient to prove that for all $\varepsilon > 0$ there exists $\varepsilon_0 \in (0,1)$ such that the following holds:

$$f(\varepsilon_0)\left(\frac{K}{\sqrt{L}}\right)^q \leqslant \left(\frac{K}{\sqrt{L}}\right)^q + \varepsilon$$

The function $f : (0,1) \to (1, +\infty)$ with $f(t) = (1+t)^{q/p}/(1-t)$ is monotonically increasing, and thus invertible, so the last inequality can be equivalently written:

$$f(\varepsilon_0) \leqslant 1 + \varepsilon \left(\frac{K}{\sqrt{L}}\right)^{-q}$$

or

$$\varepsilon_0 \leqslant f^{-1}\left(1 + \varepsilon\left(\frac{K}{\sqrt{L}}\right)^{-q}\right) \qquad (4.2)$$

From (4.1) choosing $\varepsilon_0 \in (0,1)$ such that (4.2) holds, for all $\varepsilon > 0$ and for all $u \in C^\infty_{0,G}(T)$ we obtain:

$$\left(\int_T |u|^p dV\right)^{q/p} \leqslant \left[\left(\frac{K}{\sqrt{L}}\right)^q + \varepsilon\right] \int_T |\nabla u|^q dV + B \int_T |u|^q dV.$$

The last relation allow us to use Lemma 1 of [11]. This completes the proof of this part of theorem 3.1.

Our aim in what follows is to prove that $\frac{K}{\sqrt{L}}$ in Theorem 3.1 is the best constant. For that purpose, for all $\varepsilon > 0$ we need to find a family of functions $(u_\alpha)_{\alpha>0} \subset \overset{\circ}{H}{}^q_{1,G}$ such that for any given real number E the following inequality holds:

$$\lim_{\alpha\to 0} \frac{\int\limits_T |\nabla u_\alpha|^q dV + E\int\limits_T |u_\alpha|^q dV}{\left(\int\limits_T |u_\alpha|^p\, dV\right)^{q/p}} \leqslant \left(\frac{\sqrt{L}}{K}\right)^q + \varepsilon \tag{4.3}$$

Consider the orbit O_{inf} of minimal length $2\pi(l-r)$, $\delta = \varepsilon_0(l-r) < 1$, the set $T_{j_0} = \left\{(x,y,z) \in \mathbb{R}^3 / \left(\sqrt{x^2+y^2} - (l-r)\right)^2 + z^2 < \delta^2\right\}$, and for any $\alpha > 0$ we define the function $u_\alpha \in \overset{\circ}{H}{}^q_{1,G}(T_{j_0})$ by

$$u_\alpha(Q) = \begin{cases} \left(\alpha + d^2(Q,O_{inf})\right)^{1-\frac{2}{q}} - \left(\alpha+\delta^2\right)^{1-\frac{2}{q}}, & \text{if } Q \in T\cap T_{j_0} \\ 0, & \text{if } Q \notin T \end{cases}$$

where $d(Q,O_{inf})$ denotes the distance from Q to the orbit O_{inf}. Since u_α depends only on the distance to O_{inf}, $u_\alpha \in H^q_{1,G}$. Setting $\varphi_\alpha = u_\alpha \circ \xi^{-1}_{j_0}$ according to (2.1) and (2.2) for any constant E we obtain:

$$\frac{\int\limits_T |\nabla u_\alpha|^q dV + E\int\limits_T |u_\alpha|^q dV}{\left(\int\limits_T |u_\alpha|^p\, dV\right)^{q/p}} =$$

$$\frac{2\pi\delta^{2-q}\int\limits_D |\nabla\phi_\alpha|^q\,(l-r+\delta t)\,dtds + 2\pi\delta^2 E\int\limits_D |\phi_\alpha|^q\,(l-r+\delta t)\,dtds}{\left(2\pi\delta^2\int\limits_D |\phi_\alpha|^p\,(l-r+\delta t)\,dtds\right)^{q/p}}$$

