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Resetting in a geometric approach to the control of nonminimum phase nonlinear systems

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Abstract

Input-output exact feedback linearization allows for the implementation of linear control, such as pole placement, to the control of nonlinear systems. However this approach does not take into account the system internal dynamics which would be unstable in the case of nonminimum phase nonlinear systems. The use of an auxiliary and a synthetic output provide the means of achieving maximum relative degree and thus remove the difficulty of unstable zero dynamics. This approach provides the means of controlling directly the synthetic output and indirectly the actual system output, through a resetting technique. The present paper considers the efficient implementation of resetting and extends it to cater for the case of loss of relative degree as well as to enforce the compliance with input constraints. The developed strategy has some desirable stability and asymptotic convergence properties and is shown by means of a challenging numerical example to be capable of excellent results.

Keywords: Feedback linearization, pole placement, nonminimum systems, synthetic outputs, parameter resetting

1. Introduction

Output regulation (and its extension to setpoint tracking) of SISO systems is concerned with the minimization of a cost J that consists of the integral, over infinite horizon, of a linear mix of the squares of the output y and input u. For linear systems, optimal control provides an elegant answer to this minimization in the form of a simple state feedback closed form solution. Including in J the term that places a penalty on the input has been used in the past as an indirect means of reducing control activity with the view to avoiding input constraint violations. In recent years however we have seen the emergence of host of techniques that take direct account of constraints, and which techniques therefore enable the use of costs J which penalize output activity alone; this form of cost will be adopted in this paper. For minimum phase non-linear systems, input-output state feedback linearization provides an effective means for the use of optimal control in the pursuit of optimality in regulation. In the presence of unstable inverse dynamics however, this approach, which still affords optimal output behaviour, can result in unstable internal dynamics. A way to avoid this problem is to resort [1] to minimum-phase approximation (e.g. through the use of appropriate factorisations) or alternatively [2] to regulate a synthetic output chosen so as to remove unstable zeros as well as approximate (in a suitable sense) the behaviour of the actual output. A recent paper [3] considered instead the definition of an auxiliary output, λ, which avoided nonminimum-phase difficulties through the achievement of maximum relative degree, a property also retained in the definition of a synthetic output, $\psi = \phi(\lambda)$, where $\phi(.)$ denotes a polynomial function. In this setting, input-state feedback linearization was introduced to cause the synthetic output to follow a pole placement strategy while at the same time, the vector a of coefficients of $\phi(.)$ were reset periodically to match the behaviour of the actual output (and its derivatives) to that of the synthetic output (and its derivatives). Suitable conditions that guaranteed the convergence of the vector ξ of ψ and its derivatives were developed and these in turn enable the statement of stability conditions on the actual system state vector x. The convergence conditions, though effective in avoiding nonminimum-phase difficulties, leads to ill-conditioning. This problem is avoided in this paper through the use of a singular value decomposition which achieves robustness at the cost of an insignificant degree of suboptimality. A further problem considered here is that of input constraint violations which are avoided through a non-periodic resetting of the vector of coefficients of ϕ. Resetting is achieved in an optimal manner in that the perturbation on the vector a is minimized while input constraints are respected. The paper concludes by considering an alternative approach which uses resetting of the input with the view meeting particular control objectives as closely as can be achieved within the input constraints while at the same time being mindful of the convergence properties of ξ and x. The results of the paper are illustrated by a numerical example which exhibits strongly nonlinear behaviour, is open-loop unstable and does not possess any equilibrium points which are minimum phase. Despite all these features and despite the fact that for the chosen initial condition the input constraints are active, the proposed algorithm achieves excellent closed loop behaviour.

