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Stochastic control of interest rate policy and solvency interaction within a mixed portfolio of loans

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Abstract

In this presentation, we investigate one of the basic problems of the banking system. Generally speaking, a bank offers a variety of loans with different lending rates, according to a basic interest rate and the experience of the repayment patterns. Here, we construct a stochastic control model in order to evaluate the associated credit risk and obtain the optimal strategy for the determination of the level of the lending interest rates by maximizing the accumulated profit. Each sub-portfolio of loans is treated separately during a unit interval while at the end of the each time period there is some kind of solvency interaction. We assume that the repayment pattern follows a Brownian motion and using advanced optimization techniques, we derive the optimal solutions.

Keywords: Stochastic Optimal Control, Brownian Motion, Matrix Riccati Differential Equation, Portfolio of Banking Loans, Successive approximation of Picard.

1. Introduction

In credit risk management, the loan pricing is one of the basic problems. Loan pricing is the determination of the lending interest rates for the different banking sub-portfolios of loans (for example, mortgages, overdrafts etc) which are offered to its customers, according to their risk exposure, see [1]. The respective literature is very rich although some of the approaches and the concluding results are not linked intuitively to common lending practices, see [2].

Recently, in [3] research work, the optimal loan interest rate contracts under the conditions of risky, symmetric information for multi-period (dynamic) models is analyzed. According to their work, the optimal loan interest rate depends on the volatility, and co-variation among the market interest rate, borrower collateral, and income, as well as the time horizon and the risk preferences of lenders and borrowers. Moreover,

[4] take into consideration the determination of optimal loan and deposit rates, as well as the phenomenon of loan prepayments and deposit withdrawals. In this paper, the main object is to develop a stochastic model for managing smoothly the interest rates for the different sub portfolios of loan in a way that banking managers are seeking for. Moreover, the bank should consider this problem in a more general framework, letting some kind of interaction between the different sub-portfolios. The paper connects the interest rate policy of an interacted portfolio of loans with stochastic control theory. Although optimal control theory was developed by engineers in order to investigate the properties of dynamic systems of difference or differential equations, it has also been applied to financial problems. [5] was the first to spot a possible analogy between the industrial and engineering processes and post-war macroeconomic policy-making (see [6], for further historical details).

From this point of view, a stochastic control model for a certain banking system is constructed. The bank has a certain total capacity for providing loans equal to $\Delta(t)$ at time t. We take in mind that the bank's customers are not always consistent with their repayments. So, at each time t, a different amount say $\Delta^{'}(t)$ is repaid through the installments paid by the customers. This $\Delta^{'}(t)$ is normally smaller than $\Delta(t)$ but at some exceptions may take values greater than $\Delta(t)$, whenever the policy holders pay with some time delay two, three or more installments to the bank.
Hence, we may visualize the situation above with the following theoretical Figure 1. A brief outline of the paper is as follows. Section 2 provides the incentives and the

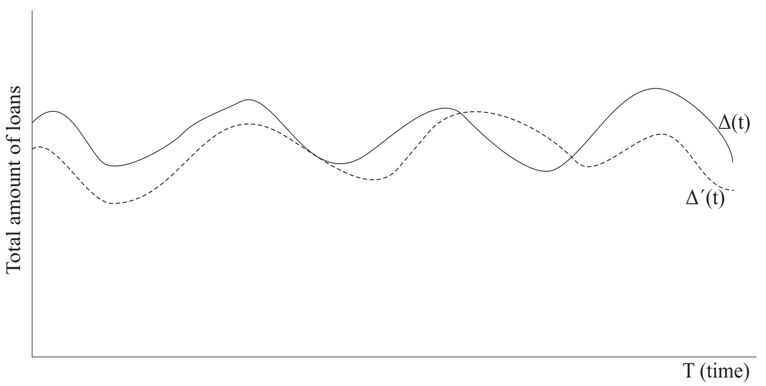

Figure 1: The total capacity for providing loans.

typical modeling features of the problem. Moreover, it is devoted to the results of stochastic calculus and control theory for standard Brownian motion and the respective linear systems driven by such a process. Section 3 provides the approximation solution for the matrix Riccati differential equation. Finally, section 4 presents a numerical example with some interesting and insightful diagrams, while section 5 concludes the whole paper.

