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# On the Shape Operator of the Hypersurfaces $M_2^3$ of $E_2^4$

G. Kaimakamis, V.J. Papantoniou and K. Petoumenos

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**Abstract**

In his paper ([3], [4]) M. Magid shows that there exist four possible canonical forms for the shape operator of Lorentz hypersurfaces of $E_1^4$. The study of submanifolds of the pseudo-Euclidean space $E_2^4$ in Differential Geometry is very interesting ([1]). For examble the problem of finding, if there exists, any relation between the minimality and the biharmonicity of a hypersurface $M_2^3$ of $E_2^4$ is open.

The shape operator $S$ plays a crucial role in the study of these hypersurfaces and in these cases is not generally, diagonalizable.

In this paper we find out all the possible canonical forms of $S$ and prove that these agree with those found by Magid.

*Keywords*: Pseudo-Euclidean space, Pseudo-Euclidean hypersurface, Shape operator, Characteristic polynomial.

## 1. Introduction

Let $M_2^3$ be a hypersurface of the pseudo-Euclidean manifold $E_2^4$ with signatures $(-,+,-)$ and $(-,+,-,+)$ respectively. Then the three dimensional tangent space $T_p(M_2^3)$ for each $P \in M_2^3$ is isomorphic to the space $E_2^3$ with the same signature $(-,+,-)$. Hence if $\overrightarrow{\xi}$ is the unit normal vector of $T_p(M_2^3)$, then $<\overrightarrow{\xi},\overrightarrow{\xi}>=+1$. As it is known ([2],[5]) a non zero vector X in $E_2^4$ is called **time-like**, **space-like**, or **light-like** (or isotropic) according to whether $<X,X>$ is negative, positive or zero. The zero vector is considered to be **space-like**.

The shape operator $S$ of the hypersurface $M_2^3$ is a symmetric endomorphism of $T_p(M_2^3)$ for each $P \in M_2^3$, satisfying the relation

$$<S(X),Y>=<X,S(Y)> \tag{1.1}$$

It is well known that the shape operator of a Riemannian submanifold is always diagonalizable, but as we prove in this paper, this is not the case for the shape operator of a pseudo-Riemannian submanifold.

In fact, since $S$ is an endomorphism, one can find all its canonical forms with respect to suitable bases.

Therefore, if we consider any basis of $T_p(M_2^3)$, then we conclude that

$$GA = A^t G, \tag{1.2}$$

where $A$ and $G$ denote the matrix representations of $S$ and $<,>$ respectively, with respect to the considered basis. The eigenvalues of $A$ are the roots of the equation

$$\det(A - \lambda I) = 0. \tag{1.3}$$

This equation has three eigenvalues in general, say $\lambda_i$, $i = 1, 2, 3$. Let $v_i$, $E_{\lambda_i}$, $i = 1, 2, 3$ be the corresponding eigenvectors and eigenspaces. These eigenvectors do not constitute necessarily a basis. In fact, if the geometric multiplicity ($\dim E_{\lambda_i}$) is less than the algebraic multiplicity of the eigenvalue, there are not enough eigenvectors to construct a basis.

In these cases we start with an eigenvector, as a starting point, and then add, appropriately, more vectors which are not eigenvectors themselves, to build a basis.

Hence, the basic question which usually arises, is what is the nature of these eigenvectors.

In the next sections we find all the possible canonical forms of $S$ and $G$ by examining the various values of $\lambda_i$ and the nature of the vectors $v_i$.

More precisely, in the second section we find the canonical form of $S$ and $G$ if the eigenvalues are different. In this case we prove that none of the vectors $v_i$ can be light-like and therefore these vectors constitute an orthonormal basis.

In the third section we assume that one of the roots of (1.3) has multiplicity 2. In this case several cases appear which are studied separately. More precisely, the eigenvectors $v_i$ either constitute an orthonormal basis or a pseudo-orthonormal basis, e.g. a basis which contains and light-like vectors that satisfy the relations

$$\begin{aligned} &< v_1, v_1 >=< v_2, v_2 >=< v_1, v_3 >=< v_2, v_3 >= 0, \\ &< v_1, v_2 >= +1, \qquad < v_3, v_3 >= -1 \end{aligned} \tag{1.4}$$

for $v_1, v_2$ being light-like vectors and $v_3$ a time-like vector.

In the fourth section we assume that the equation (1.3) has one real root of multiplicity 3 and solve the problem for the various cases which appear.

Finally, in the fifth section we assume that the equation (1.3) has one real and two complex conjugate roots. In this case the only eigenspace which we consider and study is the one that corresponds to the real root (as we do not consider complex eigenspaces). Therefore, in this case the eigenvectors constitute either an orthonormal or a pseudo-orthonormal basis and solve the problem.

By combining the results of these sections we prove the following theorem.

**Theorem 1.1** *If $S$ is the shape operator of $M_2^3$ in $E_2^4$, then there exist orthonormal or pseudo-orthonormal frames with respect to which the canonical forms of $S$ and $G$ are given by the matrices*

(I)

$$[S] = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \lambda_i \neq 0, \quad [G] = \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(II)

$$[S] = \begin{pmatrix} \lambda & \mu & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \nu \end{pmatrix}, \lambda \neq 0,\ \nu \neq 0, \quad [G] = \begin{pmatrix} 0 & +1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(III)

$$[S] = \begin{pmatrix} \lambda & \mu & \nu \\ 0 & \lambda & 0 \\ 0 & -\nu & \lambda \end{pmatrix}, \lambda \neq 0,\ \mu, \nu \in \mathbb{R}, \quad [G] = \begin{pmatrix} 0 & +1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(IV)

$$[S] = \begin{pmatrix} \mu & \nu & 0 \\ -\nu & \mu & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \nu \neq 0, \quad [G] = \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The matrices $G$, of the metric tensor for the cases (I), (IV)) are referred to an orthonormal basis of $T_p(M_2^3)$, whereas for the cases (II), (III) are referred to a pseudo-orthonormal basis.