2. Earlier work

For s a scalar function of $x \in \Re^n$, let ∇s be the column vector of first order partial derivatives, and for $v \in \Re^n$ let ∇v be the Jacobian matrix. Then for a constant vector and w, v vector functions of x it follows that

$$\nabla(a^T v) = \nabla(v^T a) = [\nabla v]^T a \tag{1}$$

$$\nabla(w^T v) = [\nabla w]^T v + [\nabla v]^T w \tag{2}$$

Consider next the SISO affine in the input state space model

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= Cx \end{aligned} \tag{3}$$

where for ease of presentation alone it will be assumed that $x \in \Re^3$; all the results to be presented here have obvious extensions to the general case. The dependence of y on x is also restricted to be linear for clarity. It will be assumed that the origin is an equilibrium point about which the inverse dynamics are unstable, but that there exists an equilibrium point x^0 in the kernel of C about which the system is minimum phase. It will be assumed that in the operating region of the state space, denoted by Ω, (3) is controllable and involutive (e.g. see [4] or [5]):

$$rank([g, \{f, g\}, \{f, \{f, g\}\}]) = 3 \tag{4}$$

$$rank([\gamma_1, \gamma_2, \gamma_3]) = 2; \quad \gamma_1 = g, \gamma_2 = \{g, f\}, \gamma_3 = \{\gamma_1, \gamma_2\} \tag{5}$$

where $\{.,.\}$ denotes the Lie bracket. Condition (5) implies that for $\psi = \phi(\lambda)$ it is possible to use input to state feedback linearization to achieve any desired linear dynamics, eg:

$$\dot{\xi} = A_p \xi; \quad A_p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}; \quad \xi = \begin{bmatrix} \psi \\ \dot{\psi} \\ \ddot{\psi} \end{bmatrix} \tag{6}$$

where in this example a pole placement (PP) strategy has been followed that places all closed loop poles at -1. The necessary and sufficient conditions for achieving this are:

$$L_g \lambda = [\nabla \lambda]^T \gamma_1 = 0 \tag{7a}$$
$$L_g L_f \lambda = [\nabla \lambda]^T \gamma_2 = 0 \tag{7b}$$
$$\phi'(\lambda) L_g L_f^2 \lambda = L_g L_f^2 \lambda \neq 0 \tag{7c}$$

where $L_{(.)}$ denotes Lie derivatives and where the third condition in (7) ensures that there is no loss of maximum relative degree. For $\phi(\lambda) = \begin{bmatrix} \lambda^4 & \lambda^3 & \lambda^2 & \lambda \end{bmatrix}$ it is easy to show that (7) implies

$$\begin{bmatrix} \xi \\ \dddot{\psi} \end{bmatrix} = Da + u\phi'(\lambda) L_g L_f^2 \lambda e_4; \quad D = MN \tag{8}$$

where e_i denotes the i^{th} column of the 4×4 identity matrix and M, N are

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & L_f \lambda & 0 & 0 \\ 0 & L_f^2 \lambda & (L_f \lambda)^2 & 0 \\ 0 & L_f^3 \lambda & 3(L_f \lambda)(L_f^2 \lambda) & (L_f \lambda)^3 \end{bmatrix} \quad N = \begin{bmatrix} \lambda^4 & \lambda^3 & \lambda^2 & \lambda \\ 4\lambda^3 & 3\lambda^2 & 2\lambda^1 & 1 \\ 12\lambda^2 & 6\lambda & 2 & 0 \\ 24\lambda & 6 & 0 & 0 \end{bmatrix} \tag{9}$$

It follows therefore that the input u that achieves the PP objective can be obtained from:

$$d^T(u)a = 0; \quad d^T(u) = v_p^T D + uL_g L_f^2 \lambda e_2^T N; \quad v_p = \begin{bmatrix} 1 & 3 & 3 & 1 \end{bmatrix}^T \tag{10}$$

The integrability of (7a,7b) is ensured by the involutivity condition (5) and the solution is

$$\nabla\lambda = sv; \tag{11a}$$
$$v^T\gamma_1 = 0, \tag{11b}$$
$$v^T\gamma_2 = 0, \tag{11c}$$
$$v^T v = 1 \tag{11d}$$

with s and v being a scalar and vector function of x. A necessary and sufficient for the solvability of (11a) is [3]:

$$[\nabla(sv)] = [\nabla(sv)]^T \tag{12}$$

and a sufficient condition for that can be shown, through use of (1), (3), (11b,11c,11d) and the involutivity condition, to be

$$\nabla s = s[\nabla v]v \tag{13}$$

where an explicit expression of ∇v can be obtained by applying the ∇ operator to (11b,11c) to get:

$$[\nabla v]^T = \begin{bmatrix} -[\nabla\gamma_1]^T v & -[\nabla\gamma_2]^T v & 0 \end{bmatrix} \begin{bmatrix} \gamma_1 & \gamma_2 & v \end{bmatrix}^{-1} \tag{14}$$