2. The framework model

We consider a portfolio of n loans. Thereafter, we define the necessary symbols and the respective notation.

$\Pi_i(t)$: Accumulated profit or loss at time t for the i-sub-portfolio of loans, $i = 1, 2, \ldots, n$.
$a_i(t)$" Rate of return for the accumulated profit or loss at time t for the sub-portfolio of loans, $i = 1, 2, \ldots, n$.
$p_i(t)$" The ratio of the total amount of loan which corresponds to the installments paid at time t over the total amount of loan which has been placed into the i loan sub-portfolio, $i = 1, 2, \ldots, n$.
$\rho_i(t)$: The ratio of the total amount placed to the i loan sub-portfolio, $i = 1, 2, \ldots, n$, over the total amount of loans. $\sum_{i=1}^{n} \rho_i(t) = 1$.
$\Delta(t)$: Total amount for the whole portfolio of loans at time t.
$\Delta_i(t)$: Total amount of loans at time t for the i loan sub portfolio, $i = 1, 2, \ldots, n$, where $\Delta_i(t) = \rho_i(t)\Delta(t)$, and
$\Delta_i^{'}(t)$: Total amount of loans that corresponds to customers who consistently pay their installments at time t for the i loan sub portfolio, $i = 1, 2, \ldots, n$, where $\Delta_i^{'}(t) = p_i(t)\Delta(t)$.
$c_i(t)$: Capital cost (including expenses, operational cost, rate of return paid to customers due to bank deposits and the desirable profit for the bank) at time t for the i sub-portfolio of loan, $i = 1, 2, \ldots, n$.
$\varepsilon_i(t)$: The interest rate at time t for the i sub-portfolio of loan, $i = 1, 2, \ldots, n$.
$\lambda_{ij}(t)$: The percentage of profit or loss (solvency) transferred from the i loan sub-portfolio to j sub-portfolio at time t.

We assume that the percentage for the i sub-portfolio of loans which is consistently repaid is driven by a standard Brownian motion. This uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The flow of information is given by the natural filtration $\mathcal{F}_{t\in[0,T]}$, i.e. the $\mathcal{P}$-augmentation of a one dimensional Brownian filtration. Without loss of generality let us assume that the $\mathcal{F}_{t\in[0,T]} = \mathcal{F}$, i.e. the observable events are eventually known. Hence,

$$dp_i(t) = m_i(t)dt + \sigma_i(t)dW_i(t) \tag{1}$$

Then the accumulated profit of the total portfolio for the bank at time t is

$$\Pi(t) = \sum_{i=1}^{n} \Pi_i(t) \tag{2}$$

Moreover, under the above notation, we may describe the system by the following stochastic differential equation

$$d\Pi_i(t) = a_i(t)\Pi i(t)dt + \epsilon_i(t)\Delta_i(t)dp_i(t) - c_i(t)\Delta_i(t)dt$$
$$+\lambda_{1i}\Pi_1(t)dt + \ldots + \lambda_{i-1i}\Pi_{i-1}(t)dt + \lambda_{ii}\Pi_i(t)dt$$
$$+\lambda_{i+1i}\Pi_{i+1}(t)dt + \ldots + \lambda_{ni}\Pi_n(t)dt \quad (3)$$

or equivalently, substituting equation (1) into (3) we obtain

$$d\Pi_i(t) = a_i(t)\Pi i(t)dt + \epsilon_i(t)\Delta_i(t)[-m_i(t)dt + \sigma_i(t)dW_i(t)] - c_i(t)\Delta_i(t)dt$$
$$+\lambda_{1i}\Pi_1(t)dt + \ldots + \lambda_{i-1i}\Pi_{i-1}(t)dt + \lambda_{ii}\Pi_i(t)dt$$
$$+\lambda_{i+1i}\Pi_{i+1}(t)dt + \ldots + \lambda_{ni}\Pi_n(t)dt \quad (4)$$

