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On the interior acoustic and electromagnetic excitation of a layered scatterer with a resistive or conductive core

Nikolaos L. Tsitsas and Christodoulos Athanasiadis

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Abstract

Concrete interesting scientific and technological applications suggest the investigation of the scattering problems, concerning the spherical wave excitation of a layered scatterer by a point-source located in its interior. In this paper we investigate scattering theorems for the acoustic and electromagnetic excitation of a layered scatterer by an interior point-source. On the scatterer's core the resistive- or conductive-transmission boundary conditions are imposed. Such boundary conditions arise mainly in scattering problems by thin, shell like structures, modelling lightweight materials. Acoustic and electromagnetic general scattering, optical and mixed scattering theorems for the total, primary and secondary fields are derived.

Keywords: Scattering theorems, layered media, spherical waves, acoustics, electromagnetics, resistive and conductive boundary conditions

1. Introduction

In scattering theory appear some general relations between the solutions of two scattering problems, due to two distinct waves simultaneously incident on the same scatterer. Such relations, commonly referred to as *scattering theorems*, reduce the solution of one of the two scattering problems to that of the other. Scattering relations for acoustic and electromagnetic plane incident waves on homogeneous obstacles first appeared in [1] and [2]. These relations were extended to layered scatterers in [3] and [4]. On the other hand, scattering relations for acoustic and electromagnetic spherical waves incident on homogeneous obstacles appeared in [5] and mainly in [6].

The papers mentioned above concern incident waves, generated by a source in the exterior of the obstacle. However, interesting scientific and technological applications motivate the study of the scattering problems, where the scatterer is excited by a spherical wave generated by a source in its interior. These include inverse scattering methods for the localization of an object buried in a layered medium [7], implantations inside the human head for hyperthermia or biotelemetry purposes [8], investigation of the activity of the human brain, due to its excitation by the electrochemically generated neurons currents [9], [10], and certain other applications discussed in [11] and [12].

The investigation of acoustic and electromagnetic scattering theorems for interior excitation of a layered obstacle by a spherical wave was initiated in our papers [11] and [12]. In this paper we enrich the techniques and extend the results of [11] and [12] for scatterers with resistive and conductive cores. Such cores are modelled by employing the boundary conditions introduced in [13] and [14], which are different to and non-comparable with those of [11] and [12]. Resistive and conductive boundary conditions arise in scattering problems by thin, shell like structures, modelling lightweight materials. Representative applications of such boundary conditions involve strips and flat plates as well as quasi-stationary models in magnetotellurics, investigating the electromagnetic induction in the earth (for further discussion see [13]-[15]).

The layered scatterer is modelled as a nested body, consisting of a finite number of homogeneous layers with constant material parameters in each layer. On the boundary surfaces of the layers, except from the core, transmission conditions are imposed. Such a scatterer constitutes an appropriate model for the investigation of specific applications, such as those reported above.

Scattering problems, concerning point-sources located outside the obstacle, deal with the effects that a discontinuity of the medium of propagation has upon a known wave. For sources of illumination inside the obstacle, and observation of the field outside it, we have a radiation and not a scattering problem. In this context, in order to formulate and prove radiation theorems of the interior excitation problem, handle the cases of sources inside the scatterers layers and unify the cases of interior and exterior illumination, we make essential use of Sommerfeld's method (see [16], Section 6.32 or [17], Section 9.28).

The layered scatterer is excited by two acoustic point-sources or two magnetic dipoles, located in any two layers. In Sections 2 and 4 we present the mathematical formulation of the layered media acoustic and electromagnetic excitation problem respectively by making essential use of Sommerfeld's method. In Sections 3 and 5 we establish acoustic and electromagnetic general scattering, optical, and mixed scattering theorems for the total, primary, and secondary fields. General scattering theorems are useful in determining low-frequency expansions of the far-field patterns [18] and in studying the spectrum of the far-field operator [4]. The optical theorem serves the efficient computation of the cross-section in specific scattering applications. Mixed scattering theorems relate the solutions of plane and spherical wave incidence problems and play a central role in inverse scattering methods [19].

