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A Non-Resonant Generalized Multi-Point BVP of Mixed Neumann-Dirichlet Type.

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Abstract

Let ϕ be an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ satisfying $\phi(0) = 0$, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function satisfying Carathéodory conditions and $e : [0,1] \to \mathbb{R}$ be a function in $L^1[0,1]$. Let $\xi_i, \tau_j \in (0,1)$, a_i, $b_j \in R$, $i = 1, 2, \cdots, m-2$, $j = 1, 2, \cdots, n-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \cdots < \tau_{n-2} < 1$ be given. We study the problem of existence of solutions for the generalized multi-point boundary value problem

$$(\phi(x'))' = f(t, x, x') + e, \; 0 < t < 1,$$
$$x'(0) = \sum_{i=1}^{m-2} a_i x(\xi_i), \; x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j), \tag{1}$$

in the non-resonance case. We say that this problem is non-resonant if the associated problem:

$$(\phi(x'))' = 0, \; 0 < t < 1,$$
$$x'(0) = \sum_{i=1}^{m-2} a_i x(\xi_i), \; x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j), \tag{2}$$

has the trivial solution as its only solution. This is the case if

$$(\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j \tau_j - 1) \neq (1 - \sum_{j=1}^{n-2} b_j)(1 - \sum_{i=1}^{m-2} a_i \xi_i).$$

Our methods consist in using topological degree and some a priori estimates.

Mathematics subject classification: 34B10, 34B15, 34L30, 34.90
Keywords: generalized multi-point boundary value problems, $p-$Laplace type operator, non-resonance, a priori estimates, topological degree.

1. Introduction

Let ϕ be an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ satisfying $\phi(0) = 0$, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function satisfying carathéodory conditions and $e : [0,1] \to \mathbb{R}$ be a function in $L^1[0,1]$. Let ξ_i,$\tau_j \in (0,1)$, a_i, $b_j \in R$, $i = 1,2,\cdots,m-2$, $j = 1,2,\cdots,n-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \cdots < \tau_{n-2} < 1$ be given. We study the problem of existence of solutions for the generalized multi-point boundary value problem

$$(\phi(x'))' = f(t,x,x') + e,\ 0 < t < 1,$$
$$x'(0) = \sum_{i=1}^{m-2} a_i x(\xi_i),\ x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j), \tag{3}$$

in the non-resonance case. We say that this problem is non-resonant if the associated problem:

$$(\phi(x'))' = 0,\ 0 < t < 1,$$
$$x'(0) = \sum_{i=1}^{m-2} a_i x(\xi_i),\ x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j), \tag{4}$$

has the trivial solution as its only solution. This is the case if

$$(\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j\tau_j - 1) \neq (1 - \sum_{j=1}^{n-2} b_j)(1 - \sum_{i=1}^{m-2} a_i\xi_i).$$

The study of multi-point boundary value problems for second order ordinary differential equations was initiated by Il'in and Moiseev in [23], [24] motivated by the works of Bitsadze and Samarskĩi on nonlocal linear elliptic boundary value problems, [2], [3], [4] and has been the subject of many papers, see for example, [5], [6], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [25], [30] and [31]. More recently multipoint boundary value problems involving a p-Lalacian type operator or the more general operator $-(\phi(x'))'$ has been studied in [1], [7], [8], [9], [10], [26] to mention a few.

We present in Section 2 some a priori estimates for functions $x(t)$ that satisfy the boundary condtions in (3). Our a priori estimates corresponding estimates in explicitly utilize the non-resonance condition for the boundary value problem (3). In section 3, we present an existence theorem for the boundary value problem (3) using degree theory.

2. A Priori Estimates

We shall assume throughout that ϕ is an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ satisfying $\phi(0) = 0$. We shall also assume that the homeomorphism ϕ satisfies the following conditions:

(a) For any constant $M > 0$,

$$\lim \sup_{z \longrightarrow \infty} \frac{\phi(Mz)}{\phi(z)} \equiv \alpha(M) < \infty. \tag{5}$$

(b) For any σ, $0 \leq \sigma < 1$,

$$\widetilde{\alpha}(\sigma) \equiv \lim \sup_{z \longrightarrow \infty} \frac{\phi(\sigma z)}{\phi(z)} < 1. \tag{6}$$