for $i = 1, 2, \ldots, n$

Thus, we take in matrix form the (non homogeneous) linear stochastic differential equation:

$$d\bar{\Pi}(t) = [A(t)\bar{\Pi}(t)dt + B(t)\bar{\varepsilon}(t) + C(t)]dt + \sum_{j=1}^{n} H_j(t)\bar{\varepsilon}(t)dW_j(t)$$
$$\bar{\Pi}(0) = \Pi_0 \quad (5)$$

where $A(t) = \begin{pmatrix} a_1(t)+\lambda_{11}(t) & \lambda_{21}(t) & \cdots & \lambda_{n1}(t) \\ \lambda_{12}(t) & a_2(t)+\lambda_{22}(t) & \cdots & \lambda_{n2}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1n}(t) & \lambda_{2n}(t) & \cdots & a_n(t)+\lambda_{nn}(t) \end{pmatrix}$

$$\bar{\Pi}(t) = \begin{pmatrix} \Pi_1(t) \\ \Pi_2(t) \\ \vdots \\ \Pi_n(t) \end{pmatrix}, \quad \bar{\varepsilon}(t) = \begin{pmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \\ \vdots \\ \varepsilon_n(t) \end{pmatrix}$$

$$B(t) = diag\{m_1(t)\rho_1(t)\Delta(t); m_2(t)\rho_2(t)\Delta(t); \cdots ; m_n(t)\rho_n(t)\Delta(t)\}$$

$$C(t) = \begin{pmatrix} -c_1(t)\rho_1(t)\Delta(t) \\ -c_2(t)\rho_2(t)\Delta(t) \\ \vdots \\ -c_n(t)\rho_n(t)\Delta(t) \end{pmatrix} \quad and \quad D(t) = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \sigma_j(t)\rho_j(t)\Delta(t) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

The functions appearing in stochastic linear equation (SLQ) satisfy:

$$A(t),\ \ B(t),\ \ H_j(t) \in L^{\infty}([0,T];\mathbb{R}^{n\times n})\ \ \textit{and}\ \ C(t) \in L^{\infty}([0,T];\mathbb{R}^{n})$$

For any $t \in [0,T]$, we denote $U^{\omega}[0,t]$ the set of all 5 -tuples $(\Omega, \mathcal{F}, \mathcal{P}, W(\cdot), \bar{\varepsilon}(\cdot))$ satisfying the following:

— $(\Omega, \mathcal{F}, \mathcal{P})$ is a complete probability space

— $\{W(t)\}_{t\geq 0}$ is a n-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathcal{P})$ over $[0,T]$ (with $W(0) = 0$ almost surely), and $\mathcal{F}_t = \sigma\{W(r) : 0 \leq r \leq t\}$ augmented by all the $\mathcal{P}$-null sets in $\mathcal{F}$.

— $\varepsilon(\cdot) \in L^2_F([0,T];\mathbb{R}^n)$.

— Under $\varepsilon(\cdot)$, for any $\bar{\Pi}(0) = \Pi_0$ equation (2.3) admits a unique solution on $\bar{\Pi}(\cdot)$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$.

In addition, we must minimize the following expression, a quadratic cost criterion:

$$\begin{aligned} J^{\varepsilon}(t,\Pi) \quad &= \quad E^{t,\Pi}\,[\beta \int_0^T (\varepsilon(t)-\varepsilon_{\tau})^T(\varepsilon(t)-\varepsilon_{\tau})dt \\ &\quad + (1-\beta)(\Pi(T)-\Pi_{\tau})^T(\Pi(T)-\Pi_{\tau})] \end{aligned} \tag{6}$$

where, $T > 0$, $R = \beta I_n$ and $G = (1-\beta)I_n$, β is a weighting factor i.e. $0 \leq \beta \leq 1$.