## 2. The eigenvalues of $S$ are different

As we mentioned earlier, if we consider any basis of the space $T_p(M_2^3)$ then the eigenvalues of the shape operator $S$ with respect to this basis are the roots of the equation (1.3), where we denote by $A$ the matrix of $S$ with respect to this basis. Assume that this equation has three real and distinct eigenvalues $\lambda_i$, $i = 1, 2, 3$ and let $v_i$, $i = 1, 2, 3$ be the corresponding eigenvectors. Let $\mathcal{B} = \{v_i\}$, $i = 1, 2, 3$ be the basis which constitute these eigenvectors. Then, $S$ with respect to this basis is diagonalised, i.e.

$$[S]_{\mathcal{B}} = A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{2.1}$$

Next, we obtain the matrix $G$ of the covariant tensor field $<,>$, which for each $P \in M_2^3$ gives rise the inner product $<,>$, which is a 2-linear form over the space $T_p(M_2^3)$, with value

$$< u, v >= -u^1v^1 + u^2v^2 - u^3v^3 \tag{2.2}$$

Hence, the matrix $G$ depends largely on the nature of the vectors $v_i$, $i = 1, 2, 3$. Applying the relation (1.1) we easily conclude that

$$< v_i, v_j >= 0, \quad i \neq j \tag{2.3}$$

Therefore, the vectors $v_i$ constitute a basis and none of them is light-like. In fact, if $v_2$ say, is light-like then from the relation (2.3) for $i = 2$ and $j = 1, 3$ we conclude that the vectors $v_1, v_3$ lie in the orthogonal complement $v_2^{\perp}$, of $v_2$. Hence they are not linearly independent. So, the only possible cases for the nature of the vectors $v_i$ are one of these has to be space-like and the remainder two, time-like. For example if $v_2$ is space-like and $v_1, v_3$ time-like, then

$$\begin{aligned} &< v_2, v_2 >= +1, \quad < v_1, v_1 >=< v_3, v_3 >= -1, \\ &< v_1, v_2 >=< v_2, v_3 >=< v_1, v_3 >= 0 \end{aligned} \tag{2.4}$$

So, the vectors $\{v_i\}$, $i = 1, 2, 3$ constitute an **orthonormal basis**, and therefore the matrix $G$ of $<, >$ with respect to this basis is

$$[G]_{\mathcal{B}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{2.5}$$

Hence we have proved the following proposition

**Proposition 2.1** *If $S$ is the shape operator of $M_2^3$ in $E_2^4$ and the equation (1.3) has three real and distinct eigenvalues, then none of the corresponding eigenvectors is light-like, and hence they constitute an orthonormal basis. Moreover, the canonical forms of $S$ and the inner product $<, >$ with respect to this basis are given by (2.1) and (2.5) respectively.*

## 3. One of the eigenvalues of $S$ has multiplicity two

Assume that $A$ has one real root $\lambda$ of multiplicity two, and one simple real root $\lambda_3$, $(\lambda \neq \lambda_3)$. Let $E_\lambda$, $E_{\lambda_3}$ be the eigenspaces corresponding to the eigenvalues $\lambda$ and $\lambda_3$. Then the dimension of $E_\lambda$ is 1, or 2 and the dimension of $E_{\lambda_3}$ is 1.

**Subcase 3.1:** Consider the case $\dim E_\lambda = 1$ and $\dim E_{\lambda_3} = 1$.

Let $v_1, v_2$ be vectors of $T_p(M_2^3)$ such that $E_\lambda = span\{v_1\}$, $E_{\lambda_3} = span\{v_2\}$. We need to determine the nature of the vectors $v_1, v_2$. Then, adding a third suitable vector, say $v_3$ we can construct a basis $\{v_1, v_2, v_3\}$ for $T_p(M_2^3)$. First, we prove that $< v_1, v_2 >= 0$. In fact applying the relation (1.1) we have that $< v_1, S(v_2) >=$ $< S(v_1), v_2 >$, or $< v_1, \lambda_3 v_2 >=< \lambda v_1, v_2 >$, and since $\lambda_3 \neq \lambda$, then $< v_1, v_2 >= 0$.

**A.** Let $v_1$ be **time-like** vector and $v_2, v_3$ be **light-like** vectors.
Then $< v_1, v_1 >= -1$, $< v_2, v_3 >= +1$, and $< v_1, v_2 >=< v_1, v_3 >=< v_2, v_2 >=$ $< v_3, v_3 >= 0$.

So we have constructed a **pseudo-orthonormal** basis $\mathcal{U} = \{v_1, v_2, v_3\}$. Hence, the matrix representation $G = (g_{ij})_{\mathcal{U}}$ where $g_{ij} = g(v_i, v_j)$ of the metric tensor is

$$[G]_{\mathcal{U}} \quad = \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & +1 & 0 \end{pmatrix} \tag{3.1}$$

Let $S : T_p(M_2^3) \to T_p(M_2^3)$, be the shape operator of $T_p(M_2^3)$. Next, we find the canonical form of $S$ with respect to this basis. Let

$$[S]_{\mathcal{U}} = A = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}$$

be this canonical form. By using the fact that $S$ is a symmetric endomorphism, we can apply the relation (1.2) and conclude that: $\lambda_{31} = -\lambda_{12}$, $\lambda_{13} = -\lambda_{21}$, $\lambda_{22} = \lambda_{33}$.