The boundary value problem (3) is a non-resonant problem if the boundary value problem (4) has only the trivial solution. This holds if and only if

$$(\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j\tau_j - 1) \neq (1 - \sum_{j=1}^{n-2} b_j)(1 - \sum_{i=1}^{m-2} a_i\xi_i). \tag{7}$$

We shall assume in the following that $\xi_i,\tau_j \in (0,1)$, a_i, $b_j \in \mathbb{R}$, $i = 1, 2, \cdots, m-2$, $j = 1, 2, \cdots, n-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \cdots < \tau_{n-2} < 1$ satisfy the condition (7). We observe that when condition (7) holds then at least one of $\sum_{i=1}^{m-2} a_i$, $1 - \sum_{j=1}^{n-2} b_j$ is non-zero. Now, for $a \in \mathbb{R}$, we note that $a^+ = \max(a, 0)$, $a^- = \max(-a, 0)$ so that $a = a^+ - a^-$ and $|a| = a^+ + a^-$. Let us define σ_1 by the following:

$$\sigma_1 \equiv \min \begin{cases} \frac{\sum_{j=1}^{n-2} b_j^+}{1+\sum_{j=1}^{n-2} b_j^-}, \frac{1+\sum_{j=1}^{n-2} b_j^-}{\sum_{j=1}^{n-2} b_j^+}\} \in [0,1), \text{ if } 1 - \sum_{j=1}^{n-2} b_j \neq 0 \text{ and } \sum_{j=1}^{n-2} b_j^+ \neq 0, \\ 0, \text{ if } 1 - \sum_{j=1}^{n-2} b_j \neq 0 \text{ and } \sum_{j=1}^{n-2} b_j^+ = 0, \\ 1, \text{ if } 1 - \sum_{j=1}^{n-2} b_j = 0. \end{cases} \tag{8}$$

Proposition 1. - *Let $\xi_i,\tau_j \in (0,1)$, a_i, $b_j \in R$, $i = 1, 2, \cdots, m-2$, $j = 1, 2, \cdots, n-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \cdots < \tau_{n-2} < 1$, with $(\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j\tau_j - 1) \neq (1 - \sum_{j=1}^{n-2} b_j)(1 - \sum_{i=1}^{m-2} a_i\xi_i)$ be given. Also let the function $x(t)$ be such that $x(t)$, $x'(t)$ be absolutely continuous on $[0,1]$ and $x'(0) = \sum_{i=1}^{m-2} a_i x(\xi_i)$, $x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j)$. Then*

$$\| x \|_\infty \leq M \| x' \|_\infty , \tag{9}$$

where

$$\begin{aligned} M &= \min\{\frac{1}{| \sum_{i=1}^{m-2} a_i |}(1 + \sum_{i=1}^{m-2} | a_i | \lambda_i), \\ &\frac{1}{| \sum_{j=1}^{n-2} b_j |}(\sum_{j=1}^{n-2} | b_j | \mu_j + \frac{\sum_{j=1}^{n-2} | b_j(1-\tau_j) |}{| 1 - \sum_{j=1}^{n-2} b_j |}), \\ &1 + \frac{\sum_{j=1}^{n-2} | b_j(1-\tau_j) |}{| 1 - \sum_{j=1}^{n-2} b_j |}, \frac{1}{1-\sigma_1}\} \end{aligned} \tag{10}$$

with $\lambda_i = \max(\xi_i, 1-\xi_i)$ *for* $i = 1, 2, \cdots, m-2$, $\mu_j = \max(\tau_j, 1-\tau_j)$ *for* $j = 1, 2, \cdots, n-2$, *and* σ_1 *as defined in (8).*

Proof.- We first observe that $M < \infty$, since at least one of $\sum_{i=1}^{m-2} a_i$, $1 - \sum_{j=1}^{n-2} b_j$ is non-zero.

Next, we see from $x(t) = x(\xi_i) + \int_0^t x'(s)ds$ for $i = 1, 2, \cdots, m-2$ and the assumption that $x'(0) = \sum_{i=1}^{m-2} a_i x(\xi_i)$ that

$$(\sum_{i=1}^{m-2} a_i)x(t) = \sum_{i=1}^{m-2} a_i x(\xi_i) + \sum_{i=1}^{m-2} a_i \int_{\xi_i}^t x'(s)ds = x'(0) + \sum_{i=1}^{m-2} a_i \int_{\xi_i}^t x'(s)ds.$$