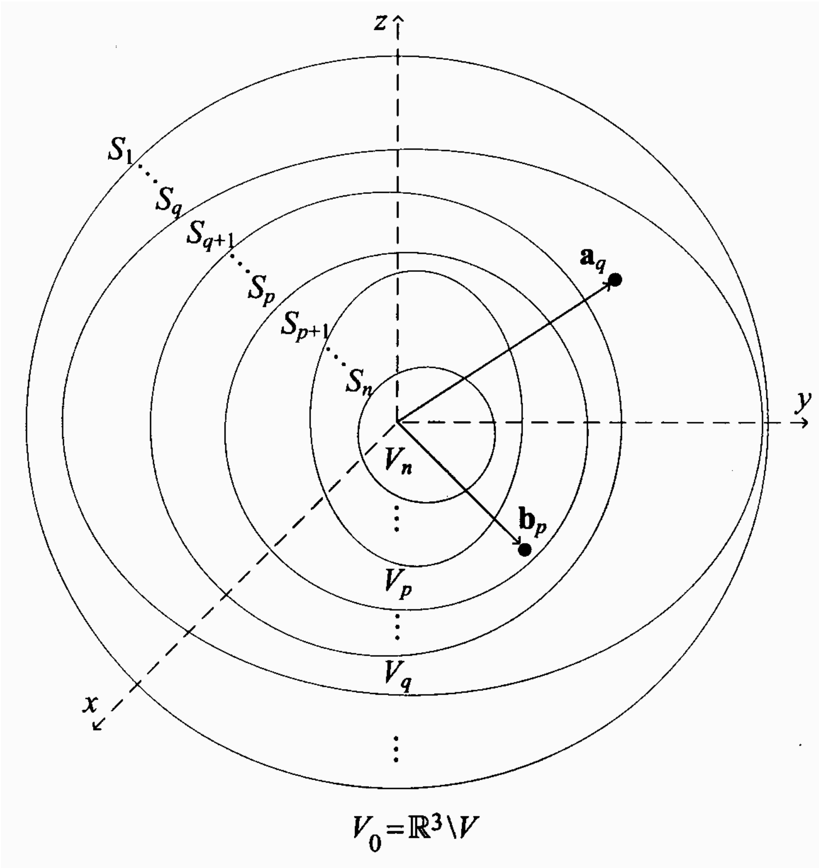

Figure 1: Typical cross-section of the layered scatterer V.

2. Mathematical Formulation: Acoustic Waves

The layered scatterer V considered here is identified by a compact subset of $\mathbb{R}^3$ with $\mathcal{C}^2$ boundary S_1, the interior of which is divided by $n-1$ surfaces S_j (j=2,...,n) into n annuli-like regions (layers) V_j (j=1,...,n) (see Fig. 1). The surfaces S_j are supposed $\mathcal{C}^2$, oriented by the outward normal unit vector $\hat{\mathbf{n}}$, with S_j including S_{j+1} and $\text{dist}(S_j, S_{j+1})>0$. The layers V_j (j=1,...,$n-1$), are homogeneous media specified by real wavenumbers k_j and mass densities ρ_j. The scatterer's core V_n is resistive or conductive. The exterior V_0 of V is a homogeneous medium with real constants k_0 and ρ_0.

The scatterer V is excited by a time-harmonic spherical acoustic wave, generated by a point-source located at $\mathbf{a}_q$ of layer V_q (q=0,...,n). The field $u^{pr}_{\mathbf{a}_q}$, radiated by this point-source, under the assumptions that the scatterer is absent and that $\mathbb{R}^3$ is filled by the material of V_q, constitutes the *primary field* of the Sommerfeld's method (see [16], [17] and the discussion of [20]). Suppressing the time dependence $\exp(-i\omega t)$, this primary spherical field is expressed by

$$u^{pr}_{\mathbf{a}_q}(\mathbf{r}) = a_q \exp(-ik_q a_q) \frac{\exp(ik_q|\mathbf{r}-\mathbf{a}_q|)}{|\mathbf{r}-\mathbf{a}_q|}, \qquad \mathbf{r} \in \mathbb{R}^3 \backslash \{\mathbf{a}_q\}, \tag{1}$$

reducing to the plane wave

$$\lim_{a_q \to \infty} u^{pr}_{\mathbf{a}_q}(\mathbf{r}) = \exp(-ik_q \hat{\mathbf{a}}_q \cdot \mathbf{r})$$

of unit amplitude and propagation direction $-\hat{\mathbf{a}}_q$, for point-sources receding to infinity.