It follows therefore that PP will be achieved so long as u satisfies (10) and PP evolve according to

$$\begin{aligned} \dot{s} &= [\nabla s]^T \dot{x} = sv^T[\nabla v]^T(f + ug) \\ \dot{\lambda} &= sv^T\dot{x} = sv^T(f + ug) = sv^T f \end{aligned} \tag{15}$$

Of necessity these will have to be implemented in discrete time, e.g. according to simple Euler integration:

$$s_{k+1} = s_k + \tau\dot{s}_k; \quad \lambda_{k+1} = \lambda_k + \tau\dot{\lambda}_k \tag{16}$$

where τ represents the sampling interval. Correspondingly u will be kept constant over the interval and thus the evolution of ξ will be given by

$$\xi_{k+1} = e^{A_p\tau}\xi_k \tag{17}$$

where for sufficiently small τ, the error in the above can be made to be negligible.

Such a strategy would cause ψ to follow the PP objective but will not allow direct control of the actual output y. To overcome this difficulty it is possible to reset a

periodically, say every $T = r\tau$ sec where r is a positive integer, in order to match ξ, the vector of ψ and its derivatives, to w the vector of y and its derivatives; for obvious reasons this resetting will be referred to as the Matching Resetting. Without resetting, the function

$$V_k = \xi_k^T P_p \xi_k, \quad P_p A_p + A_p^T P_p < 0 \tag{18}$$

will act like a Lyapunov function thereby guaranteeing the convergence of ξ to the origin. In order to retain this guarantee it is necessary to ensure that the reset value of ξ_k, say ξ_k^+, results in a reset value of V, V^+, which may be larger than V_k, but not by more than the reduction in V due to the application of PP until the next resetting. Thus resetting can be posed as the optimisation

$$\min_a \quad \|\xi_k^+ - w\| \tag{19}$$

such that

$$\|\xi_k^+\|_{P_T}^2 \leq \bar{V}^2; \quad \bar{V}^2 = \|\xi_k\|_{P_p}^2 - \varepsilon \|\xi_k\|_{P_p - P_T}^2, \, 0 < \varepsilon \leq 1; \quad P_T = e^{A_p^T T} p e^{A_p T} \tag{20}$$

In addition it is possible to ensure that resetting does not result in discontinuity of u (as this would invalidate the feedback linearization equations) by requiring that the reset value of a should satisfy (10).

Algorithm 2.1
Evolve s,λ through (15), (16) and every T seconds reset a using (19), (20). Compute the current control move from (9), (10).

The overall strategy of Algorithm 2.1 is such that ξ converges to zero, whereas x converges to a limit, with y itself going to zero under the conditions of the theorem below.

Theorem 2.2 *The implementation of the PP strategy with resetting described in this section will cause: (i) ξ to converge asymptotically to zero; (ii) λ to converge asymptotically to λ_p, a root of $\phi(\lambda)$; and (iii) x to converge to a limit. Moreover, if as $t \to \infty$, (20) remains inactive, then y itself will converge to zero. In addition, for sufficiently fast resetting, the origin of the state space will be locally attractive.*

The proof of this result can be found in [3].

3. Online implementation

The strategy described above has the significant advantage of ease of computation which makes the approach particularly attractive for online control of systems with fast dynamics. Between resetting times, all that is required is the computation of the current control move, u_k, from (10) which is easy to solve, given the values of M, and N. These in turn are easy to compute given the values of s, λ which can be obtained in a straightforward manner from (16). At resetting times, one has to perform the optimisation of (19) first subject to (20), which is of the general form:

$$\min_z \|Sz + \alpha\| \quad \text{s.t.}\ \|Qz + B\| \le 1 \tag{21}$$

It has been shown [6] that for the special case when $\beta = 0$ and both S, Q are square and full rank (with S also being real symmetric), this problem can be solved efficiently through a Newton-Raphson iteration that has guaranteed convergence. However, in (19) $\beta = 0$ and the dimension of w and ξ do not match that of α. It is therefore necessary to generalize the result of [6] and this is done in theorem below.