Accordingly,

$$| \sum_{i=1}^{m-2} a_i || x(t) | \leq | x'(0) | + \sum_{i=1}^{m-2} | a_i || \int_{\xi_i}^t x'(s)ds \ | ,$$

$$\leq (1 + \sum_{i=1}^{m-2} \lambda_i \mid a_i \mid) \parallel x' \parallel_\infty ,$$

where $\lambda_i = \max(\xi_i, 1-\xi_i)$ for $i = 1, 2, \cdots, m-2$. It is now immediate that

$$\parallel x \parallel_\infty \leq \frac{1}{| \sum_{i=1}^{m-2} a_i |}(1 + \sum_{i=1}^{m-2} \lambda_i \mid a_i \mid) \parallel x' \parallel_\infty . \tag{11}$$

Now, we shall assume in the following that $1 - \sum_{j=1}^{n-2} b_j \neq 0$, since we see from (10) that $M = \dfrac{1}{| \sum_{i=1}^{m-2} a_i |}(1 + \sum_{i=1}^{m-2} \mid a_i \mid \lambda_i)$ when $1 - \sum_{j=1}^{n-2} b_j = 0$.

Now, we see from $x(1) - x(\tau_j) = \int_{\tau_j}^1 x'(s)ds$, for $j = 1, 2, \cdots, n-2$, and the assumption $x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j)$ that $(\sum_{j=1}^{n-2} b_j - 1)x(1) = \sum_{j=1}^{n-2} b_j \int_{\tau_j}^1 x'(s)ds$. It, then, follows that

$$|x(1)| \leq \frac{\sum_{j=1}^{n-2} \mid b_j(1-\tau_j) \mid}{\mid 1 - \sum_{j=1}^{n-2} b_j \mid} \parallel x' \parallel_\infty . \tag{12}$$

Next, we use the equations $x(t) - x(\tau_j) = \int_{\tau_j}^t x'(s)ds$, for $t \in [0,1]$, $j = 1, 2, \cdots, n-2$, and the assumption $x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j)$ to get

$$x(t) = \frac{1}{\sum_{j=1}^{n-2} b_j}(x(1) + \sum_{j=1}^{n-2} b_j \int_{\tau_j}^t x'(s)ds), \text{ for } t \in [0,1]. \tag{13}$$

It, now, follows from (12), (13) and with $\mu_j = \max(\tau_j, 1-\tau_j)$ for $j = 1, 2, \cdots, n-2$ that

$$\| x \|_\infty \leq \frac{1}{| \sum_{j=1}^{n-2} b_j |}(\frac{\sum_{j=1}^{n-2} | b_j(1-\tau_j) |}{| 1 - \sum_{j=1}^{n-2} b_j |} + \sum_{j=1}^{n-2} \mu_j | b_j |) \| x' \|_\infty . \tag{14}$$

Similarly. starting from the equation $x(t) = x(1) - \int_t^1 x'(s)ds$, we obtain the estimate

$$\| x \|_\infty \leq (\frac{\sum_{j=1}^{n-2} | b_j(1-\tau_j) |}{| 1 - \sum_{j=1}^{n-2} b_j |} + 1) \| x' \|_\infty . \tag{15}$$

Next, since $x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j)$ we see that

$$x(1) + \sum_{j=1}^{n-2} b_j^- x(\tau_j) = \sum_{j=1}^{n-2} b_j^+ x(\tau_j).$$

It follows that there must exist χ_1, χ_2 in $[0,1]$ such that

$$(1 + \sum_{j=1}^{n-2} b_j^-)x(\chi_1) = (\sum_{i=1}^{m-2} b_j^+)x(\chi_2). \tag{16}$$

If, now, one of $x(\chi_1)$, $x(\chi_2)$ is zero, we see using one of the two equations

$$x(t) = x(\chi_k) + \int_{\tau_k}^t x'(s)ds, \; k = 1,2; \; t \in [0,1] \tag{17}$$

that

$$\| x \|_\infty \leq \| x' \|_\infty . \tag{18}$$

If both $x(\chi_1)$, $x(\chi_2)$ are non-zero it is easy to see from (16) that $x(\chi_1) \neq x(\chi_2)$, since we have assumed that $1 - \sum_{j=1}^{n-2} b_j \neq 0$, so that $1 + \sum_{j=1}^{n-2} b_j^- \neq \sum_{j=1}^{n-2} b_j^+$. It then follows easily from (16) and (17) that

$$\| x \|_\infty \leq \frac{1}{1-\sigma_1} \| x' \|_\infty , \tag{19}$$

where σ_1 as defined in (8).