This criterion requires a stable interest rate policy $\varepsilon(t)$, near to the target rate which is fully desirable by the customers of the banking system while also a small final value for the surplus fund $\Pi(T)$ obtained from this operation. The above (SLQ) problem at $(0,G) \in [0,T]\times\mathbb{R}^n$, where G is solvable if there exists a control $(\Omega, \mathcal{F}, \mathcal{P}, W(\cdot), \bar{\varepsilon}^*(\cdot))$ such that

$$J(0,G;\varepsilon^*(\cdot)) = inf_{\bar{\varepsilon}(\cdot)\in U^{\omega}[0,T]} J(0,G;\bar{\varepsilon}(\cdot)) \triangleq V(0,G) \tag{7}$$

We should stress that in the case where $\bar{\varepsilon}^*(\cdot)$ is an optimal control; the corresponding $\bar{\Pi}^*(\cdot)$ and $(\bar{\Pi}^*(\cdot), \bar{\varepsilon}^*(\cdot))$ are called an optimal state process and an optimal pair, respectively, to our problem. Finally, closing this section, we provide the basic theorem from [7] with respect to the necessary optimal control framework.

Theorem 2.1 Let $P(\cdot) \in C([0,T];\mathbb{R}^{n\times n})$ and $\phi(\cdot) \in C([0,T];\mathbb{R}^n)$ be the solution of (11) and (12) respectively, then:

$$B\Psi, D\Psi \in L^{\infty}(0,T;\mathbb{R}^{n\times n}),\ \ \textit{where}\ \ \Psi = (R + \sum_{j=1}^{n} H_j^T P H_j)^{-1} B^T P \tag{8}$$

and

$$B\psi, D\psi \in L^2(0,T;\mathbb{R}^n), \quad where \;\; \psi = (R + \sum_{j=1}^{n} H_j^T P H_j)^{-1} B^T \phi \tag{9}$$

The (SLQ) problem is solvable with the optimal control $\bar{\varepsilon}^*(\cdot)$ being of a state feedback form:

$$\bar{\varepsilon}^*(t) = \bar{\varepsilon}_\tau - \Psi(t)(\bar{\Pi}(t) - \bar{\Pi}_\tau) - \psi(t), \quad for \;\; t \in [0,T] \tag{10}$$

Proof: see [7].
Where $P(\cdot) \in C([0,T];\mathbb{R}^{n\times n})$ is symmetric and $\phi(\cdot) \in C([0,T];\mathbb{R}^n)$ are matrix-stochastic equations of the following form:

$$\dot{P} + PA + A^T P - PB(R + \sum_{j=1}^{n} H_j^T P H_j)^{-1} BP = 0; \; where \;\; P(T) = G, \;\; a.e. \;\; t \in [0,T] \tag{11}$$

and

$$\dot{\phi} + \{A - B(R + \sum_{j=1}^{n} H_j^T P H_j)^{-1} BP\}^T \phi + PC = 0; \; where \;\; \phi(T) = 0, \;\; a.e. \;\; t \in [0,T] \tag{12}$$

To find the optimal control (10) we need the solution of the non linear differential matrix equation (11), which is discussed extensively in the next section.
Moreover, the solution of equation (12) has the following form

$$\phi(t) = \zeta(t,0) + \int_0^t \zeta(t,r) P(r) C(r) dr$$

we define

$$\tilde{A} = [A - B(R + \sum_{j=1}^{n} H_j^T P H_j)^{-1} BP]$$

where by using Picard's successive approximation, the state transition matrix is given by the following expression, which is called the Peano-Baker series [8].

$$\zeta(t,r) = I + \int_r^t \tilde{A}(s) ds + \ldots + \int_r^t \tilde{A}(s_1) \ldots \int_r^{s_n} \tilde{A}(s_n) ds_n \ldots ds_1 + \ldots \tag{13}$$

Taking into consideration the above expressions, the optimal feedback control (10) can be substituted into equation (5) and after some algebra we obtain

$$d\bar{\Pi}(t) = [A^*(t)\bar{\Pi}(t)dt + b^*(t)]dt - \sum_{j=1}^{n} [C^*(t)\bar{\Pi}(t) + d^*(t)] dW_j(t) \tag{14}$$

where

$$A^* = A - B\Psi, \;\; b^* = B(\Psi\bar{\Pi}_\tau - \psi) + C$$

$$C^* = H_j\Psi, \quad and \quad d^* = H_j(\Psi\bar{\Pi}_\tau - \psi) \tag{15}$$

Theorem 2.2 For $A^*(t), C^*(t) \in L^\infty([0,T];\mathbb{R}^{n\times n})$ and $b^*(t), d^*(t) \in L^\infty([0,T];\mathbb{R}^n)$ the linear equation