Theorem 3.1 *The solution to (21) is given by*

$$z = R_0\Lambda_0^{-1}R(\mu I - \Lambda)^{-1}R^T\Lambda_0^{-1}R_0^T(Q^T\beta - S^T\alpha) \tag{22}$$

where μ is the only real negative root of

$$\begin{aligned} p(\mu) = &\sum_i \prod_{j\neq i} p_i(\mu)(\mu - \lambda_j)^2 + \\ &2\sum_i \prod_{j\neq i} p_i(\mu)(\mu - \lambda_i)(\mu - \lambda_j)^2 e_j^T R^T\beta + (\beta^T\beta - 1)\prod_i(\mu - \lambda_i)^2 \end{aligned} \tag{23}$$

where $R_0, \Lambda_0, R, \Lambda$ are defined from the eignevalue/vector decompositions

$$S^TS = R_0\Lambda_0^2R_0^T; \quad \Lambda_0^{-1}R_0^TQ^TQR_0\Lambda_0^{-1} = R\Lambda R^T \tag{24}$$

where e_j denotes the j^{th} column of the identity matrix (of conformal dimensions) and

$$p_i(\mu) = -e_i^TR^T\alpha\mu + e_i^TR^T\beta \tag{25}$$

Proof. The details of the proof are straightforward and will be omitted, but an outline of the argument is as follows. First simplify the statement of the problem through use of the transformation $\tilde{z} = \Lambda_0R_0^Tz$ and deploy the fact that the optimum will occur at a z which makes the inequality constraint of (21) hold with strict equality. Then the optimality condition implies that the gradient of the cost must be parallel to the gradient of $\|QR_0\Lambda_0^{-1}\tilde{z} + \beta\| - 1$, and in particular the gradient of the latter must be equal to μ times of the former, with μ negative. Solving the optimality condition for $\tilde{z}$ and substituting into the equality constraint gives the result.

Returning now to the optimisation of (21), it is noted that

$$\xi = D_0^Ta \tag{26}$$

where D_0^T is the matrix D with its last row omitted. In addition however, in order to ensure that the differential properties of Section 2 hold true, the input u must be continuous, which implies that the reset value of a must be such that (10) where u denotes the control move given by PP before resetting:

$$d^T(u)a = 0 \Leftrightarrow a = K\gamma$$

where K is a matrix representation of the kernel of $d^T(u)$.

$$\min_{\gamma} \|D_0^T K\gamma - w\| \quad \text{s. t.} \quad \|\frac{1}{V} P_T^{1/2} D_0^T K\gamma\| \leq 1 \tag{27}$$

However, as the procedure of Section 2 converges towards a steady state, so ξ converges to zero which in turn implies that as we approach steady state, at resetting $D_0^T K$ becomes rank deficient and since this matrix is square, it will become ill-conditioned, with some singular values being much larger than the remainder. A conformal partition of the singular value decomposition then will be given as:

$$D_0^T K = U_1 \Sigma_1 V_1^T + U_2 \Sigma_2 V_2^T \tag{28}$$

where $(.)_{1,2}$ identifies the part of the decomposition associated with the large and small singular values, respectively. Then the projection

$$\gamma = V_1\gamma_1 + V_2\gamma_2 \tag{29}$$

together with (28) convert (27) into the equivalent problem:

$$\min_{\gamma} \|U_1\Sigma_1\gamma_1 + U_2\Sigma_2\gamma_2 - w\| \quad \text{s. t.} \|\frac{1}{V} P_T^{1/2} U_1\Sigma_1\gamma_1 + \frac{1}{V} P_T^{1/2} U_2\Sigma_2\gamma_2\| \leq 1 \tag{30}$$

The presence of Σ_2 causes the polynomial p of (23) to be ill-conditioned and this precludes the possibility of optimising simultaneously over$\gamma_{1,2}$, but the problem is overcome by the sequential solution of the following two problems: (A) optimise over γ_1 for $\gamma_2 = 0$; (B) optimise over γ_2 for $\gamma_1 = \gamma_1^*$, the optimiser to problem (A). Both Problems (A) and (B) are of the form of (21) and hence admit the explicit solution of the form of Theorem 3.1 (which can be computed efficiently). Clearly the solution thus obtained is sub-optimal, but the degree of sub-optimality will be commensurate to the size of the elements of Σ_2. The exact optimal can be obtained under the assumption that Σ_2, though small, can be computed reliably. In this case optimisation can be performed (as per Theorem 3.1 over $\Sigma_1\gamma_1$ and $\Sigma_2\gamma_2$ (simultaneously); absorbing Σ_2 into the variable $\Sigma_2\gamma_2$ avoids the ill-conditioning of the p polynomial but does not affect the processes of retrieving the optimal values for $\gamma_{1,2}$ (since $\Sigma_{1,2}$ are both diagonal).