The proposition is now immediate from (11), (14), (15), (18), (19) and the definition of σ_1 as given in (8).//

Lemma 2. - *Let us set*

$$\begin{aligned} A &= (1 - \sum_{j=1}^{n-2} b_j)^+ + \sum_{j=1}^{n-2} [b_j(1-\tau_j)(\sum_{i=1}^{m-2} a_i)]^+ \\ &\quad + \sum_{i=1}^{m-2} [a_i(1-\xi_i)(1 - \sum_{j=1}^{n-2} b_j)]^+ \end{aligned} \tag{20}$$

and

$$B = (1-\sum_{j=1}^{n-2} b_j)^- + \sum_{j=1}^{n-2}[b_j(1-\tau_j)(\sum_{i=1}^{m-2} a_i)]^- + \sum_{i=1}^{m-2}[a_i(1-\xi_i)(1-\sum_{j=1}^{n-2} b_j)]^- \tag{21}$$

Then

$$A \neq B,$$

when the non-resonance assumption (7) holds.

Proof.- We note that

$$\begin{aligned} A-B &= (1-\sum_{j=1}^{n-2} b_j) + \sum_{j=1}^{n-2}[b_j(1-\tau_j)(\sum_{i=1}^{m-2} a_i)] + \sum_{i=1}^{m-2}[a_i(1-\xi_i)(1-\sum_{j=1}^{n-2} b_j)] \\ &= 1-\sum_{j=1}^{n-2} b_j + (\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j - \sum_{j=1}^{n-2} b_j\tau_j) \\ &\quad +(1-\sum_{j=1}^{n-2} b_j)(\sum_{i=1}^{m-2} a_i - \sum_{i=1}^{m-2} a_i\xi_i) \\ &= 1-\sum_{j=1}^{n-2} b_j - (\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j\tau_j) + \sum_{i=1}^{m-2} a_i - (1-\sum_{j=1}^{n-2} b_j)(\sum_{i=1}^{m-2} a_i\xi_i) \\ &= (1-\sum_{j=1}^{n-2} b_j)(1-\sum_{i=1}^{m-2} a_i\xi_i) - (\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j\tau_j - 1) \\ &\neq 0, \end{aligned}$$

in view of the non-resonance assumption (7). Hence $A \neq B$. This completes the proof of the lemma.//

Let us define σ^* by

$$\sigma^* = \min\{\frac{A}{B}, \frac{B}{A}\} \in [0,1), \tag{22}$$

where A, B are as defined in Lemma 2.. Accordingly, we see that

$$\widetilde{\alpha}(\sigma^*) = \limsup_{z\to\infty} \frac{\phi(\sigma^* z)}{\phi(z)} < 1,$$

in view of our assumption (6). Let $\varepsilon > 0$ be such that $\widetilde{\alpha}(\sigma^*) + \varepsilon < 1$ and the constant C_ε be such that

$$\phi(\sigma^* z) \leq (\widetilde{\alpha}(\sigma^*) + \varepsilon)\phi(z) + C_\varepsilon, \text{ for every } z \in \mathbb{R}. \tag{23}$$

Proposition 3. - *Let* $\xi_i, \tau_j \in (0,1)$, a_i, $b_j \in R$, $i = 1, 2, \cdots, m-2$, $j = 1, 2, \cdots, n-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \cdots < \tau_{n-2} < 1$, *with* $(\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j\tau_j - 1) \neq (1 - \sum_{j=1}^{n-2} b_j)(1 - \sum_{i=1}^{m-2} a_i\xi_i)$ *be given. Also let the function* $x(t)$ *be such that* $x(t)$, $x'(t)$ *be absolutely continuous on* $[0,1]$ *with* $(\phi(x'))' \in L^1(0,1)$ *and* $x'(0) = \sum_{i=1}^{m-2} a_i x(\xi_i)$, $x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j)$. *Then*

$$\| \phi(x') \|_\infty \leq \frac{1}{1 - \widetilde{\alpha}(\sigma^*) - \varepsilon} \| (\phi(x'))' \|_{L^1(0,1)} + \frac{C_\varepsilon}{1 - \widetilde{\alpha}(\sigma^*) - \varepsilon}, \tag{24}$$

where ε *and* C_ε *are as in (23).*

Proof.- For $j = 1, 2, \cdots, n-2$ we see using mean value theorem that there exist λ_j in $[0,1]$ such that

$$x(1) - x(\tau_j) = (1 - \tau_j)x'(\lambda_j),$$

and we see using $x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j)$ that

$$(\sum_{j=1}^{n-2} b_j - 1)x(1) = \sum_{j=1}^{n-2} b_j(1 - \tau_j)x'(\lambda_j). \tag{25}$$