Because the range of O_{inf} is $l-r$ from the last relation for $\delta = \varepsilon_0(l-r)$ we get

$$\frac{\int\limits_T |\nabla u_\alpha|^q dV + E\int\limits_T |u_\alpha|^q dV}{\left(\int\limits_T |u_\alpha|^p\, dV\right)^{q/p}} \leq$$

$$\frac{(1+\varepsilon_0)\,2\pi\,(l-r)\left(\delta^{2-q}\int\limits_D |\nabla\phi_\alpha|^q dtds + \delta^2 E\int\limits_D |\phi_\alpha|^q dtds\right)}{\left[(1-\varepsilon_0)\,2\pi\,(l-r)\,\delta^2\int\limits_D |\phi_\alpha|^p\,dtds\right]^{q/p}} \leqslant$$

$$\frac{(1+\varepsilon_0)\left[\sqrt{2\pi(l-r)}\right]^q\left(\int\limits_D |\nabla\phi_\alpha|^q\, dtds+\delta^q E\int\limits_D |\phi_\alpha|^q\, dtds\right)}{(1-\varepsilon_0)^{q/p}\left(\int\limits_D |\phi_\alpha|^p\, dtds\right)^{q/p}}$$

Thus

$$\lim_{\alpha\to 0}\frac{\int\limits_T |\nabla u_\alpha|^q dV+E\int\limits_T |u_\alpha|^q dV}{\left(\int\limits_T |u_\alpha|^p\, dV\right)^{q/p}}\leq$$

$$\frac{1+\varepsilon_0}{(1-\varepsilon_0)^{q/p}}\left(\sqrt{L}\right)^q\lim_{\alpha\to 0}\frac{\int\limits_D |\nabla\phi_\alpha|^q\, dtds+E\int\limits_D |\phi_\alpha|^q\, dtds}{\left(\int\limits_D |\phi_\alpha|^p\, dtds\right)^{q/p}} \tag{4.4}$$

Since $(x,y,z)=\xi_{j_0}^{-1}(\omega,t,s)=((l-r+\delta t)\cos\omega,(l-r+\delta t)\sin\omega,\delta s))$, for any $Q\in T_{j_0}$ we have

$$d^2\left(Q,O_{\inf}\right)=\left[\sqrt{x^2+y^2}-(l-r)\right]^2+z^2=\delta^2\left(t^2+s^2\right)=\delta^2 d_D^2(\xi_{j_0}(Q),O_D),$$

where d_D denotes the distance on D and O_D is the center of D. Consequently we have

$$\phi_\alpha\left(\xi_{j_0}(Q)\right)=\left[\alpha+\delta^2 d_D^2(\xi_{j_0}(Q),\,O_D)\right]^{1-2/q}-\left[\alpha+\delta^2\right]^{1-2/q}$$

On the other hand according to [1] and [14] for all $\upsilon_\alpha\in\overset{\circ}{H}{}^q_{1,G}(D_\delta)$ with $\upsilon_\alpha(y)=\left(\alpha+\|y\|^2\right)^{1-2/q}-\left(\alpha+\delta^2\right)^{1-2/q}$ the following is valid:

$$\lim_{\alpha\to 0}\frac{\int\limits_D |\nabla\upsilon_\alpha|^q\, dtds+E\int\limits_D |\upsilon_\alpha|^q\, dtds}{\left(\int\limits_D |\upsilon_\alpha|^p\, dtds\right)^{q/p}}=\frac{1}{K^q} \tag{4.5}$$

From (4.4) because of (4.5) we obtain

$$\lim_{\alpha\to 0}\frac{\int\limits_T |\nabla u_\alpha|^q\, dV+E\int\limits_T |u_\alpha|^q\, dV}{\left(\int\limits_T |u_\alpha|^p\, dV\right)^{q/p}}\leq\frac{1+\varepsilon_0}{(1-\varepsilon_0)^{q/p}}\left(\frac{\sqrt{L}}{K}\right)^q \tag{4.6}$$

For the completion of the proof of this part of theorem 3.1 it suffices for all $\varepsilon>0$ an $\varepsilon_0\in(0,1)$ to exist such that

$$\frac{1+\varepsilon_0}{(1-\varepsilon_0)^{q/p}}\left(\frac{\sqrt{L}}{K}\right)^q\leq\left(\frac{\sqrt{L}}{K}\right)^q+\varepsilon \tag{4.7}$$

The function $g : (0,1) \to (1,+\infty)$ with $g(t) = (1+t)/(1-t)^{q/p}$ is monotonically increasing and then (4.7) can be written

$$g(\varepsilon_0) \leq 1 + \varepsilon \left(\frac{K}{\sqrt{L}}\right)^q$$

or

$$\varepsilon_0 \leq g^{-1}\left(1 + \varepsilon \left(\frac{K}{\sqrt{L}}\right)^q\right)$$

We proved that for all $\varepsilon > 0$ there exists an $\varepsilon_0 > 0$ with

$$\varepsilon_0 < min\left\{f^{-1}\left(1 + \varepsilon \left(\frac{K}{\sqrt{L}}\right)^{-q}\right),\ g^{-1}\left(1 + \varepsilon \left(\frac{K}{\sqrt{L}}\right)^q\right)\right\}$$

such that (4.3) holds for all $u_\alpha \in C^\infty_{0,G}(T)$ and this completes the proof of the first part of the theorem.