4. Input constraint handling through resetting the vector a

It was assumed in Section 2 that in the operating region Ω, the system dynamics are controllable and involutive. This however does not preclude the existence of $x \in \Omega$ where the system from u to ψ loses relative degree, a condition that will arise when

$$L_g L_f^2 \lambda = 0 \tag{31}$$

or

$$\phi'(\lambda) = 0 \tag{32}$$

The second of these depends on x but also on a which can be reset in order to restore maximum relative degree; for this reason loss of relative degree due to (32) will be referred to as latent loss of relative degree (LLRD). In contrast, if x lies on the manifold defined by (31), then loss of relative degree cannot be avoided and hence this condition will be referred to as intrinsic loss of relative degree (ILRD).

It can be shown that (31, 32) define a set of manifolds in state space which will be referred to as Ξ. From (10) it follows that as x approaches Ξ the input u becomes unbounded which in practice will come into conflict with input constraints, which for simplicity will be take to be of the form:

$$|u| \leq u \tag{33}$$

To avoid input constraint violation it becomes necessary to reset the value of a so that

$$d^T(\widehat{u})a = 0 \quad \Leftrightarrow \quad a = \widehat{K}\gamma \tag{34}$$

with

$$\widehat{u} = \mathrm{sat}(u) = \begin{cases} u \text{ when } u > u \\ u \text{ when } -u \leq u \leq u \\ -u \text{ when } u < u \end{cases} \tag{35}$$

where $\widehat{K}$ is a full rank matrix representation of the kernel of $d^T(\widehat{u})$. Clearly resetting must be done in a way that has minimal effect on the PP objective and this suggests the following optimisation problem:

$$\min_{\gamma} \|\widehat{K}\gamma - \alpha\| \text{ s.t. } \|P^{1/2} D_0^T \widehat{K}\gamma\| \leq \|P^{1/2}\xi\| \tag{36}$$

where a, ξ denote the values of these vectors prior to resetting.

On account of (33), it is still the case that ξ will not evolve as per (17), but resetting adjusts the value of ξ_k which therefore implies the need for the constraint in (36) which ensures that the reset value of $V_k = \xi_k^T P \xi_k$ is no greater than its value before resetting. Once again, the minimization problem in hand is of the form of (21) and can therefore be solved efficiently and robustly.

Algorithm 4.1
Step 1: Evolve s, λ through (15), (16), and every T seconds reset a using (19), (20), and compute u from (9), (10).
Step 2: Implement u if $|u| \leq u$, increment k and return to Step 1.
Step 3: If $|u| > u$, reset a through (34, 35, 36), implement $\widehat{u}$, increment k and return to Step 1.

Theorem 4.2 *Algorithm 4.1 has the properties stated in Theorem 2.2 provided that as $k \to \infty$, Step 3 becomes inactive.*

Proof. First note optimisation (36) is always feasible since γ can always be scaled so that the constraint of (36) is met with equality. This constraint ensures that V_k behaves as a Lyapunov function which in turn implies that $\xi \to 0$. Under the conditions of the theorem, and the resetting (every T seconds) of (19), (20), it follows that a will tend to zero if and only if x tends to zero. On the other hand, if a does not tend to zero, λ will tend to λ_p, a root of $\phi(\lambda)$, and hence x will tend to a limit. Moreover if (20) remains inactive, then for sufficiently fast resetting the arguments of [3] will hold and thus the origin of the state space will be locally attractive.

The condition of Theorem 4.2 that Algorithm 4.1 asymptotically does not enter Step 3 may appear restrictive but it becomes necessary in the case of open loop unstable systems as a natural consequence of PP objective in the presence of constraints. Even for linear constrained systems, PP will fail if the constraints remain asymptotically active. The proviso of the theorem is an assertion of the fact that it is unreasonable to assume that constrained systems are globally stabilizable by PP. The results of the next section allow for a relaxation of this condition.

