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On Bonciocat's congruences involving the sum of divisors function

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Abstract
We give an elementary proof of some congruences of Bonciocat.

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1. Introduction

Our notation is classical (see e. g. [4]). For a positive integer n we denote by $\sigma_k(n)$ the sum of all k-th powers of positive divisors of n and $\sigma(n)$ denotes $\sigma_1(n)$.

Let $n > 0$ be a positive integer. Set

$$S_0(n) = \sum_{k=1}^{n-1} \sigma(k)\ \sigma(n-k).$$

The main result of Bonciocat's paper [1] is the following:
If $n \equiv 1 \pmod 5$ then $S_0(n) \equiv 0 \pmod 5$ and If $n \equiv -1 \pmod 7$ then $S_0(n) \equiv 0 \pmod 7$.

The proofs use some elegant techniques of Kolberg involving formal power series, (see [6]).

2. Main result

Bonciocat's results are obtained immediately from the following classical formula:

$$12S_0(n) = 5\sigma_3(n) - (6n-1)\sigma(n). \tag{1}$$

These formula first appeared in [3] and appears also in [2, p. 300] ; it is formula (3.10) in [5] where the complete history of the formula is described.

Theorem 2.1 *a) If* $n \equiv 1 \pmod 5$ *then* $S_0(n) \equiv 0 \pmod 5$.

b) If $n \equiv -1 \pmod 7$ *then* $S_0(n) \equiv 0 \pmod 7$.

Proof.
The congruence modulo 5 is immediate by taking $n \equiv 1 \pmod 5$ and by reducing (1) modulo 5. While, the congruence modulo 7, is obtained by reducing (1) modulo 7 so that we require only to observe that for $n \equiv -1 \pmod 7$ (so that n is not a square) one has

$$\sigma_3(n) \equiv \sum_{d|n, d<\sqrt{n}} d^3 + n/d^3 \pmod 7$$

so that each term $d^3 + n/d^3 = d^3 - 1/d^3 = (d^6 - 1)/d^3 \equiv 0 \pmod 7$. So, the reduction modulo 7 of $\sigma_3(n)$ gives also 0 finishing the proof that the two congruences hold.

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