$$d\bar{\Pi}(t) = [A^*(t)\bar{\Pi}(t)dt + b^*(t)]dt - \sum_{j=1}^{n}[C^*(t)\bar{\Pi}(t) + d^*(t)]dW_j(t) \tag{16}$$

and $\Phi(t)$ is the solution of the following:

$$d\Phi(t) = A^*(t)\Phi(t)dt - \sum_{j=1}^{n} C_j^*(t)\Phi(t)dW_j(t), \quad where \quad \Phi(0) = I \tag{17}$$

then the strong solution of $\bar{\Pi}$ of (16) can be represented as

$$d\bar{\Pi}(t) = \Phi(t)\bar{\Pi}_0 + \Phi(t)\int_0^t \Phi(s)^{-1}[b^*(s) - \sum_{j=1}^{n} C_j^*(s)d_j^*(s)]ds + \sum_{j=1}^{n}\Phi(t)\int_0^t \Phi(s)^{-1}d_j^*(s)dW_j(s) \tag{18}$$

where

$$d(\Phi(t)^{-1}) = \Phi(t)^{-1}[\sum_{j=1}^{n}(C_j^*(t))^2(t) - A^*(t)]dt - \sum_{j=1}^{n}\Phi(t)^{-1}C_j^*(t)dW_j(t) \tag{19}$$

Proof: see [7].

So, using the above theorems the solution of the (non homogeneous) linear stochastic differential equation is obtained.

3. The analytic solution of the non linear differential matrix equation, $P(t)$

In this section, we describe the solution of the non linear differential matrix equation of $P(\cdot) \in C([0,T];\mathbb{R}^{n\times n})$, which is symmetric and has the following form:

$$\dot{P}(t) + P(t)A(t) + A(t)P(t) - P(t)B(t)(R + \sum_{j=1}^{n} H(t)_j^T P(t)H_j(t))^{-1}B(t)P(t) = 0$$

$$where \quad P(T) = G, \quad a.e. \quad t \in [0,T] \tag{20}$$

We define $P(t)$ as follows:

$$P(t) = \begin{pmatrix} P_{11}(t) & P_{12}(t) & \cdots & P_{1n}(t) \\ P_{12}(t) & P_{22}(t) & \cdots & P_{n2}(t) \\ \vdots & \vdots & \ddots & \vdots \\ P_{1n}(t) & P_{2n}(t) & \cdots & P_{nn}(t) \end{pmatrix} \tag{21}$$

where $P_{ij}(t)$, $i \leq j = 1, 2, \ldots n$ are scalars continuous functions. In a case the matrix $A(t)$ is symmetric, i.e. $\lambda_{ij} = \lambda_{ji}$,for $i \neq j$ and taking into consideration the following approximations for the simplicity of calculations, i.e.:
$a_i = a(t)$,the same rate of return for the accumulated profit or loss at time t for the each sub-portfolio of loans, $i = 1, 2, \ldots, n$,
$\lambda_{ij}(t) = \lambda(t)$, the percentage of profit or loss transferred from the i loan sub-portfolio to j sub-portfolio at time t, $\lambda_{ij} = 1 - (n-1)\lambda(t)$,and
$\rho_i(t) = \rho(t)$, the ratio of the total amount placed to the i loan sub-portfolio,$i = 1, 2, \ldots, n$, over the total amount of loans.
The matrix $A(t)$ takes the following format

$$A(t) = \begin{pmatrix} a(t)+(1-(n-1)\lambda(t)) & \lambda(t) & \cdots & \lambda(t) \\ \lambda(t) & a(t)+(1-(n-1)\lambda(t)) & \cdots & \lambda(t) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda(t) & \lambda(t) & \cdots & a(t)+(1-(n-1)\lambda(t))) \end{pmatrix} \tag{22}$$