2. Let $\mathcal{A}_j = \{(\Omega_{ij}, \xi_{ij})\,/i = 1,2\}$ be an atlas on T_j and $(\Omega_j, \xi_j) \in \mathcal{A}_j$. Then, by the definition of (Ω_j, ξ_j), every Ω_j is homeomorphic either to $I \times D$, if $T_j \subset T$ or to $I \times D_+$, if $T_j \cap \partial T \neq \emptyset$, where $D_+ = \{(t,s) \in D/s \geqslant 0\}$ (see theorem 2.30 of [1]). Let $u \in H^q_{1,G}$. Then $\eta_j u$ has support in T_j thus $(\eta_j u) \in H^q_{1,G}(T_j)$ and according to lemma $IX.5$ of [4], $(\eta_j u) \in \overset{\circ}{H}{}^q_{1,G}(T_j)$.

Now we distinguish the cases:

(a) If $T_j \subset T$ we proved in lemma 2.1 that

$$\left(\int_{T_j} |\eta_j u|^p \, dV\right)^{1/p} \leqslant \frac{(1+\varepsilon_0)^{1/p}}{(1-\varepsilon_0)^{1/q}} \frac{K}{\sqrt{2\pi\, l_j}} \left(\int_{T_j} |\nabla(\eta_j u)|^q dV\right)^{1/q} \tag{4.8}$$

(b) If $T_j \cap \partial T \neq \emptyset$, as in (4.1), we have

$$\left(\int_{T_j} |\eta_j u|^p \, dV\right)^{1/p} \leqslant \left(2\pi\, l_j \delta^2\right)^{1/p} (1+\varepsilon_0)^{1/p} \left(\int_D |\phi_j|^p dtds\right)^{1/p}$$

where $\phi_j = (\eta_j u) \circ \xi_j^{-1}$.
From the last inequality by theorem 2.14 and lemma 2.31 of [1] we obtain that

$$\left(\int_{T_j} |\eta_j u|^p \, dV\right)^{1/p} \leqslant \left[\left(2\pi\, l_j \delta^2\right)(1+\varepsilon_0)\right]^{1/p} \sqrt{2} K \left(\int_D |\nabla\phi_j|^q dtds\right)^{1/q}$$

and because of (4.3), $(1/p) = (1/q) - (1/2)$ and $infl_j = l - r$ from the last inequality we obtain

$$\left(\int_{T_j} |\eta_j u|^p \, dV\right)^{1/p} \leqslant \frac{(1+\varepsilon_0)^{1/p}}{(1-\varepsilon_0)^{1/q}} \frac{K}{\sqrt{L/2}} \left(\int_{T_j} |\nabla(\eta_j u)|^q dV\right)^{1/q} \tag{4.9}$$

Finally from (4.9) and according to lemma 1 of [11], for all $\varepsilon > 0$ and for all $u \in H^q_{1,G}$ we have:

$$\left(\int_T |u|^p \, dV\right)^{q/p} \leqslant \left[\frac{(1+\varepsilon_0)^{q/p}}{1-\varepsilon_0}\left(\frac{K}{\sqrt{L/2}}\right)^q + \varepsilon\right] \int_T |\nabla u|^q dV + B \int_T |u|^q dV.$$

Then, the rest of the proof follows in a way similar to the proof of the first part of this theorem. □

Proof of Theorem 3.2. We prove the theorem by contradiction. Assuming that the inequality (3.3) is false, for any $\alpha > 0$ we may build a positive function u_α, which is a weak solution of the equation

$$\Delta_q u_\alpha + \alpha u_\alpha^{q-1} = \lambda_\alpha u_\alpha^{p-1}$$

where $\Delta_q u = -div\left(|\nabla u|^{q-2}\nabla u\right)$ is the $q-$Laplacian of u.