Also, for $i = 1, 2, \cdots, m-2$ we see using mean value theorem that there exist χ_i in $[0,1]$ such that

$$x(1) - x(\xi_i) = (1 - \xi_i)x'(\chi_i),$$

and we see using $x'(0) = \sum_{i=1}^{m-2} a_i x(\xi_i)$ that

$$(\sum_{i=1}^{m-2} a_i)x(1) - x'(0) = \sum_{i=1}^{m-2} a_i(1 - \xi_i)x'(\chi_i). \tag{26}$$

Now, we see from equations (25), (26) that

$$\begin{aligned} & (\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j(1 - \tau_j)x'(\lambda_j)) \\ = & (\sum_{j=1}^{n-2} b_j - 1)(x'(0) + \sum_{i=1}^{m-2} a_i(1 - \xi_i)x'(\chi_i)) \\ = & (\sum_{j=1}^{n-2} b_j - 1)x'(0) + (\sum_{j=1}^{n-2} b_j - 1)(\sum_{i=1}^{m-2} a_i(1 - \xi_i)x'(\chi_i)). \end{aligned}$$

It follows that

$$\begin{aligned} & (\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j(1 - \tau_j)x'(\lambda_j)) \\ & + (1 - \sum_{j=1}^{n-2} b_j)x'(0) + (1 - \sum_{j=1}^{n-2} b_j)(\sum_{i=1}^{m-2} a_i(1 - \xi_i)x'(\chi_i)) = 0. \end{aligned}$$

Using, next, the intermediate value theorem we see that there exist υ_1, υ_2 in $[0,1]$ such that

$$Ax'(\upsilon_1) - Bx'(\upsilon_2) = 0, \tag{27}$$

where A, B are as defined in (20), (21). Suppose, now, one of $x'(\upsilon_1)$, $x'(\upsilon_2)$ is zero. We then see from one of the following equations

$$\phi(x'(t)) = \phi(x'(\upsilon_k)) + \int_{\upsilon_k}^{t} (\phi(x'))'(s)ds,\ k = 1, 2;\ t \in [0,1] \tag{28}$$

that

$$\| \phi(x') \|_\infty \leq \| (\phi(x'))' \|_{L^1(0,1)} . \tag{29}$$

Let us, next, suppose that both $x'(\upsilon_1)$, $x'(\upsilon_2)$ are non-zero. Since, now, $A \neq B$, in view of Lemma 2. we see from equation (27) that

$$x'(\upsilon_1) \neq x'(\upsilon_2).$$

We now use the equations

$$\begin{aligned} \phi(x'(t)) &= \phi(x'(\upsilon_1)) + \int_{\upsilon_k}^{t} (\phi(x'))'(s)ds \\ &= \phi(\frac{B}{A}x'(\upsilon_2)) + \int_{\upsilon_k}^{t} (\phi(x'))'(s)ds, \end{aligned}$$

$$\begin{aligned} \phi(x'(t)) &= \phi(x'(\upsilon_2)) + \int_{\upsilon_k}^{t} (\phi(x'))'(s)ds \\ &= \phi(\frac{A}{B}x'(\upsilon_1)) + \int_{\upsilon_k}^{t} (\phi(x'))'(s)ds, \end{aligned}$$

along with the definition of σ^*, as given in (22), (23) and the estimate (29) to obtain the estimate (24). This completes the proof of the proposition.

3. Existence Theorem

Let ϕ be an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ satisfying $\phi(0) = 0$, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function satisfying carathéodory conditions and $e : [0,1] \to \mathbb{R}$ be a function in $L^1[0,1]$. Let ξ_i,$\tau_j \in (0,1)$, a_i, $b_j \in R$, $i = 1, 2, \cdots, m-2$, $j = 1, 2, \cdots, n-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $0 < \tau_1 < \tau_2 < \cdots < \tau_{n-2} < 1$ with $(\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j\tau_j - 1) \neq (1 - \sum_{j=1}^{n-2} b_j)(1 - \sum_{i=1}^{m-2} a_i\xi_i)$ be given.