Now each element of (19) can be written as:

$$A(t)P(t) + P(t)A(t) = \begin{pmatrix} Q_{11}(t) & Q_{12}(t) & \cdots & Q_{1n}(t) \\ Q_{12}(t) & Q_{22}(t) & \cdots & Q_{n2}(t) \\ \vdots & \vdots & \ddots & \vdots \\ Q_{1n}(t) & Q_{2n}(t) & \cdots & Q_{nn}(t) \end{pmatrix} \tag{23}$$

where, the above matrix is symmetric, $(AP + PA)^T = AP + PA$, as A, P are $n \times n$-symmetric matrices. Thus,

$$\begin{aligned} Q_{ij}(t) &= 2[a(t) + (1-(n-1)\lambda(t))]P_{ij}(t) \\ &+\lambda(t)[\sum_{l=1}^{i} P_{li}(t) + \sum_{l=i}^{n} P_{il}(t) - P_{ij}(t) - P_{ii}(t)] \\ &+\lambda(t)[\sum_{l=1}^{j} P_{lj}(t) + \sum_{l=j}^{n} P_{jl} - P_{ij}(t) - P_{jj}(t)] \end{aligned} \tag{24}$$

Before we go further, we calculate the

$$P(t)B(t)(R+\sum_{j=1}^{n} H_j^T(t)P(t)H_j(t))^{-1})B(t) = \begin{pmatrix} S_{11}(t) & S_{12}(t) & \cdots & S_{1n}(t) \\ S_{12}(t) & S_{22}(t) & \cdots & S_{n2}(t) \\ \vdots & \vdots & \ddots & \vdots \\ S_{1n}(t) & S_{2n}(t) & \cdots & S_{nn}(t) \end{pmatrix} \tag{25}$$

We easily prove that $PB(R+\sum_{j=1}^{n} H_j^T PH_j)^{-1})B$ is also symmetric, as P is symmetric and B, R and H_j, for $j = 1, 2, \ldots, n$ are diagonal matrices. Thus, for $i = 1, 2, \ldots, n$

$$S_{ij}(t) = \sum_{l=1}^{i} \frac{(m_l(t)\rho(t)\Delta(t))^2}{\beta + \sigma_l(t)\rho(t)\Delta(t)P_{ll}(t)} P_{li}(t)P_{lj}(t) + \sum_{l=i}^{j} \frac{(m_l(t)\rho(t)\Delta(t))^2}{\beta + \sigma_l(t)\rho(t)\Delta(t)P_{ll}(t)} P_{il}(t)P_{lj}(t)$$

$$+ \sum_{l=j}^{n} \frac{(m_l(t)\rho(t)\Delta(t))^2}{\beta + \sigma_l(t)\rho(t)\Delta(t)P_{ll}(t)} P_{il}(t)P_{jl}(t)$$

$$- \frac{(m_i(t)\rho(t)\Delta(t))^2}{\beta + \sigma_i(t)\rho(t)\Delta(t)P_{ii}(t)} P_{ii}(t)P_{ij}(t) - \frac{(m_j(t)\rho(t)\Delta(t))^2}{\beta + \sigma_j(t)\rho(t)\Delta(t)P_{jj}(t)} P_{ij}(t)P_{jj}(t) \tag{26}$$

Substituting the above expressions (24) and (26) to (20) we obtain the family of the ordinary non linear differential equations. With that expression, we succeed in transferring the non homogeneous matrix Riccati differential equation (20), into a Cauchy problem for a system of first order differential equations, where $P(T) = G$, $a.et \in [0, T]$.
Consider the Cauchy problem of the first - order differential equation:

$$\dot{P}_k = f_k(t, P_{ij}), \quad for \;\; i \leq j, \quad i, j, k = 1, 2, \ldots, n \tag{27}$$

or equivalently,

$$\dot{P} = f(t, P) \tag{28}$$

where, $P = (P_{11}, P_{12}, \ldots, P_{1n}, \ldots, P_{ij}, \ldots, P_{nn})$ and

$$f(t, P_{11}, P_{12}, \ldots, P_{1n}, \ldots, P_{ij}, \ldots, P_{nn}) =$$

$$(f_1(t, P_{11}, P_{12}, \ldots, P_{1n}, \ldots, P_{ij}, \ldots, P_{nn}), \ldots, f_n(t, P_{11}, P_{12}, \ldots, P_{1n}, \ldots, P_{ij}, \ldots, P_{nn})) \tag{29}$$

with the initial condition, after a change of variable,

$$P(t) = P(T - t) \tag{30}$$

$$so, \quad P_0 = P(0) = G, \;\; a.e. \;\; t \in [0, T] \tag{31}$$

where $G = (1 - \beta)I_n$, β is a weighting factor i.e. $0 \leq \beta \leq 1$.