The primary field $u^{pr}_{\mathbf{a}_q}$ excites the scatterer V, generating the *secondary fields* $u^j_{\mathbf{a}_q}$ and $u^{sec}_{\mathbf{a}_q}$ in layers V_j $(j \neq q)$ and V_q. Applying Sommerfeld's method, the total field $u^q_{\mathbf{a}_q}$ in V_q is defined as the superposition of the primary and the secondary field

$$u^q_{\mathbf{a}_q}(\mathbf{r}) = u^{pr}_{\mathbf{a}_q}(\mathbf{r}) + u^{sec}_{\mathbf{a}_q}(\mathbf{r}), \quad \mathbf{r} \in V_q \setminus \{\mathbf{a}_q\}. \tag{2}$$

Note that the total field in V_j $(j \neq q)$ coincides with the secondary field $u^j_{\mathbf{a}_q}$.

The total field $u^j_{\mathbf{a}_q}$ satisfies the Helmholtz equation

$$\Delta u^j_{\mathbf{a}_q}(\mathbf{r}) + k_j^2 u^j_{\mathbf{a}_q}(\mathbf{r}) = 0, \tag{3}$$

for $\mathbf{r} \in V_j$ if $j \neq q$ and $\mathbf{r} \in V_q \backslash \{\mathbf{a}_q\}$ if $j = q$.

On the surfaces S_j $(j \neq n)$ the total fields satisfy the transmission boundary conditions

$$\begin{aligned} u^{j-1}_{\mathbf{a}_q}(\mathbf{r}) &= u^j_{\mathbf{a}_q}(\mathbf{r}), \quad \mathbf{r} \in S_j \\ \frac{1}{\rho_{j-1}} \frac{\partial u^{j-1}_{\mathbf{a}_q}(\mathbf{r})}{\partial n} &= \frac{1}{\rho_j} \frac{\partial u^j_{\mathbf{a}_q}(\mathbf{r})}{\partial n}, \quad \mathbf{r} \in S_j \end{aligned} \tag{4}$$

The total field on the surface S_n of the core satisfies respectively
the resistive transmission

$$\begin{aligned} \mu_{n-1} u^{n-1}_{\mathbf{a}_q}(\mathbf{r}) - \mu_n u^n_{\mathbf{a}_q}(\mathbf{r}) &= 0, \quad \mathbf{r} \in S_n \\ \frac{\partial u^{n-1}_{\mathbf{a}_q}(\mathbf{r})}{\partial n} - \frac{\partial u^n_{\mathbf{a}_q}(\mathbf{r})}{\partial n} + \frac{2ik_{n-1}}{\eta} u^{n-1}_{\mathbf{a}_q}(\mathbf{r}) &= 0, \quad \mathbf{r} \in S_n \end{aligned} \tag{5}$$

or the conductive transmission boundary conditions

$$\begin{aligned} \mu_{n-1} u^{n-1}_{\mathbf{a}_q}(\mathbf{r}) - \mu_n u^n_{\mathbf{a}_q}(\mathbf{r}) - \frac{2i\eta}{k_{n-1}} \frac{\partial u^{n-1}_{\mathbf{a}_q}(\mathbf{r})}{\partial n} &= 0, \quad \mathbf{r} \in S_n \\ \frac{\partial u^{n-1}_{\mathbf{a}_q}(\mathbf{r})}{\partial n} - \frac{\partial u^n_{\mathbf{a}_q}(\mathbf{r})}{\partial n} &= 0, \quad \mathbf{r} \in S_n \end{aligned} \tag{6}$$

Physical interpretations of the parameters μ_{n-1}, μ_n and η in the above boundary conditions are discussed in [13].