Theorem 4. - *Let $f : [0,1] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function satisfying carathéodory's conditions such that there exist non-negative functions $d_1(t)$, $d_2(t)$, and $r(t)$ in $L^1(0,1)$ such that*

$$|f(t,u,v)| \leq d_1(t)\phi(|u|) + d_2(t)\phi(|v|) + r(t),$$

for a. e. $t \in [0,1]$ *and all* $u,\ v \in \mathbb{R}$. *Suppose, further,*

$$\alpha(M)||d_1||_{L^1(0,1)} + ||d_2||_{L^1(0,1)} < 1 - \widetilde{\alpha}(\sigma^*) \tag{30}$$

where M *is as defined in Proposition 1.,* $\alpha(M)$ *is as defined in (5),* σ^* *and* $\widetilde{\alpha}(\sigma^*)$ *are as defined in (22), (23). Then, for every given function* $e(t) \in L^1[0,1]$, *the boundary value problem (3) has at least one solution* $x(t) \in C^1[0,1]$.

Proof.- We consider the family of boundary value problems

$$\begin{aligned} (\phi(x'))' &= \lambda f(t,x,x') + \lambda e,\ 0<t<1,\ \lambda\in[0,1] \\ x'(0) &= \sum_{i=1}^{m-2} a_i x(\xi_i),\ x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j). \end{aligned} \tag{31}$$

Also, we define an operator $\Psi : C^1[0,1]\times[0,1] \longrightarrow C^1[0,1]$ by setting for $(x,\lambda) \in C^1[0,1]\times[0,1]$

$$\begin{aligned} \Psi(x,\lambda)(t) &= x(0) + \int_0^t \phi^{-1}(\phi(x'(0) + \lambda\int_0^s (f(\tau,x(\tau),x'(\tau)) + e(\tau))d\tau)ds \\ &\quad +(x'(0) - \sum_{i=1}^{m-2} a_i x(\xi_i)) + t(x(1) - \sum_{j=1}^{n-2} b_j x(\tau_j)) \end{aligned} \tag{32}$$

Let us, suppose that $x(t) \in C^1[0,1]$ is a solution to the operator equation, for some $\lambda \in [0,1]$,

$$\begin{aligned} x &= \Psi(x,\lambda) \\ &= x(0) + \int_0^t \phi^{-1}(\phi(x'(0) + \lambda\int_0^s (f(\tau,x(\tau),x'(\tau)) + e(\tau))d\tau)ds \\ &\quad +(x'(0) - \sum_{i=1}^{m-2} a_i x(\xi_i)) + t(x(1) - \sum_{j=1}^{n-2} b_j x(\tau_j)) \end{aligned} \tag{33}$$

Evaluating the equation (33) at $t=0$ we see that $x(t)$ satisfies the boundary condition

$$x'(0) = \sum_{i=1}^{m-2} a_i x(\xi_i).$$

Next, we differentiate the equation (33) with respect to t to get

$$\begin{aligned} x'(t) &= \phi^{-1}(\phi(x'(0) + \lambda\int_0^t (f(\tau,x(\tau),x'(\tau)) + e(\tau))d\tau \\ &\quad +x(1) - \sum_{j=1}^{n-2} b_j x(\tau_j). \end{aligned} \tag{34}$$

Evaluating,now, the equation (34) at $t = 0$ we see that $x(t)$ satisfies the boundary condition

$$x(1) = \sum_{j=1}^{n-2} b_j x(\tau_j),$$

and on differentiating the equation (34) with respect to t we get

$$(\phi(x'))' = \lambda f(t, x, x') + \lambda e,\ 0 < t < 1,\ \lambda \in [0, 1].$$

Thus we see that if $x(t) \in C^1[0, 1]$ is a solution to the operator equation $x = \Psi(x, \lambda)$ for some $\lambda \in [0, 1]$ then $x(t)$ is a solution to the boundary value problems (31) for the corresponding $\lambda \in [0, 1]$. Conversely, it is easy to see that if $x(t) \in C^1[0, 1]$ is a solution to the boundary value problems (31) for some $\lambda \in [0, 1]$ then $x(t) \in C^1[0, 1]$ is a solution to the operator equation $x = \Psi(x, \lambda)$ for the corresponding $\lambda \in [0, 1]$.