The method of successive approximations obtains the solution $P(T - t)$ as the limit of a sequence of functions $P^{(n)}(T - t)$ which are determined by the recurrence formula.

$$P^{(n)}(T - t) = P^{(0)} + \int_{T-t}^{T} f(r, P^{(n-1)}(T - r))dr \tag{32}$$

It has been shown by [9] that, if the right-hand term in the domain $Q \in \mathbb{R}^{n+1}\{|t| \leq k_1, |P - P_0| \leq k_2\}$ satisfies the Lipschitz condition with respect to $\mathcal{P}$

$$|f(t, P^{(1)}) - f(t, P^{(2)})| \leq K|P^{(1)} - P^{(2)}|, \quad for \;\; K > 0 \tag{33}$$

then, irrespective of the choice of the initial function, the consecutive approximations $P^{(n)}(T-t)$ converge on some interval $[0,h]$ to the solution of this Cauchy problem. Moreover, if $f(t,P)$ is continuous in a rectangle $Q \in \mathbb{R}^{n+1}\{|t| \leq k_1, |P-P_0| \leq k_2\}$, then the error of the approximate solution $P^{(n)}(T-t)$ on the interval $[0,h]$ is estimated by the inequality:

$$\epsilon_n = |P(T-t) - P^{(n)}(T-t)| \leq MK^n \frac{(T-t)^{n+1}}{(n+1)!} \tag{34}$$

where, $M = max\{|f(t,P) : (t,P) \in \mathbb{R}^{n+1}|\}$ and h is determined by $h = min(k_1, \frac{k_2}{M})$.

4. Numerical Example for the portfolio of two loans

The numerical application is subject to the basic parameters $a = 5\%$, $\lambda = 0.3$, $\rho = 0.5$ and $\Delta = 1$ billion EUROS set out in the following table and the other subsidiary variables $m_1 = 70\%$, $m_2 = 85\%$, $s_1 = 0.08$, $s_2 = 0.07$, $c_1 = 0.03$, $c_2 = 0.02$, $T = 15$, $\beta = 0.5$.

The matrix $P(\cdot)$ has the following form:

$$P_{11}(t) = 1.5612 + 0.8112t - 0.7975t^2$$

$$P_{12}(t) = 1.5738 + 0.8299t - 0.8450t^2, \quad P_{22}(t) = 1.2842 + 0.3000t - 0.5135t^2 \tag{35}$$

For convenience the other coefficients are equivalently small, for instance

$$k_4 = 0.0631 << K_3, \quad l_4 = 0.0805 << l_3 \;\; and \;\; c_4 = 0.0463 << c_3$$

Moreover,

$$\zeta(t,0) = I + \Lambda_1 t + \Lambda_2 t^2 + \Lambda_3 t^3$$

where, the desirable capital cost is $c_1 = 0.03$ and $c_2 = 0.02$, respectively.

$$\Lambda_1 = \begin{pmatrix} 0.3694 & -0.0131 \\ -0.1094 & 0.2483 \end{pmatrix} \Lambda_2 = \begin{pmatrix} -0.0984 & -0.0362 \\ -0.0474 & 0.1318 \end{pmatrix} \Lambda_3 = \begin{pmatrix} 0.0647 & 0.0415 \\ 0.0543 & 0.0896 \end{pmatrix} \tag{36}$$