Furthermore, $u^0_{\mathbf{a}_q}$ satisfies the Sommerfeld radiation condition [21]

$$\frac{\partial u^0_{\mathbf{a}_q}(\mathbf{r})}{\partial n} - ik_0 u^0_{\mathbf{a}_q}(\mathbf{r}) = o(r^{-1}), \quad r \to \infty \tag{7}$$

uniformly for all directions $\hat{\mathbf{r}}$ of $\mathbb{R}^3$, i.e. $\hat{\mathbf{r}} \in S^2 = \{\mathbf{x} \in \mathbb{R}^3, |\mathbf{x}| = 1\}$.

By combining the techniques of [13] and [22], we see that the above two problems are well posed.

Besides, the total field $u^0_{\mathbf{a}_q}$ in layer V_0 has the asymptotic expression

$$u^0_{\mathbf{a}_q}(\mathbf{r}) = g^0_{\mathbf{a}_q}(\hat{\mathbf{r}}) h_0(k_0 r) + \mathcal{O}(r^{-2}), \quad r \to \infty, \tag{8}$$

where $h_0(x)=\exp(ix)/ix$ is the zero-th order spherical Hankel function of the first kind. The primary spherical acoustic wave (1) satisfies the radiation condition (7) and hence has the asymptotic expression

$$u^{pr}_{\mathbf{a}_q}(\mathbf{r}) = g^{pr}_{\mathbf{a}_q}(\hat{\mathbf{r}}) h_0(k_q r) + \mathcal{O}(r^{-2}), \quad r \to \infty. \tag{9}$$

In the sequel, the functions $g^0_{\mathbf{a}_q}$ and $g^{pr}_{\mathbf{a}_q}$ will be referred to as the *total* and *primary far-field pattern.* By [11] we have

$$g^{pr}_{\mathbf{a}_q}(\hat{\mathbf{r}}) = \frac{\exp(-ik_q \mathbf{a}_q \cdot \hat{\mathbf{r}})}{h_0(k_q a_q)}.$$

For a point-source in the exterior of the layered scatterer the secondary field $u^{sec}_{\mathbf{a}_0}$ in V_0 has also the asymptotic expression

$$u^{sec}_{\mathbf{a}_0}(\mathbf{r}) = g^{sec}_{\mathbf{a}_0}(\hat{\mathbf{r}}) h_0(k_0 r) + \mathcal{O}(r^{-2}), \quad r \to \infty. \tag{10}$$

The function $g^{sec}_{\mathbf{a}_0}$ will be referred to as the *secondary far-field pattern.* Now, Eqs. (2) and (8)-(10) imply the total far-field pattern superposition

$$g^0_{\mathbf{a}_0}(\hat{\mathbf{r}}) = g^{pr}_{\mathbf{a}_0}(\hat{\mathbf{r}}) + g^{sec}_{\mathbf{a}_0}(\hat{\mathbf{r}}).$$

3. Acoustic Scattering Relations

The layered scatterer V is excited by two point-sources located at $\mathbf{a}_q \in V_q$ and $\mathbf{b}_p \in V_p$ $(p,q=0,\ldots,n)$ $(p \geq q)$, generating primary $u^{pr}_{\mathbf{a}_q}$, $u^{pr}_{\mathbf{b}_p}$, secondary $u^{sec}_{\mathbf{a}_q}$, $u^{sec}_{\mathbf{b}_p}$ and total fields $u^q_{\mathbf{a}_q}$, $u^p_{\mathbf{b}_p}$ in layers V_q, V_p.