Next, it is easy to show, following standard arguments, that $\Psi : C^1[0, 1] \times [0, 1] \longrightarrow C^1[0, 1]$ is a completely continuous operator.

We shall next show that there is a constant $R > 0$, independent of $\lambda \in [0, 1]$, such that if $x(t) \in C^1[0, 1]$ is a solution to (33), equivalently to the boundary value problems (31), for some $\lambda \in [0, 1]$ then $||x||_{C^1[0,1]} < R$.

We note first that if $x(t) \in C^1[0, 1]$ satisfies

$$x = \Psi(x, 0), \tag{35}$$

then $x(t) = 0$ for all $t \in [0, 1]$. Indeed, from the definition of Ψ or from the boundary value problem (31), it follows that $x(t) = x(0) + x'(0)t$. It then follows from the two boundary conditions in (31) and the non-resonance assumption (7) we find that $x(0) = x'(0) = 0$, implying that $x(t) = 0$ for all $t \in [0, 1]$.

We shall assume, in the following, that $\lambda \in (0, 1]$. We shall also assume that σ^*, as defined in (22) is positive, since the proof for the case $\sigma^* = 0$ is simpler. Let us choose $\varepsilon > 0$ such that $\widetilde{\alpha}(\sigma^*) + \varepsilon < 1$ and

$$(\alpha(M) + \varepsilon)||d_1||_{L^1(0,1)} + ||d_2||_{L^1(0,1)} < 1 - \widetilde{\alpha}(\sigma^*) - \varepsilon, \tag{36}$$

which is possible to do, in view of our assumption (30). Here M is as defined in Propostion 1. and $\alpha(M)$ is as defined in (5) so that for the $\varepsilon > 0$, chosen above, there exists a constant $C^1_\varepsilon > 0$ such that

$$\phi(Mz) \leq (\alpha(M) + \varepsilon)\phi(z) + C^1_\varepsilon, \text{ for every } z \in \mathbb{R}. \tag{37}$$

Also, from Proposition 3. we see that there is a constant $C^2_\varepsilon > 0$, for the chosen $\varepsilon > 0$, such that

$$\phi(||x'||_\infty) \leq \frac{1}{1 - \widetilde{\alpha}(\sigma^*) - \varepsilon}||(\phi(x'))'||_{L^1(0,1)} + C^2_\varepsilon. \tag{38}$$

We, now, see from the equation in (31), using our assumptions on the function f, Propostion 1., and estimates (37), (38) that

$$\begin{aligned} ||(\phi(x'))'||_{L^1(0,1)} &\le \phi(||x||_\infty)||d_1||_{L^1(0,1)} + \phi(||x'||_\infty)||d_2||_{L^1(0,1)} \\ &\quad +||r||_{L^1(0,1)} + ||e||_{L^1(0,1)} \\ &\le \phi(M||x'||_\infty)||d_1||_{L^1(0,1)} + \phi(||x'||_\infty)||d_2||_{L^1(0,1)} \\ &\quad +||r||_{L^1(0,1)} + ||e||_{L^1(0,1)} \\ &\le ((\alpha(M)+\varepsilon)||d_1||_{L^1(0,1)} + ||d_2||_{L^1(0,1)})\phi(||x'||_\infty) \\ &\quad +||r||_{L^1(0,1)} + ||e||_{L^1(0,1)} + C_\varepsilon^1||d_1||_{L^1(0,1)} \\ &\le \frac{(\alpha(M)+\varepsilon)||d_1||_{L^1(0,1)} + ||d_2||_{L^1(0,1)}}{1-\widetilde{\alpha}(\sigma^*)-\varepsilon}||(\phi(x'))'||_{L^1(0,1)} + C\varepsilon, \end{aligned}$$

where $C\varepsilon = ||r||_{L^1(0,1)} + ||e||_{L^1(0,1)} + C_\varepsilon^1||d_1||_{L^1(0,1)} + C_\varepsilon^2[(\alpha(M)+\varepsilon)||d_1||_{L^1(0,1)} + ||d_2||_{L^1(0,1)}]$. It, now, follows from (36) that there exists a constant R_0, independent of $\lambda \in [0,1]$, such that if $x(t) \in C^1[0,1]$ is a solution to the boundary value problems (31) for some $\lambda \in [0,1]$ then

$$||(\phi(x'))'||_{L^1(0,1)} \le R_0.$$

This combined with (38) and (9) give that there exists a constant $R > 0$ such that

$$||x||_{C^1[0,1]} < R.$$

This then implies that $\deg_{LS}(I - \Psi(\cdot,\lambda), B(0,R), 0)$ is well-defined for all $\lambda \in [0,1]$, where $B(0,R)$ is the ball with center 0 and radius R in $C^1[0,R]$.