When $\alpha \to +\infty$, we show that the functions u_α concentrate on the orbit of minimum length. This concentration phenomenon leads to a contradiction and this fact completes the proof of theorem. We define now the concentration orbit.

Definition 4.1 (Concentration orbit).([11]) Set O_P a $G-$orbit of T. O_P is an orbit of concentration of the sequence (u_α) if for any $\delta > 0$, the following holds: $\lim_{\alpha\to\infty} sup \int_{O_{P,\delta}} u_\alpha^p dv(g) > 0$, where $O_{P,\delta} = \{Q \in T / d(Q, O_P) < \delta\}$.

We give now a sketch of the proof of theorem 3.2. Following the same arguments as in [12] we prove that for all subsequences (u_α) of (u_α), there is only one orbit O_{P_0} of concentration, this orbit is of minimum length $2\pi(l-r)$ and for any compact set K of $T \backslash O_{P_0}$, $\lim_{\alpha\to\infty} \sup_K u_\alpha = 0$ holds. In addition we need the following lemma:

Lemma 4.1 *For all $u \in C^\infty_{0,G}(T)$ and for all $p, q \in \mathbb{R}$, with $1 \leq q < 2$ and $1/p = (1/q) - (1/2)$ there exists $B > 0$ such that the following inequality holds:*

$$\|u\|_p^q \leq \left(\frac{K}{\sqrt{L}}\right)^q \|\nabla u\|_q^q + B\,\|u\|_q^q \tag{4.10}$$

Proof of Lemma 4.1. Consider the conformal metric $\hat{e} = f^4 e$ of Euclidian e of the unitary disk D, with $f(t) = (l + rt)^{1/4} > 0$, then we have:

$$\begin{aligned} dv(\hat{e}) &= \sqrt{\det(\hat{e})} dtds = \sqrt{\det(f^4 e)} dtds \\ &= \sqrt{(f^4)^2 \det(e)} dtds = f^4 \sqrt{\det(e)}\, dtds \\ &= f^4 dv(e) = (l + rt) dv(e) \end{aligned}$$

Thus

$$\left(\int_D |\varphi|^p dv(\hat{e}) \right)^{q/p} = \left(\int_D |\varphi(t,s)|^p (l + rt) dv(e) \right)^{q/p} \tag{4.11}$$

We also have:

$$\begin{aligned} &|\nabla\varphi|^q_{\hat{e}} = \left(|\nabla\varphi|^2_{\hat{e}} \right)^{q/2} = \left[\nabla\varphi \cdot (\hat{e}^{ij}) \cdot \nabla\varphi \right]^{q/2} = \\ &\left[\nabla\varphi \cdot \left(f^{-4} e^{ij} \right) \cdot \nabla\varphi \right]^{q/2} = \left(f^{-4} \right)^{q/2} \left[\nabla\varphi \cdot \left(e^{ij} \right) \cdot \nabla\varphi \right]^{q/2} = \\ &\left(f^{-4} \right)^{q/2} \left(|\nabla\varphi|^2_e \right)^{q/2} = \left(f^{-4} \right)^{q/2} |\nabla\varphi|^q_e = \frac{1}{(l + rt)^{q/2}} |\nabla\varphi|^q_e \end{aligned}$$

So we obtain

$$\int_D |\nabla\varphi|^q_{\hat{e}} dv(\hat{e}) = \int_D \frac{1}{(l + rt)^{q/2}} |\nabla\varphi|^q (l + rt) dv(e) \tag{4.12}$$

From Theorem 10 of [2] because of (2.1), (4.11), (4.12) and (2.2) we obtain

$$\begin{aligned} \left(\int_T |u|^p \, dV \right)^{q/p} &= \left(2\pi r^2 \int_D |\varphi(t,s)|^p (l + rt) dtds \right)^{q/p} \\ &= \left(2\pi r^2 \right)^{q/p} \left(\int_D |\varphi(t,s)|^p (l + rt) dv(e) \right)^{q/p} \\ &= \left(2\pi r^2 \right)^{q/p} \left(\int_D |\varphi|^p dv(\hat{e}) \right)^{q/p} \\ &\leqslant \left(2\pi r^2 \right)^{q/p} \left(K^q \int_D |\nabla\varphi|^q_{\hat{e}} dv(\hat{e}) + \mathrm{B} \int_D |\varphi|^q dv(\hat{e}) \right) \\ &= \left(2\pi r^2 \right)^{q/p} K^q \int_D \frac{1}{(l + rt)^{q/2}} |\nabla\varphi|^q_e (l + rt) dv(e) \end{aligned}$$