5. Input constraint handling through input resetting

From (9), (31) and (32) it is apparent that as x approaches the manifolds Ξ where the system from u to ψ loses relative degree (irrespective of whether the loss is latent or intrinsic), the continuous time $u(t)$ that achieves PP is likely to change significantly over the sampling interval τ, even when τ is chosen to be small. Another way of saying the same thing, is that as the state approaches Ξ, so the magnitude of $\dot{u}$ increases. Under such circumstances, it is no longer obvious that the best choice for the piecewise constant input, u_k, which meets the PP objective of (17) will be given by the value of the $u(t)$ obtained from (10) at the beginning of the sampling interval.

Close to Ξ therefore one requires an alternative means of choosing u_k.

The remedy is provided through the computation of the change in x

$$\delta x_k = \tau[f(x_k) + g(x_k)u_k] \tag{37}$$

which then can be related to the change in ξ

$$\delta \xi_k = J_{\xi x}(x_k)\delta x_k \tag{38}$$

where $J_{\xi x}$ is the Hessian of ξ with respect to x and its rows are given by

$$\begin{aligned} \left[\nabla\psi\right]^T &= [\nabla\phi(\lambda)]^T \\ \left[\nabla\dot{\psi}\right]^T &= [\nabla\{\phi'(\lambda)L_f\lambda\}]^T \\ \left[\nabla\ddot{\psi}\right]^T &= [\nabla\{\phi''(\lambda)(L_f\lambda)^2 + \phi'(\lambda)L_f^2\lambda\}]^T \end{aligned} \tag{39}$$

all of which can be computed through iterative formulae based on the results presented in Section 2. It is pointed out that both (37) and (38) are subject to an error but that is of order $O(\tau^2)$ and $O(\tau^4)$, respectively. The combination of (37) and (38) gives

$$\xi_{k+1} = \xi_k + \tau J_{\xi x}(x_k)[f(x_k) + g(x_k)u_k] \tag{40}$$

which implies that the optimal PP choice for u_k is given by the minimization

$$\min_{u_k} \|e^{A_p T}\xi_k - \xi_{k+1}\| = \min_{u_k} \|(e^{A_p\tau} - I)\xi_k - \tau J_{\xi x}(x_k)[f(x_k) + g(x_k)u_k]\| \tag{41}$$

This is a univariate optimisation problem which has an obvious explicit solution. Moreover, it is still possible to obtain an explicit expression for the optimiser when the minimization of (41) is to be performed subject to the input constraint of (33):

$$\min_{u_k} \|(e^{A_p\tau} - I)\xi_k - \tau J_{\xi x}(x_k)[f(x_k) + g(x_k)u_k]\| \quad \text{s.t.} -u \le u_k \le u \tag{42}$$

Algorithm 5.1
Step 1: Evolve s, λ through (15), (16), and every T seconds reset a using (19), (20), and compute u from (9), (10).
Step 2: Implement u if $|u| \le u$, increment k and return to Step 1.
Step 3: If $|u| > u$, reset a through (42), implement u, increment k and return to Step 1.

Theorem 5.2 *Algorithm 4.1 has the properties stated in Theorem 2.2 provided that as $k \to \infty$, Step 3 becomes inactive.*

Proof: This is similar to the proof of Theorem 4.2. The proviso of Theorem 5.2, as in Theorem 4.2, may appear restrictive, but is a consequence of the PP objective which, for the given input constraints and initial condition must be assumed to be stabilizing. A relaxation of this condition could be achieved if instead of minimizing the PP deviation (as is done in (42)) one minimized the value of the Lyapunov function at the next time instant, namely if (42) were replaced by

$$\min_{u_k} \|P^{1/2}\{\xi_k + \tau J_{\xi x}(x_k)[f(x_k) + g(x_k)u_k]\}\| \quad \text{s. t.} -u \le u_k \le u \tag{43}$$