Therefore, the matrix $A$ takes the simplest form

$$[S]_{\mathcal{U}} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & -\lambda_{21} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ -\lambda_{12} & \lambda_{32} & \lambda_{22} \end{pmatrix}$$

On the other hand, the characteristic polynomial of $A$ is:

$$f(t) = t^3 - (\lambda_{11} + 2\lambda_{22})t^2 + (2\lambda_{11}\lambda_{22} - 2\lambda_{12}\lambda_{21} - \lambda_{23}\lambda_{32} + \lambda_{22}^2)t$$
$$-(\lambda_{11}\lambda_{22}^2 - \lambda_{11}\lambda_{23}\lambda_{32} - 2\lambda_{12}\lambda_{21}\lambda_{22} - \lambda_{12}^2\lambda_{23} - \lambda_{21}^2\lambda_{32}) \tag{3.2}$$

Since $A$ has one real root $\lambda$ of multiplicity two, and one simple real root $\lambda_3$, then

$$f(t) = t^3 - (2\lambda + \lambda_3)t^2 + (\lambda^2 + 2\lambda\lambda_3)t - \lambda^2\lambda_3 \tag{3.3}$$

Comparing now the polynomials (3.2), (3.3) we obtain that

$$\lambda_{11} + 2\lambda_{22} = 2\lambda + \lambda_3 \tag{3.4}$$
$$\lambda_{22}^2 + 2\lambda_{11}\lambda_{22} - 2\lambda_{12}\lambda_{21} - \lambda_{23}\lambda_{32} = \lambda^2 + 2\lambda\lambda_3 \tag{3.5}$$
$$\lambda_{11}\lambda_{22}^2 - \lambda_{11}\lambda_{23}\lambda_{32} - 2\lambda_{12}\lambda_{21}\lambda_{22} - \lambda_{12}^2\lambda_{23} - \lambda_{21}^2\lambda_{32} = \lambda^2\lambda_3 \tag{3.6}$$

Since the eigenspace $E_\lambda$ is spanned by the eigenvector $v_1 = (1, 0, 0)$ and $E_{\lambda_3}$ is spannd by $v_2 = (0, 1, 0)$, we have the following systems

$$(A - \lambda I)v_1 = 0 \quad , \quad (A - \lambda_3 I)v_2 = 0$$

thus $\lambda_{11} = \lambda$, $\lambda_{22} = \lambda_3$, $\lambda_{12} = \lambda_{21} = \lambda_{32} = 0$.
By using the relation (3.4), we run into a contradiction $\lambda = \lambda_3$, whereas we considered $\lambda \neq \lambda_3$.

**B.** Let $v_1, v_3$ be **light-like** vectors and $v_2$ be a **time-like** vector.
Then $< v_2, v_2 >= -1$, $< v_1, v_3 >= +1$, and $< v_1, v_2 >=< v_2, v_3 >=< v_1, v_1 >=$ $< v_3, v_3 >= 0$. So we have constructed a **pseudo-orthonormal** basis $\mathcal{U}$. Hence, the matrix representation of the metric tensor with respect to this basis is

$$[G]_{\mathcal{U}} = \begin{pmatrix} 0 & 0 & +1 \\ 0 & -1 & 0 \\ +1 & 0 & 0 \end{pmatrix} \tag{3.7}$$

Using the fact that $S$ is symmetric we easily conclude that

$$[S]_{\mathcal{U}} = A = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & -\lambda_{12} \\ \lambda_{31} & -\lambda_{21} & \lambda_{11} \end{pmatrix}.$$

The characteristic polynomial of $A$ is:

$$\begin{aligned} f(t) = t^3 - (2\lambda_{11} + \lambda_{22})t^2 + (\lambda_{11}^2 + 2\lambda_{11}\lambda_{22} - 2\lambda_{12}\lambda_{21} - \lambda_{13}\lambda_{31})t \\ -(\lambda_{22}\lambda_{11}^2 - 2\lambda_{11}\lambda_{12}\lambda_{21} - \lambda_{22}\lambda_{13}\lambda_{31} - \lambda_{12}^2\lambda_{31} - \lambda_{21}^2\lambda_{13}). \end{aligned} \tag{3.8}$$

Comparing now the polynomials (3.3), (3.8) we obtain

$$2\lambda_{11} + \lambda_{22} = 2\lambda + \lambda_3 \tag{3.9}$$

$$\lambda_{11}^2 + 2\lambda_{11}\lambda_{22} - 2\lambda_{12}\lambda_{21} - \lambda_{13}\lambda_{31} = \lambda^2 + 2\lambda\lambda_3 \tag{3.10}$$

$$\lambda_{11}^2\lambda_{22} - 2\lambda_{11}\lambda_{12}\lambda_{21} - \lambda_{22}\lambda_{13}\lambda_{31} - \lambda_{12}^2\lambda_{31} - \lambda_{21}^2\lambda_{13} = \lambda^2\lambda_3. \tag{3.11}$$

Since the eigenspace $E_\lambda$ is spanned by the eigenvector $v_1 = (1, 0, 0)$ and $E_{\lambda_3}$ by $v_2 = (0, 1, 0)$ we have the following systems

$$(A - \lambda I)v_1 = 0 \quad , \quad (A - \lambda_3 I)v_2 = 0$$

Hence $\lambda_{11} = \lambda$, $\lambda_{22} = \lambda_3$, and $\lambda_{12} = \lambda_{21} = \lambda_{31} = 0$.
Relations (3.9),(3.10) and (3.11) are satisfied identically. Therefore the matrix $A$ takes the form

$$[S]_{\mathcal{U}} = A = \begin{pmatrix} \lambda & 0 & \lambda_{13} \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \tag{3.12}$$