$$
\begin{aligned}
& + \left(2\pi r^2\right)^{q/p} \mathrm{B} \int_D |\varphi|^q (l+rt) dv(e) \\
= \quad & \left(2\pi r^2\right)^{q/p} K^q \int_D \frac{1}{(l+rt)^{q/2}} |\nabla \varphi|^q (l+rt) dt ds \\
& + \left(2\pi r^2\right)^{q/p} \mathrm{B} \int_D |\varphi|^q (l+rt) dt ds \\
= \quad & (2\pi)^{(q/p)-1} r^{(2q/p)+q-2} K^q 2\pi r^{2-q} \\
& \times \int_D \frac{1}{(l+rt)^{q/2}} |\nabla \varphi|^q (l+rt) dt ds \\
& + \left(2\pi r^2\right)^{(q/p)-1} 2\pi r^2 \mathrm{B} \int_D |\varphi|^q (l+rt) dt ds \\
= \quad & \left(\frac{K}{\sqrt{2\pi}}\right)^q \int_T \frac{1}{\left(\sqrt{x^2+y^2}\right)^{q/2}} |\nabla u|^q \, dV + \mathrm{B}' \int_T |u|^q dV \\
\leqslant \quad & \left(\frac{K}{\sqrt{L}}\right)^q \int_T |\nabla u|^q dV + \mathrm{B}' \int_T |u|^q dV
\end{aligned}
$$

5. Proofs of the theorem concerning the problem

Proof of Theorem 3.3. We are interested in the existence of positive solutions $u \in H^q_{1,G}$ of the problem

$$\Delta_q u + a(x) u^{q-1} = f(x) u^{p-1}, \ u > 0 \ \text{ on } T, \ u|_{\partial T} = 0, \tag{5.1}$$

where $p = 2q/(2-q), \ 3/2 < q < 2$. Namely, we focusing our interest in the critical of supercritical case.

Since the operator $L_q(u) = \Delta_q u + a(x) u^{q-1}$ is coercive, a necessary condition for the existence of positive solutions of the problem is the function f to be somewhere positive. Indeed, if we multiply by u each term of the equation (5.1) and integrate we obtain

$$\int_T (|\nabla u|^q + a(x) u^q) dV = \int_T f(x) u^p dV$$

and thus

$$\int_T f(x) u^p dV > 0$$

The last inequality is false if $f \leq 0$. In the next we assume that f is somewhere positive.

Let

$$I(u) = \int_T |\nabla u|^q dV + \int_T a(x) u^q dV$$

and

$$\mathcal{H}_p = \left\{ u \in H^q_{1,G},\ u > 0 \,/ \int_T f(x) u^p dV = 1 \right\}$$

We carry through the proof of the theorem in several steps.

Step 1. In the first step, one gets solutions for equations, of the type

$$\Delta_q u + a(x) u^{q-1} = f(x) u^{\bar{p}-1},\ u > 0 \ \text{ on } T,\ u|_{\partial T} = 0, \tag{5.2}$$

where $\bar{p} < 2q/(2-q)$ and $3/2 < q < 2$, (i.e. subcritical of the critical of supercritical case).

In this case we consider the set

$$\mathcal{H}_{\bar{p}} = \left\{ u \in H^q_{1,G},\ u > 0 \,/ \int_T f(x) u^{\bar{p}} dV = 1,\ 6 < \bar{p} < p \right\}$$

and let

$$\mu_{\bar{p}} = \inf_{u \in \mathcal{H}_{\bar{p}}} I(u)$$

Since f is somewhere positive we have $\mathcal{H}_{\bar{p}}\,(T) \neq \emptyset$.

For any $\bar{p} < 2q/(2-q)$ the imbedding $H^q_{1,G} \hookrightarrow L^{\bar{p}}_G$ is compact and then the proof of this step is obtained by using the variation method. (See [1], [15]).

Step 2. In this step, one gets solutions for the critical of supercritical equation, that is of the initial equation.

Let

$$\mu = \inf_{u \in \mathcal{H}_p} I(u)$$

The general idea, (see [25], [23], [1] and [15]), is to get the solution u of (5.1) as the limit of (a subsequence of) $(u_{\bar{p}})$ as $\bar{p} \to p$, where $(u_{\bar{p}})$ is the solution of (5.2).