Let, now, X denote the two-dimensional subspace of $C^1[0,1]$ given by

$$X = \{A + Bt \mid \text{for } A,\ B \in \mathbb{R}\}. \tag{39}$$

Let us define the isomorphism $i : \mathbb{R}^2 \longrightarrow X$ by

$$i\begin{pmatrix} A \\ B \end{pmatrix} = i_{\begin{pmatrix} A \\ B \end{pmatrix}} \in X, \text{ for } \begin{pmatrix} A \\ B \end{pmatrix} \in \mathbb{R}^2, \tag{40}$$

where

$$i_{\begin{pmatrix} A \\ B \end{pmatrix}}(t) = A + Bt, \text{ for } t \in [0,1]. \tag{41}$$

Also, we define a 2×2 matrix $\mathbb{A}$ by setting

$$\mathbb{A} = \begin{pmatrix} \sum_{i=1}^{m-2} a_i & \sum_{i=1}^{m-2} a_i\xi_i - 1 \\ -(1 - \sum_{j=1}^{n-2} b_j) & \sum_{j=1}^{n-2} b_j\tau_j - 1 \end{pmatrix}. \tag{42}$$

We note that $\det \mathbb{A} = (\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j\tau_j - 1) - (1 - \sum_{j=1}^{n-2} b_j)(1 - \sum_{i=1}^{m-2} a_i\xi_i) \neq 0$, in view of the non-resonance assumption (7).

Next, we define a function $G : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by setting

$$\begin{aligned} G\begin{pmatrix} A \\ B \end{pmatrix} &= \mathbb{A}\cdot \begin{pmatrix} A \\ B \end{pmatrix} \\ &= \begin{pmatrix} (\sum_{i=1}^{m-2} a_i)A - (1 - \sum_{i=1}^{m-2} a_i\xi_i)B \\ -(1 - \sum_{j=1}^{n-2} b_j)A + (\sum_{j=1}^{n-2} b_j\tau_j - 1)B \end{pmatrix}, \end{aligned} \tag{43}$$

for $\begin{pmatrix} A \\ B \end{pmatrix} \in \mathbb{R}^2$.

We note that for $v(t) = A + Bt \in X$ we have

$$(I - \Psi(\cdot, 0))(v) = i_{G\begin{pmatrix} A \\ B \end{pmatrix}},$$

and it follows that

$$G = i^{-1} \circ ((I - \Psi(\cdot, 0))|_X \circ i.$$

Now, we see from the homotopy invariance property of the Leray-Schauder degree that

$$\begin{aligned} \deg_{LS}(I - \Psi(\cdot, 1), B(0, R), 0) &= \deg_{LS}(I - \Psi(\cdot, 0), B(0, R), 0) \\ &= \deg_B(I - \Psi(\cdot, 0)|_X, X \cap B(0, R), 0) \\ &= \deg_B(G, \mathbb{B}(0, R), 0), \end{aligned}$$

where $\mathbb{B}(0, R)$ denotes the ball of radius R in $\mathbb{R}^2$ with center at the origin. Finally, we have, using standard results for Brouwer degree, (see [27], [28], [29]) that

$$\deg_B(G, \mathbb{B}(0, R), 0) = \begin{cases} 1, \text{ if } \det \mathbb{A} > 0 \\ -1, \text{ if } \det \mathbb{A} < 0. \end{cases}$$

Next, we see from the non-resonance assumption (7) that

$$\det \mathbb{A} = (\sum_{i=1}^{m-2} a_i)(\sum_{j=1}^{n-2} b_j\tau_j - 1) - (1 - \sum_{j=1}^{n-2} b_j)(1 - \sum_{i=1}^{m-2} a_i\xi_i) \neq 0,$$

and hence $\deg_{LS}(I - \Psi(\cdot, 1), B(0, R), 0) \neq 0$. Accordingly, there exists $x(t) \in B(0, R) \subset C^1[0, 1]$ such that

$$x = \Psi(x, 1),$$

equivalently $x(t)$ is a solution to the boundary value (3). This completes the proof of the theorem.//

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