**C.** Let $v_1, v_3$ be **time-like** vectors and $v_2$ be a **space-like** vector.
Then, $< v_2, v_2 >= +1$, $< v_1, v_1 >=< v_3, v_3 >= -1$, and $< v_1, v_2 >=< v_2, v_3 >=$ $< v_3, v_1 >= 0$.
So we have constructed an **orthonormal** basis $\mathcal{B}$. Hence the matrix representation $G$ of the metric tensor is given by (2.5). We work as in cases $A$, $B$ and we find that the matrix $A$ takes the form

$$[S]_{\mathcal{B}} = A = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ -\lambda_{12} & \lambda_{22} & \lambda_{23} \\ \lambda_{13} & -\lambda_{23} & \lambda_{33} \end{pmatrix}.$$

he characteristic polynomial of $A$ is:

$$f(t) = t^3 - (\lambda_{11} + \lambda_{22} + \lambda_{33})t^2 + (\lambda_{11}\lambda_{22} + \lambda_{11}\lambda_{33} + \lambda_{22}\lambda_{33} + \lambda_{12}^2 - \lambda_{13}^2 + \lambda_{23}^2)t$$
$$-(\lambda_{11}\lambda_{22}\lambda_{33} + \lambda_{11}\lambda_{23}^2 + \lambda_{12}^2\lambda_{33} - \lambda_{13}^2\lambda_{22} + 2\lambda_{12}\lambda_{13}\lambda_{23}). \tag{3.13}$$

Comparing now the polynomials (3.3), (3.13) we obtain

$$\lambda_{11} + \lambda_{22} + \lambda_{33} = 2\lambda + \lambda_3 \tag{3.14}$$
$$\lambda_{11}\lambda_{22} + \lambda_{11}\lambda_{33} + \lambda_{22}\lambda_{33} + \lambda_{12}^2 - \lambda_{13}^2 + \lambda_{23}^2 = \lambda^2 + 2\lambda\lambda_3 \tag{3.15}$$
$$\lambda_{11}\lambda_{22}\lambda_{33} + \lambda_{11}\lambda_{23}^2 + \lambda_{12}^2\lambda_{33} - \lambda_{13}^2\lambda_{22} + 2\lambda_{12}\lambda_{13}\lambda_{23} = \lambda^2\lambda_3 \tag{3.16}$$

Since the eigenspace $E_\lambda$ is spanned by the eigenvector $v_1 = (1,0,0)$, and the eigenspace $E_{\lambda_3}$ by $v_2 = (0,1,0)$, we have the following systems

$$(A - \lambda I)v_1 = 0 \ , \ (A - \lambda_3 I)v_2 = 0.$$

Hence $\lambda_{11} = \lambda$, $\lambda_{22} = \lambda_3$, and $\lambda_{12} = \lambda_{13} = \lambda_{23} = 0$.
The relation (3.14) gives that $\lambda_{33} = \lambda$, and the relations (3.15), (3.16) are satisfied identically. Therefore, the matrix $A$ takes the form

$$[S]_{\mathcal{B}} = A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \tag{3.17}$$

This matrix is a special case of (II) for $\mu = 0$.

Let $v_1, v_3$ be vectors of $T_p(M_2^3)$ such that $E_\lambda = span\{v_1\}$, $E_{\lambda_3} = span\{v_3\}$. We need to determine the nature of the vectors $v_1, v_3$. Then, adding a third suitable vector, say $v_2$ we can construct a basis $\{v_1, v_2, v_3\}$ for $T_p(M_2^3)$. First, we prove that $< v_1, v_3 >= 0$. In fact applying the relation (1.1) we have that $< v_1, S(v_3) >=$ $< S(v_1), v_3 >$, or $< v_1, \lambda_3 v_3 >=< \lambda v_1, v_3 >$, and since $\lambda_3 \neq \lambda$, then $< v_1, v_3 >= 0$.

**D.** Let $v_3$ be **time-like** vector and $v_1, v_2$ be **light-like** vectors.
Then $< v_3, v_3 >= -1$, $< v_1, v_2 >= +1$, and $< v_1, v_3 >=< v_2, v_3 >=< v_1, v_1 >=$ $< v_2, v_2 >= 0$.

So we have constructed a **pseudo-orthonormal** basis $\mathcal{U} = \{v_1, v_2, v_3\}$. Hence, the matrix representation $G = (g_{ij})_{\mathcal{U}}$ where $g_{ij} = g(v_i, v_j)$ of the metric tensor is

$$[G]_{\mathcal{U}} \ = \begin{pmatrix} 0 & +1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3.18}$$

Using the fact that $S$ is symmetric endomorphism we easily conclude that

$$[S]_{\mathcal{U}} = A = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{11} & -\lambda_{31} \\ \lambda_{31} & -\lambda_{13} & \lambda_{33} \end{pmatrix}.$$

On the other hand the characteristic polynomial of $A$ is:

$$f(t) = t^3 - (2\lambda_{11} + \lambda_{33})t^2 + (\lambda_{11}^2 + 2\lambda_{11}\lambda_{33} - 2\lambda_{13}\lambda_{31} - \lambda_{12}\lambda_{21})t$$
$$-(\lambda_{33}\lambda_{11}^2 - 2\lambda_{11}\lambda_{13}\lambda_{31} - \lambda_{33}\lambda_{12}\lambda_{21} - \lambda_{13}^2\lambda_{21} - \lambda_{31}^2\lambda_{12}). \tag{3.19}$$