Following [15], as a first result, one can prove that

$$\lim_{\bar{p} \to p} \sup \mu_{\bar{p}} \leqslant \mu$$

For such an assertion, let $\varepsilon > 0$ be given and let $\upsilon \in \mathcal{H}_p$, υ nonnegative and such that

$$I(\upsilon) \leqslant \mu + \varepsilon$$

For $\bar{p}$ close to p, $\int_T f(x) \upsilon^{\bar{p}} dV \neq 0$ and so $\upsilon_{\bar{p}} = \left(\int_T f(x) \upsilon^{\bar{p}} dV\right)^{-1/\bar{p}} \upsilon$ make sense and belongs to $\mathcal{H}_{\bar{p}}$.

Hence

$$\mu_{\bar{p}} \leqslant I(\upsilon_{\bar{p}})$$

Moreover

$$\lim_{\bar{p}\to p} I(\upsilon_{\bar{p}}) = I(\upsilon)$$

As $\bar{p} \to p$, one gets that

$$\lim_{\bar{p}\to p} \sup \mu_{\bar{p}} \leqslant \mu + \varepsilon$$

The fact that such an inequality holds for any $\varepsilon > 0$ proves the above claim.

In what follows, up to extraction of a subsequence, we assume that the $\lim\limits_{\bar{p}\to p} \mu_{\bar{p}}$ exists. Let

$$\mu = \lim_{\bar{p}\to p} \mu_{\bar{p}}$$

Additionally to the hypothesis of the theorem we assume that a subsequence of $(u_{\bar{p}})$ converges in some $L^k(T), k > 1$ to a function $u \not\equiv 0$. Since L_q is coercive, $(u_{\bar{p}})$ is bounded in $H^q_{1,G}$. Thus there exists a subsequence $u_{\bar{p}}$ and a function u such that
(a) $(u_{\bar{p}}) \rightharpoonup u$ on $H^q_{1,G}$, (by Banach's theorem),
(b) $(u_{\bar{p}}) \to u$ on L^p, (by Kondrakov's theorem) and
(c) $(u_{\bar{p}}) \to u$ a.e., (by proposition 3.43 of [1]).
¿From (c) arises that $u \geqslant 0$ and $G-$invariant. Moreover, since $|\nabla u_{\bar{p}}|$ is bounded in L^q, we can assume that for $\bar{p} \to p$

$$\left(|\nabla u_{\bar{p}}|^{q-2} \nabla u_{\bar{p}}\right) \rightharpoonup F$$

in $L^{p/(p-1)}$.
Additionally, since $u_{\bar{p}}^{\bar{p}-1}$ is bounded in $L_G^{p/(\bar{p}-1)} \subset L_G^{p/(p-1)} \subset L^{p/(p-1)}$, we can assume that

$$(u_{\bar{p}}^{\bar{p}-1}) \rightharpoonup u^{p-1}$$

By passing to the limit as $\bar{p} \to p$ in the equation satisfied by $u_{\bar{p}}$' s, that is equation (5.2), we obtain

$$-divF + a(x)u^{q-1} = \mu f(x)u^{p-1}$$

Since $\left(\mu f(x)u^{p-1} - a(x)u^{q-1}\right)$ is bounded in L^1 we can prove that $F = |\nabla u|^{q-2} \nabla u$, (see [8]). Hence u is a solution of

$$\Delta_q u + a(x)u^{q-1} = \mu f(x)u^{p-1} \tag{5.3}$$

By maximum principles ([24]) and regularity results ([14]) we get that $u > 0$ and $u \in C^{1,\alpha}, \alpha \in (0,1)$. Moreover multiplying the equation (5.3) by u and integrating over T, we get that $\mu > 0$.

Now we have to prove that, $u \in \mathcal{H}_p$ and $\mu = I(u)$.
Multiplying the equation (5.3) by u and integrating over T, we get that

$$\mu \int_T f(x)\, u^p dV = \int_T (\,|\nabla u|^q + a(x)u^q)\, dV$$
$$\leqslant \liminf_{\bar p \to p} \int_T \left(|\nabla u_{\bar p}|^q + a(x)u_{\bar p}^q\right) dV$$
$$= \liminf_{\bar p \to p} \mu_{\bar p}$$

hence, $\int_T f(x)u^p dV \leqslant 1$.
Let $\upsilon = \left(\int_T f(x)u^p dV\right)^{-1/p} u$. Then $\upsilon \in \mathcal{H}_p$ and, according to what has been said above, we have

$$\mu \leqslant I(\upsilon) = \mu \left(\int_T f(x)u^p dV\right)^{1-q/p}$$

Thus $\int_T f(x)u^p dV \geqslant 1$, so that $\int_T f(x)u^p dV = 1$ and $\mu = \inf\limits_{u \in \mathcal{H}_p} I(u)$.