The implication of this modification is that the relevant proviso for the corresponding theorem (which is obvious and therefore not stated) would be that the constrained system be stabilizable by a control law which aims at making V behave like a Lyapunov function (through PP when that is feasible, or through the mininimization of (43) otherwise). Clearly such an assumption is eminently sensible. Finally it is noted that if at any given k, $x_k \in \Xi$, then (42) would be independent of u_k since then:

$$J_{\xi x}(x_k)g(x_k) = \phi'(\lambda)\begin{bmatrix} L_g\lambda \\ L_gL_f\lambda \\ L_gL_f^2\lambda \end{bmatrix} = 0 \tag{44}$$

since the first two elements are zero by construction (namely by the evolution equations for s, λ whereas the third would be zero either due to latent or intrinsic loss of relative degree (see (31), (32)). If per chance at a particular x (44) holds true, then u will be taken to be zero and the algorithm can be resumed at the next k.

The efficacy of the results developed in this paper, and especially of Algorithm 5.1 will now be illustrated in a very convincing manner in the section below.

6. Illustrative Example

The following involutive but particularly challenging bilinear modelling is selected:

$$\dot{x} = Ax + (Fx + B)u; \quad y = Cx; \quad x_0 = \begin{bmatrix} 0.3785 & 0.3785 & 0.3244 \end{bmatrix}^T \tag{45}$$

where

$$A = \begin{bmatrix} 3.4913 & -6.8235 & -0.4605 \\ 1.0264 & -3.4706 & 0.5256 \\ 2.9812 & -5.5750 & -0.4193 \end{bmatrix}; \quad B = \begin{bmatrix} -0.144 \\ -0.036 \\ -0.108 \end{bmatrix};$$
$$F = \begin{bmatrix} 1.7295 & 2.8625 & -2.9561 \\ 0.4324 & 0.7156 & -0.7390 \\ 1.2971 & 2.1469 & -2.2171 \end{bmatrix}; \quad C = \begin{bmatrix} 0.7258 & 2.6840 & -1.7693 \end{bmatrix}; \tag{46}$$

and with input constraints:

$$-2.5 \le u \le 2.5 \tag{47}$$

The features that make this model challenging are that: (i) it is strongly nonlinear (as demonstrated by the elements of F whose magnitude is at least 10 times that of the corresponding elements of B); (ii) it is open loop unstable; (iii) it possesses no equilibrium point on the kernel of C which is minimum phase; (iv) the input constraints for the given initial condition become active under PP.

Despite all these difficulties, application of Algorithm 5.1, for a sampling interval of $\tau = 0.01$ and a Matching Resetting interval of $T = 50 \times \tau = 0.5$, produces remarkably good results shown in Figures 1, 2 and 3. The first of these depicts the time behaviour of the Lyapunov Function, V, and the input u. It is seen that V is monotonically decreasing, despite the periodic Matching Resetting of a (which cause the discontinuities in the plot of V every 50 samples). Clearly as expected the input remains within the limits imposed by the constraint (47) and reaches a steady state. The performance of the actual output is compared to that of the synthetic output in Figure 2. From this figure it is seen clearly that the Matching Resetting has the desired effect of causing the synthetic output and its derivatives to match those of the actual output at each one of the resetting times. As expected the synthetic output tends to zero and due to the successful resetting so does the actual output. The PP objective adopted in this example is that all the closed loop poles of the linearized system should be placed at -1. The success of the algorithm in achieving this for the actual output is illustrated in Figure 3 which compares the closed loop behaviour of y to that would be obtained by exact PP, with all the discrete time closed loop poles 0.986; the exact value for the discrete closed loop poles for the given sampling interval is 0.99. It is noted that for this example the vector v does not depend on x and therefore s can be chosen to be 1 throughout the entire simulation. The λ however was computed by (16) and the initial value for it was chosen as $\lambda_0 = -0.5$; different initial values result in different output steady state values, but due to the efficiency of the computation demanded by Algorithm 5.1, it is possible to fine tune the choice λ_0 online; all that is required is a univariate search to ensure that the output gets sufficiently close to 0. Once x approaches a small enough neighbourhood of the origin, it is possible to revert to linear state feedback to steer x to the origin.