Comparing now the polynomials (3.3), (3.19) we obtain

$$2\lambda_{11} + \lambda_{33} = 2\lambda + \lambda_3 \tag{3.20}$$
$$\lambda_{11}^2 + 2\lambda_{11}\lambda_{33} - 2\lambda_{13}\lambda_{31} - \lambda_{12}\lambda_{21} = \lambda^2 + 2\lambda\lambda_3 \tag{3.21}$$
$$\lambda_{11}^2\lambda_{33} - 2\lambda_{11}\lambda_{13}\lambda_{31} - \lambda_{33}\lambda_{12}\lambda_{21} - \lambda_{13}^2\lambda_{21} - \lambda_{31}^2\lambda_{12} = \lambda^2\lambda_3. \tag{3.22}$$

Since the eigenspace $E_\lambda$ is spanned by the eigenvector $v_1 = (1,0,0)$ and $E_{\lambda_3}$ is spanned by $v_3 = (0,0,1)$ we have the following systems

$$(A - \lambda I)v_1 = 0 \quad , \quad (A - \lambda_3 I)v_3 = 0$$

From this system we find that: $\lambda_{11} = \lambda$, $\lambda_{33} = \lambda_3$, and $\lambda_{21} = \lambda_{13} = \lambda_{31} = 0$. Relations (3.20),(3.21) and (3.22) are satisfied identically. Therefore the matrix $A$ takes the form

$$[S]_{\mathcal{U}} = A = \begin{pmatrix} \lambda & \lambda_{12} & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{3.23}$$

**Remark.** Supposing that the vectors $v_1, v_2, v_3$ have the same nature but $E_\lambda = span\{v_2\}$, and $E_{\lambda_3} = span\{v_3\}$ then we easily get that $S$ takes the form

$$[S]_{\mathcal{U}} = A = \begin{pmatrix} \lambda & 0 & 0 \\ \lambda_{21} & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{3.24}$$

with respect to the same **pseudo-orthonormal** basis $\mathcal{U} = \{v_1, v_2, v_3\}$ and the corresponding for the inner products $<,>$, are given by (3.18).

**Subcase 3.2:** Consider the case $\dim E_\lambda = 2$ and $\dim E_{\lambda_3} = 1$.
In this case the shape operator is diagonalizable. In fact, let $v_1, v_2, v_3$ be vectors of $T_p(M_2^3)$ such that $E_\lambda = span\{v_1, v_3\}$, $E_{\lambda_3} = span\{v_2\}$. Then we easily conclude that $< v_1, v_2 >= 0$, $< v_3, v_2 >= 0$.

**A.** Let $v_1, v_3$ be **time-like** vectors and $v_2$ be a **space-like** vector.
In this case $< v_2, v_2 >= +1$, $< v_1, v_1 >=< v_3, v_3 >= -1$, and $< v_1, v_2 >=< v_1, v_3 >=< v_2, v_3 >= 0$. So we have constructed an **orthonormal** basis $\mathcal{B}$. Hence the matrix representation of the metric tensor is given by (2.5). Since the eigenspace $E_\lambda$ is spanned by the eigenvectors $v_1 = (1,0,0)$, $v_3 = (0,0,1)$ and $E_{\lambda_3}$ by $v_2 = (0,1,0)$ we have the following system, $(A - \lambda I)(k_1 v_1 + k_3 v_3) = 0$, $(A - \lambda_3 I)v_2 = 0$, where $k_1, k_3 \in \mathbb{R}$.

From these system we find that: $\lambda_{11} = \lambda_{33} = \lambda$, $\lambda_{22} = \lambda_3$, and $\lambda_{12} = \lambda_{13} = \lambda_{23} = 0$. Hence the relations (3.14), (3.15), (3.16) become identities. Therefore the matrix $A$ takes also the form (3.17)

**B.** Let $v_1, v_3$ be **light-like** vectors and $v_2$ be a **time-like** vector.
In this case $< v_2, v_2 >= -1$, $< v_1, v_3 >= +1$, and $< v_1, v_2 >=< v_2, v_3 >=$ $< v_1, v_1 >=< v_3, v_3 >= 0$. So we have constructed a **pseudo-orthonormal** basis $\mathcal{U}$. Hence the matrix representation of the metric tensor is given by (3.7).

Since the eigenspace $E_\lambda$ is spanned by the eigenvectors $v_1 = (1,0,0), v_3 = (0,0,1)$ and $E_{\lambda_3}$ by $v_2 = (0,1,0)$ we have the following system $(A - \lambda I)(k_1 v_1 + k_3 v_3) = 0$, $(A - \lambda_3 I)v_2 = 0$, where $k_1, k_3 \in \mathbb{R}$.
If we work similarly then we conclude that the matrix $A$ takes the form (3.17). Hence we have proved the following proposition.

**Proposition 3.1** *If the eigenvalues of the shape operator $S$ of $M_2^3$ in $E_2^4$ are real and one of them is of multiplicity 2, then the canonical forms of $S$ are given by the matrices (II) and the corresponding for the inner products $<,>$, are given by (2.5), (3.7), (3.18) with respect to suitable bases.*

## 4. Three equal eigenvalues of $S$

Assume that $A$ has one real root $\lambda$ of multiplicity three. Then $\dim E_\lambda = 1$, $\dim E_\lambda = 2$, or $\dim E_\lambda = 3$.

**Subcase 4.1:** Let $\dim E_\lambda = 1$. Then there exists a vector, say $v_1$ such that $E_\lambda = span\{v_1\}$. We need to determine the nature of this vector $v_1$, and then adding two suitable vectors $v_2, v_3$ we can build a basis $\{v_1, v_2, v_3\}$ for $T_p(M_2^3)$.