By $S_R(\mathbf{0})$ we denote a large sphere centered at $\mathbf{0}$ with radius R, containing V and the points $\mathbf{a}_q$ and $\mathbf{b}_p$, and by $S_\varepsilon(\mathbf{a}_q)$ a small sphere centered at $\mathbf{a}_q$ with radius ε. We also use the notation, introduced by Twersky in [1]

$$\{u, v\}_{S_j} = \int_{S_j} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds.$$

First, we need the relations (see lemmas 3.1 and 4.1 of [11])

$$\lim_{\varepsilon\to 0}\{u^{pr}_{\mathbf{a}_q}, u^{q}_{\mathbf{b}_p}\}_{S_\varepsilon(\mathbf{a}_q)} = 4\pi a_q e^{-ik_q a_q} u^{q}_{\mathbf{b}_p}(\mathbf{a}_q), \tag{11}$$

$$\lim_{R\to\infty}\{u^{0}_{\mathbf{a}_q}, u^{0}_{\mathbf{b}_p}\}_{S_R(\mathbf{0})} = \frac{2i}{k_0}\int_{S^2} g^{0}_{\mathbf{a}_q}(\hat{\mathbf{r}}) g^{0}_{\mathbf{b}_p}(\hat{\mathbf{r}}) ds(\hat{\mathbf{r}}). \tag{12}$$

The overbar denotes complex conjugation.

Now, we establish the *total acoustic field general scattering theorem.*

Theorem 3.1 *The total fields $u^q_{\mathbf{b}_p}$, $u^p_{\mathbf{a}_q}$ and the far-field patterns $g^0_{\mathbf{a}_q}$, $g^0_{\mathbf{b}_p}$, corresponding to the interior excitation of V by point-sources at $\mathbf{a}_q \in V_q$ and $\mathbf{b}_p \in V_p$, satisfy*

$$\frac{k_0\rho_0}{k_q\rho_q}\frac{u^{q}_{\mathbf{b}_p}(\mathbf{a}_q)}{h_0(k_q a_q)} + \frac{k_0\rho_0}{k_p\rho_p}\frac{u^{p}_{\mathbf{a}_q}(\mathbf{b}_p)}{h_0(k_p b_p)} = \frac{1}{2\pi}\int_{S^2} g^{0}_{\mathbf{a}_q}(\hat{\mathbf{r}}) g^{0}_{\mathbf{b}_p}(\hat{\mathbf{r}}) ds(\hat{\mathbf{r}}) + \mathcal{A}(\mathbf{a}_q,\mathbf{b}_p), \tag{13}$$

where

$$\mathcal{A}(\mathbf{a}_q,\mathbf{b}_p) = \frac{ik_0\rho_0}{4\pi\rho_{n-1}}\{u^{n-1}_{\mathbf{a}_q}, u^{n-1}_{\mathbf{b}_p}\}_{S_n}. \tag{14}$$

The function $\mathcal{A}$ depends on the physical properties of the scatterer's core. More precisely, for resistive- and conductive-transmission boundary conditions on S_n hold

$$\mathcal{A}(\mathbf{a}_q,\mathbf{b}_p) = \frac{k_0\rho_0 k_{n-1}\mu_n^2}{\pi\rho_{n-1}\eta\mu_{n-1}^2}\int_{S_n} u^{n}_{\mathbf{a}_q}(\mathbf{r}) u^{n}_{\mathbf{b}_p}(\mathbf{r}) ds$$

$$\mathcal{A}(\mathbf{a}_q,\mathbf{b}_p) = \frac{k_0\rho_0\eta}{\pi\rho_{n-1}k_{n-1}\mu_{n-1}}\int_{S_n} \frac{\partial u^{n}_{\mathbf{a}_q}(\mathbf{r})}{\partial n}\frac{\partial u^{n}_{\mathbf{b}_p}(\mathbf{r})}{\partial n} ds.$$

Proof. The main idea lies on two alternative computations of the integral $\{u^{p}_{\mathbf{a}_q}, u^{p}_{\mathbf{b}_p}\}_{S_p}$. First, the total field superposition (2) in V_q implies

$$\{u^{q}_{\mathbf{a}_q}, u^{q}_{\mathbf{b}_p}\}_{\partial V_q} = \{u^{pr}_{\mathbf{a}_q}, u^{q}_{\mathbf{b}_p}\}_{\partial V_q} + \{u^{sec}_{\mathbf{a}_q}, u^{q}_{\mathbf{b}_p}\}_{\partial V_q}.$$