7. Conclusion

Feedback linearization is an elegant tool of mathematics that provides the means of applying linear control to nonlinear systems. However, it cannot be applied directly to the output of nonminimum phase systems and thus necessitates the use of synthetic outputs. Matching resetting then closes the gap between the behaviour of the actual and synthetic output allowing for the indirect control of systems with unstable inverse dynamics. Sensitivity issues concerning the resetting procedure were considered and remedied efficiently. Moreover, extensions of the resetting strategy affecting both the definition of the synthetic output and the input that achieves pole placement were proposed with the view to overcoming problems relating to the loss of relative degree and taking account of input constraints. The approach was applied to a challenging bilinear problem and were shown to produce excellent results.

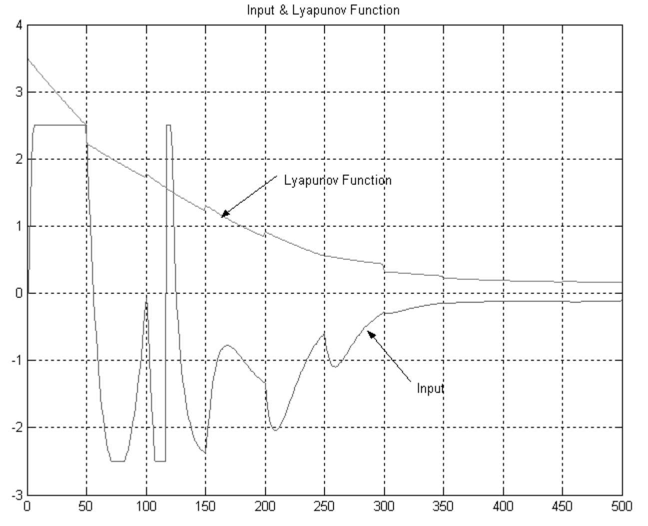

Figure 1: Closed loop response of the Lyapunov Function and the input

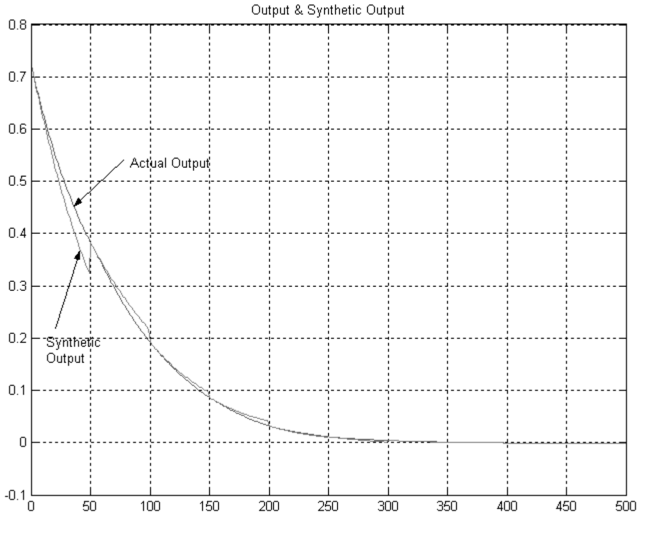

Figure 2: Closed loop responses of the actual and synthetic outputs

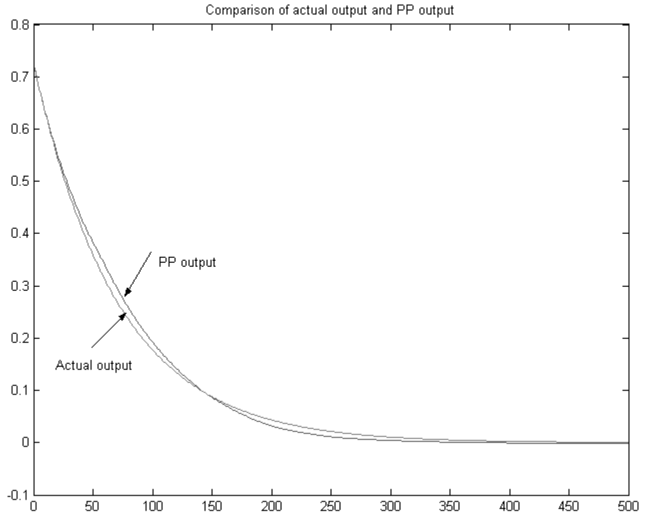

Figure 3: Comparison of the closed loop output behaviour with a PP output

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