**A.** Let $v_1, v_3$ be **time-like** vectors and $v_2$ be a **space-like** vector.
Then $< v_2, v_2 >= +1$, $< v_1, v_1 >=< v_3, v_3 >= -1$, and $< v_1, v_2 >=< v_1, v_3 >=$ $< v_2, v_3 >= 0$. So we have constructed an **orthonormal** basis $\mathcal{B}$. Hence the complementary space of $v_1$ in $E_2^3$ has signature $(+,-)$ and the matrix representation of the metric tensor $G = (g_{ij})_{\mathcal{B}}$, where $g_{ij} = g(v_i, v_j)$ is given by (2.5).

Let $S : T_p(M_2^3) \to T_p(M_2^3)$, be the shape operator of $T_p(M_2^3)$. Then $S(v_i) = \lambda_{ij} v_j$; $i, j = 1, 2, 3$ where, $\lambda_{ij} \in \mathbb{R}$, and applying the relation (1.2), we find that the matrix $A$ takes the form

$$[S]_{\mathcal{B}} = A = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ -\lambda_{12} & \lambda_{22} & \lambda_{23} \\ \lambda_{13} & -\lambda_{23} & \lambda_{33} \end{pmatrix}$$

In this case, since $A$ has a real root $\lambda$ of multiplicity three, we have that

$$f(t) = (t - \lambda)^3 = t^3 - 3\lambda t^2 + 3\lambda^2 t - \lambda^3 \tag{4.1}$$

Comparing the polynomials from the relations (3.13), (4.1) we obtain

$$\lambda_{11} + \lambda_{22} + \lambda_{33} = 3\lambda \tag{4.2}$$

$$\lambda_{11}\lambda_{22} + \lambda_{11}\lambda_{33} + \lambda_{22}\lambda_{33} + \lambda_{12}^2 - \lambda_{13}^2 + \lambda_{23}^2 = 3\lambda^2 \tag{4.3}$$

$$\lambda_{11}\lambda_{22}\lambda_{33} + \lambda_{11}\lambda_{23}^2 + \lambda_{12}^2\lambda_{33} - \lambda_{13}^2\lambda_{22} + 2\lambda_{12}\lambda_{13}\lambda_{23} = \lambda^3 \tag{4.4}$$

Since the eigenspace $E_\lambda$ is spanned by the eigenvector $v_1 = (1, 0, 0)$ we have that

$$(A - \lambda I)v_1 = 0$$

and using the relations (4.2), (4.3), and (4.4) the matrix representation of $S$ takes finally the form

$$[S]_{\mathcal{B}} = A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda \pm \lambda_{23} & \lambda_{23} \\ 0 & -\lambda_{23} & \lambda \mp \lambda_{23} \end{pmatrix} \tag{4.5}$$

**B.** Let $v_2, v_3$ be **light-like** vectors, and $v_1$ be a **time-like** vector. Then $< v_1, v_1 >= -1$, $< v_1, v_2 >=< v_1, v_3 >=< v_2, v_2 >=< v_3, v_3 >= 0$, and $< v_2, v_3 >= +1$. So we have constructed a **pseudo-orthonormal** basis $\mathcal{U}$. In this case the complementary space of $v_1$ in $E_2^3$ has also the same, as in case $A$, signature $(+, -)$ and the matrix representation of the metric tensor $G = (g_{ij})_{\mathcal{U}}$, where $g_{ij} = g(v_i, v_j)$ is given by (3.1).
Following similar analysis we easily get for $A$ the following form

$$[S]_{\mathcal{U}} = A = \begin{pmatrix} \lambda_{11} & \lambda_{12} & -\lambda_{21} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ -\lambda_{12} & \lambda_{32} & \lambda_{22} \end{pmatrix}$$

Comparing the polynomials (3.2), (4.1) we obtain

$$\lambda_{11} + 2\lambda_{22} = 3\lambda \tag{4.6}$$

$$\lambda_{22}^2 + 2\lambda_{11}\lambda_{22} - 2\lambda_{12}\lambda_{21} - \lambda_{23}\lambda_{32} = 3\lambda^2 \tag{4.7}$$

$$\lambda_{11}\lambda_{22}^2 - \lambda_{11}\lambda_{23}\lambda_{32} - 2\lambda_{12}\lambda_{21}\lambda_{22} - \lambda_{12}^2\lambda_{23} - \lambda_{21}^2\lambda_{32} = \lambda^3 \tag{4.8}$$

Since the eigenspace $E_\lambda$ is spanned by the eigenvector $v_1 = (1, 0, 0)$ then we have

$$(A - \lambda I)v_1 = 0$$

from which

$$\lambda_{11} = \lambda, \quad \lambda_{12} = \lambda_{21} = 0$$

and using the relations (4.6), (4.7), and (4.8) we obtain

$$\lambda_{22} = \lambda, \qquad \lambda_{23}\lambda_{32} = 0$$

So we distinguish the following cases:

**(i)** : If $\lambda_{23} = 0, \lambda_{32} \neq 0$, then

$$[S]_{\mathcal{U}} = A_1 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & \lambda_{32} & \lambda \end{pmatrix} \tag{4.9}$$

**(ii)** : If $\lambda_{23} \neq 0, \lambda_{32} = 0$, then

$$[S]_{\mathcal{U}} = A_2 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & \lambda_{23} \\ 0 & 0 & \lambda \end{pmatrix} \tag{4.10}$$

**(iii)** : If $\lambda_{23} = \lambda_{32} = 0$, then

$$[S]_{\mathcal{U}} = A_3 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \tag{4.11}$$

**Remark.** Both cases $A$ and $B$ lead to a same splitting and a subspace of signature $(+,-)$ in $E_2^3$ can have either a basis consisting of a **space-like** and a **time-like** vector or consisting of a **light-like** and a **light-like** vector. Therefore the cases $A$ and $B$ of this section cover the same situation only with respect different bases. Hence the matrix (4.5) represents the same endomorphisms with respect to a basis which does not lead to their most transparant matrix representation. Otherwise stated what (4.5) represents, is already present in case $B$ just with respect to a more appropriate basis.