And the matrix equation $\phi(\cdot)$ is given by the expression

$$\phi(t) = [I + \Lambda_1 t + \Lambda_2 t^2 + \Lambda_3 t^3]\{\phi_0 - \begin{pmatrix} 0.0363t + 0.0076t^2 - 0.0057t^3 \\ 0.0350t + 0.0064t^2 - 0.0054t^3 \end{pmatrix}\}$$

$$+ \begin{pmatrix} -0.0363t + 0.0140t^2 - 0.0050t^3 - 0.0011t^4 - 0.0010^5 - 0.0001t^6 \\ -0.0350t + 0.0087t^2 - 0.0070t^3 - 0.0001t^4 - 0.0009^5 - 0.0004t^6 \end{pmatrix} \tag{37}$$

Then, the optimal control (10) being of a state feedback form can have the following format, where $SR_1 = \frac{\Pi_1}{\Pi_{\tau 1}}$ and $SR_2 = \frac{\Pi_2}{\Pi_{\tau 2}}$ is the solvency ratio for the 1^{st} and the 2^{nd} portfolio of loans, respectively.

$$\varepsilon^*(t) = \varepsilon_\tau - \Psi(t) \begin{pmatrix} \Pi_{\tau 1} & 0 \\ 0 & \Pi_{\tau 2} \end{pmatrix} \begin{pmatrix} SR_1 \\ SR_2 \end{pmatrix} \tag{38}$$

In order to make insightful implementations, we stress out two important parameters:
a) the capital cost (including expenses, operational cost, rate of return paid to customers due to bank deposits and the desirable profit for the bank) is far below the accumulated profit (see table 1) at time t for both sub-portfolios of loans and
b) the borrowers are not consistent with the repayments, as it is clear form table 1. Moreover, the first loan is most vulnerable to changes, in all the following cases. Now, let have the case study, where $\varepsilon_1 = 7\%$, and $\varepsilon_2 = 6\%$.
At time $t = 1.5$, for rate of loans $\Pi_{\tau 1} = 0.06$ and $\Pi_{\tau 2} = 0.05$ respectively, and same Solvency Ratio, between 0.6 to 1.5, for each of the two sub-portfolio of loans(Figure 2). In that case, the solvency ratio is above 1, so the financial managers may

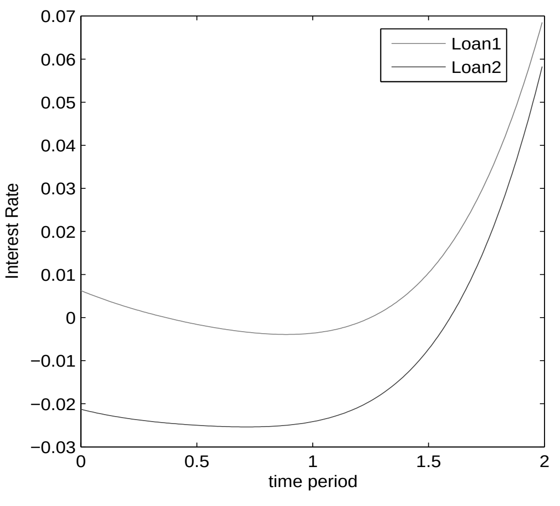

Figure 2: A portfolio of two loans with the same Solvency Ratio, between 0.6 and 1.5.

diminish or even "equalize to zero" the respective loan interest rate in order to provide cheaper loans attracting more customers for other profitable business. On the other hand, when the solvency ratio is below 1, the decreasing of loan interest rate may be equivalently less.

5. Conclusions

The paper provides a powerful theoretical model for the loan pricing problem using a stochastic dynamic approach. The proposed model coincides better (than other static models) with the common lending practices and procedures used by the banking system and may be easily linked with the standard figures and results of the annual balance sheet and profit/loss account. The assumption that the repayment pattern (i.e the proportion of persons who properly repay their loans) follows a Brownian motion also upgrades the realism of the model.

At the end, the full model is proved to be quite complicated but using advanced optimization techniques of stochastic control theory we manage to obtain the solution of the stochastic differential equations in closed form. The solution is actually an automatic controller which determines the level of lending interest rate for each sub-portfolio of loans. Then standard approximation procedures (as the method of successive approximations of Picard) are employed in order to obtain analytical solution in open form.

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