Step 3. By step 2, the proof of the theorem reduces to the proof that $u \not\equiv 0$. By theorem 3.1, for all $\varepsilon > 0$, there exists a constant $B = B(\varepsilon, q) > 0$ such that, the following holds

$$\left(\int_T |u|^p dV\right)^{q/p} \leqslant \left[\left(\frac{K}{\sqrt{L}}\right)^q + \varepsilon\right] \int_T |\nabla u|^q\, dV + B \int_T |u|^q\, dV$$

By assumption, we have

$$\mu < \left(\frac{K}{\sqrt{L}}\right)^{-q} (sup_{x\in T} f)^{-q/p}$$

Thus, there exists $\varepsilon > 0$ such that

$$\left[\left(\frac{K}{\sqrt{L}}\right)^q + \varepsilon\right] \inf_{u \in \mathcal{H}_p} I(u) < \frac{1}{(\sup_{x\in T} f(x))^{q/p}}$$

Fix such an ε. Then for any $\bar p$, we have

$$\left(\int_T |u_{\bar p}|^p dV\right)^{q/p} \leqslant \left[\left(\frac{K}{\sqrt{L}}\right)^q + \varepsilon\right] \mu_{\bar p} + \tilde{B}_\varepsilon \int_T u_{\bar p}^q dV$$

for some $\tilde{B}_\epsilon$ independent of $\bar p$.
Moreover we have

$$\frac{1}{\sup_{x\in T} f(x)} = \frac{1}{\sup_{x\in T} f(x)} \int_T f(x) u_{\bar p}^{\bar p} dV$$
$$\leqslant \int_T u_{\bar p}^{\bar p} dV \leqslant \left(\int_T u_{\bar p}^p dV\right)^{\bar p/p} Vol\,(T)^{1-\bar p/p}$$

Thus

$$\left(\int_T u_{\bar{p}}^{p} dV\right)^{q/p} \geqslant \frac{1}{Vol\,(T)^{(1-(\bar{p}/q))q/\bar{p}}} \frac{1}{(\sup_{x\in T} f\,(x))^{q/\bar{p}}}$$

and then

$$\frac{1}{Vol\,(T)^{(q/\bar{p})-(q/p)}} \frac{1}{(\sup_{x\in T} f\,(x))^{q/\bar{p}}} \leqslant \left[\left(\frac{K}{\sqrt{L}}\right)^q + \varepsilon\right] \mu_{\bar{p}} + \tilde{B}_\varepsilon \int_T u_{\bar{p}}^q dV$$

Since

$$\lim_{\bar{p}\to p} \sup \mu_{\bar{p}} \leqslant \inf_{u\in\mathcal{H}_p} I(u)$$

passing to the limit as $\bar{p} \to p$ we obtain

$$\frac{1}{(\sup_{x\in T} f\,(x))^{q/p}} \leqslant \left[\left(\frac{K}{\sqrt{L}}\right)^q + \varepsilon\right] \inf_{u\in\mathcal{H}_p} I(u) + \tilde{B}_\varepsilon \int_T u_p^q dV$$

or

$$\frac{1}{(\sup_{x\in T} f\,(x))^{q/p}} \leqslant \left[\left(\frac{K}{\sqrt{L}}\right)^q + \varepsilon\right] \mu + \tilde{B} \int_T u_p^q dV$$

or

$$\frac{1}{(\sup_{x\in T} f\,(x))^{q/p}} - \left[\left(\frac{K}{\sqrt{L}}\right)^q + \varepsilon\right] \mu \leqslant \tilde{B} \int_T u_p^q dV$$

According to the choice of ε, one gets that

$$\mu < \left(\frac{\sqrt{L}}{K}\right)^q \frac{1}{(\sup_{x\in T} f\,(x))^{(2-q)/2}}$$

and then $\int_T u^q dV > 0$. So $u \not\equiv 0$. □

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