**C.** Let $v_1, v_2$ be **light-like** vectors, and $v_3$ be a **time-like** vector. Then $< v_3, v_3 >= -1$, $< v_1, v_1 >=< v_2, v_2 >=< v_1, v_3 >=< v_2, v_3 >= 0$, and $< v_1, v_2 >= +1$. So we also have constructed a **pseudo-orthonormal** basis $\mathcal{U}$. Hence the matrix representation of the metric tensor $G = (g_{ij})_{\mathcal{U}}$, is given by (3.18).

By similar analysis we obtain that

$$[S]_{\mathcal{U}} = A = \begin{pmatrix} \lambda & \lambda_{12} & \lambda_{13} \\ 0 & \lambda & 0 \\ 0 & -\lambda_{13} & \lambda \end{pmatrix} \tag{4.12}$$

**Subcase 4.2:** Let $\dim E_\lambda = 2$. Hence there exist two linearly independent vectors, say $v_1, v_2$ such that $E_\lambda = span\{v_1, v_2\}$. We need to determine the possible nature of theses vectors $v_1, v_2$, and then adding a suitable vector $v_3$ we can build a basis $\{v_1, v_2, v_3\}$ for $T_p(M_2^3)$.

Following now similar analysis as in the previous subcase, we can construct either an **orthonormal** basis $\mathcal{B}$ by considering that the vectors $v_1, v_2$ spanning the space $E_\lambda$ are **time-like** and the third one $v_3$ is **space-like** or a **pseudo-orthonormal** basis $\mathcal{U}$ by choosing both vectors $v_1, v_2$ to be **light-like** and the third one **time-like** or $v_1$ to be **time-like** and $v_2, v_3$ **light-like**.

Then we prove that in each of these three cases, the matrix $G$ is given, either by (2.5) or by (4.12) and the corresponding for $S$, is given by (III).

**Subcase 4.3:** Let dim$E_\lambda = 3$. Since the eigenvalue $\lambda$ has multiplicity three it is known that every vector $v \in T_p(M_2^3)$ is an eigenvector of $S$ with the same eigenvalue. Therefore, the matrix representation of the shape operator $S$ with respect to an **orthonormal**, or a **pseudo-orthonormal** basis is given by (4.11). Hence we have proved the following proposition.

**Proposition 4.1** *If the shape operator $S$ of $M_2^3$ in $E_2^4$ has one real eigenvalue of multiplicity three, then its canonical forms, are given by the matrices (III) and the corresponding for the inner product $G$, by the matrices (2.5), (3.1), and (3.18), with respect to suitable bases.*

## 5. Complex eigenvalues of $S$

Suppose that $A$ has two complex conjugate roots $\lambda_1 = \mu + i\nu$, $\lambda_2 = \mu - i\nu$, $\nu \neq 0$, $\mu \in \mathbb{R}$ and one real root $\lambda = \lambda_3$. Then dim$E_{\lambda_3} = 1$. Let $v_3$ be a vector which span $E_{\lambda_3}$, e.g. $E_{\lambda_3} = span\{v_3\}$. As before, we need to determine the character of $v_3$. Then we add two suitable vectors, say $v_1, v_2$ such that they could construct a basis $\{v_1, v_2, v_3\}$ for $T_p(M_2^3)$.

**A.** Let $v_1, v_3$ be **time-like** vectors and $v_2$ be a **space-like** vector.
In this case $< v_2, v_2 >= +1$, $< v_1, v_1 >=< v_3, v_3 >= -1$, and $< v_1, v_2 >=< v_1, v_3 >=< v_2, v_3 >= 0$. So we have constructed an **orthonormal** basis $\mathcal{B}$. Hence the matrix representation of the metric tensor $G = (g_{ij})_\mathcal{B}$, where $g_{ij} = g(v_i, v_j)$ is given by (2.5), namely

$$G = \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

By applying the relation (1.2), we find that the matrix $A$ takes the form

$$[S]_\mathcal{B} = A = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ -\lambda_{12} & \lambda_{22} & \lambda_{23} \\ \lambda_{13} & -\lambda_{23} & \lambda_{33} \end{pmatrix}$$

In this case, since $A$ has one real root $\lambda_3$ and two complex conjugate roots $\lambda_1 = \mu + i\nu$, $\lambda_2 = \mu - i\nu$, $\nu \neq 0, \mu \in \mathbb{R}$ then,

$$f(t) = t^3 - (2\mu + \lambda_3)t^2 + (\mu^2 + \nu^2 + 2\lambda_3\mu)t - \lambda_3(\mu^2 + \nu^2) \tag{5.1}$$

Comparing the polynomials (3.13), and (5.1) we obtain that

$$\lambda_{11} + \lambda_{22} + \lambda_{33} = \lambda_3 + 2\mu \tag{5.2}$$

$$\lambda_{11}\lambda_{22} + \lambda_{11}\lambda_{33} + \lambda_{22}\lambda_{33} + \lambda_{12}^2 - \lambda_{13}^2 + \lambda_{23}^2 = \mu^2 + \nu^2 + 2\lambda_3\mu \tag{5.3}$$

$$\lambda_{11}\lambda_{22}\lambda_{33} + \lambda_{11}\lambda_{23}^2 + \lambda_{12}^2\lambda_{33} - \lambda_{13}^2\lambda_{22} + 2\lambda_{12}\lambda_{13}\lambda_{23} = \lambda_3(\mu^2 + \nu^2) \tag{5.4}$$

Since the eigenspace $E_{\lambda_3}$ is spanned by the eigenvector $v_3 = (0,0,1)$, we obtain the following equation

$$(A - \lambda_3 I)v_3 = 0$$

It follows that $\lambda_{13} = \lambda_{23} = 0$, and $\lambda_{33} = \lambda_3$, so relations (5.2), (5.3), and (5.4) give that $\lambda_{11} = \lambda_{22} = \mu$, $\lambda_{12} = \nu$, and $\lambda_{21} = -\nu$. Therefore, the matrix representation of $S$ takes the form

$$[S]_{\mathcal{B}} = A = \begin{pmatrix} \mu & \nu & 0 \\ -\nu & \mu & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \nu \neq 0, \quad \mu \in \mathbb{R} \tag{5.5}$$

**B.** Let $v_1, v_2$ be **light-like** vectors and $v_3$ be a **time-like** vector.
In this case $< v_1, v_2 >= +1$, $< v_3, v_3 >= -1$, and $< v_1, v_1 >=< v_2, v_2 >=< v_1, v_3 >=< v_2, v_3 > 0$. So we have constructed a **pseudo-orthonormal** basis $\mathcal{U}$. Hence the matrix representation of the metric tensor $G = (g_{ij})_{\mathcal{U}}$, where $g_{ij} = g(v_i, v_j)$ is given by (3.18).

By applying the relation (1.2), we find that the matrix $A$ takes the form

$$[S]_{\mathcal{U}} = A = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{11} & -\lambda_{31} \\ \lambda_{31} & -\lambda_{13} & \lambda_{33} \end{pmatrix}$$

Comparing the polynomials (3.19), and (5.1) we obtain that

$$2\lambda_{11} + \lambda_{33} = \lambda_3 + 2\mu \tag{5.6}$$

$$\lambda_{11}^2 + 2\lambda_{11}\lambda_{33} - 2\lambda_{13}\lambda_{31} - \lambda_{12}\lambda_{21} = \mu^2 + \nu^2 + 2\lambda_3\mu \tag{5.7}$$

$$\lambda_{11}^2\lambda_{33} - 2\lambda_{11}\lambda_{13}\lambda_{31} - \lambda_{33}\lambda_{12}\lambda_{21} - \lambda_{13}^2\lambda_{21} - \lambda_{31}^2\lambda_{12} = \lambda_3(\mu^2 + \nu^2). \tag{5.8}$$

Since the eigenspace $E_{\lambda_3}$ is spanned by the eigenvector $v_3 = (0,0,1)$, we obtain the following equation

$$(A - \lambda_3 I)v_3 = 0$$

It follows that $\lambda_{13} = \lambda_{31} = 0$, and $\lambda_{33} = \lambda_3$. Also from relations (5.6), (5.7) we obtain that $\lambda_{11} = \mu$, $\lambda_{12} = \nu$, and $\lambda_{21} = -\nu$, whereas relation (5.8) is satisfied identically. After that, the matrix representation of $S$ takes the form (5.5).

**Remark.** If we assume that $v_1$ is a **time-like** vector and $v_2, v_3$ are **light-like** vectors or, $v_1, v_3$ are **light-like** vectors and $v_2$ a **time-like** vector or, $v_1, v_2$ are **time-like** vectors and $v_3$ a **space-like** vector, then, we run into a contradiction $\nu = 0$, whereas we considered that $\nu \neq 0$.
Hence we have proved the following proposition.

**Proposition 5.1** *If the the shape operator $S$ of $M_2^3$ in $E_2^4$ has one real and two complex conjugate eigenvalues, then its canonical form $S$ is given by the matrix (5.5) and the corresponding for the inner product $G$ by the matrices (2.5), or (3.18) with respect to suitable bases.*

Finally combining the Propositions (2.1), (3.1), (4.1), and (5.1) we obtain the Theorem 1.1 stated in the introduction.

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## References

[1] F. Defever, G. Kaimakamis and V. Papantoniou, *Biharmonic hypersurfaces of the four-dimensional semi-Euclidean space* $E^4_s$, J. Math. Anal. Appl. 315:1 (2006), 276-286.

[2] A. Arvanitoyeorgos, F. Defever, G. Kaimakamis and V. Papantoniou, *Biharmonic Lorentz hypersurfaces in* $E^4_1$, Pacific Journal of Mathematics, Vol.229, No 2, (2007), 293-305.

[3] M. A. Magid, *Isometric immersions of Lorentz space with parallel second fundamental forms*, Tsukuba J. Math. 8:1 (1984), 31-54.

[4] M. A. Magid, *Lorentzian isoparametric hypersurfaces*, Pacific J. Math. 118 (1985), 165-198.

[5] A. Z. Petrov, *Einstein spaces*, Pergamon Press, Oxford, (1969)

◇ G. Kaimakamis
Hellenic Army Academy
GR-16673 Vari,
Attica, Greece
miamis@math.upatras.gr

◇ V.J. Papantoniou
Department of Mathematics
University of Patras
GR-26500 Patras, Greece
bipapant@math.upatras.gr

◇ K. Petoumenos
Department of Mathematics
University of Patras
GR-26500 Patras, Greece
copetoum@math.upatras.gr