Since $u^j_{\mathbf{a}_q}$ and $u^j_{\mathbf{b}_p}$ constitute regular solutions of (3) in V_j (j=0,...,$q-1$), by applying successively Green's second theorem, using in each step the boundary conditions (4), taking into account that the respective triple integrals vanish, and utilizing (12), we get

$$\{u^{q}_{\mathbf{a}_q}, u^{q}_{\mathbf{b}_p}\}_{S_q} = \frac{2i}{k_0}\frac{\rho_q}{\rho_0}\int_{S^2} g^{0}_{\mathbf{a}_q}(\hat{\mathbf{r}}) g^{0}_{\mathbf{b}_p}(\hat{\mathbf{r}}) ds(\hat{\mathbf{r}}).$$

In a similar way we obtain

$$\{u^{q}_{\mathbf{a}_q}, u^{q}_{\mathbf{b}_p}\}_{S_{q+1}} = (\rho_q/\rho_p)\{u^{p}_{\mathbf{a}_q}, u^{p}_{\mathbf{b}_p}\}_{S_p}.$$

By Green's theorem $\{u^{sec}_{\mathbf{a}_q}, u^q_{\mathbf{b}_p}\}_{\partial V_q}=0$ and $\{u^{pr}_{\mathbf{a}_q}, u^q_{\mathbf{b}_p}\}_{\partial V_q}$ coincides with $\{u^{pr}_{\mathbf{a}_q}, u^q_{\mathbf{b}_p}\}_{S_{\varepsilon_1}(\mathbf{a}_q)}$. Thus, (11) for $\varepsilon_1 \to 0$, combined with the above three relations, give

$$\{u^p_{\mathbf{a}_q}, u^p_{\mathbf{b}_p}\}_{S_p} = \frac{2i}{k_0}\frac{\rho_p}{\rho_0}\int_{S^2} g^0_{\mathbf{a}_q}(\hat{\mathbf{r}}) g^0_{\mathbf{b}_p}(\hat{\mathbf{r}}) ds(\hat{\mathbf{r}}) - 4\pi \frac{\rho_p}{\rho_q} a_q e^{ik_q a_q} u^q_{\mathbf{b}_p}(\mathbf{a}_q). \tag{15}$$

Second, for the alternative evaluation of $\{u^p_{\mathbf{a}_q}, u^p_{\mathbf{b}_p}\}_{S_p}$, superposition (2) in V_p gives

$$\{u^p_{\mathbf{a}_q}, u^p_{\mathbf{b}_p}\}_{\partial V_p} = \{u^p_{\mathbf{a}_q}, u^{pr}_{\mathbf{b}_p}\}_{\partial V_p} + \{u^p_{\mathbf{a}_q}, u^{sec}_{\mathbf{b}_p}\}_{\partial V_p}.$$

Now, by taking into account that $\{u^p_{\mathbf{a}_q}, u^{sec}_{\mathbf{b}_p}\}_{\partial V_p}=0$ and that $\{u^p_{\mathbf{a}_q}, u^{pr}_{\mathbf{b}_p}\}_{\partial V_p}$ coincides with $\{u^p_{\mathbf{a}_q}, u^{pr}_{\mathbf{b}_p}\}_{S_{\varepsilon_2}(\mathbf{b}_p)}$, (11) for $\varepsilon_2 \to 0$ and the last relation imply

$$\{u^p_{\mathbf{a}_q}, u^p_{\mathbf{b}_p}\}_{S_p} = (\rho_p/\rho_{n-1})\{u^{n-1}_{\mathbf{a}_q}, u^{n-1}_{\mathbf{b}_p}\}_{S_n} - 4\pi b_p e^{-ik_p b_p} u^p_{\mathbf{a}_q}(\mathbf{b}_p).$$

By imposing on the surface S_n the resistive (5) and the conductive transmission boundary condition (6) and applying the Green's second theorem in V_n we obtain for a resistive and conductive core

$$\{u^{n-1}_{\mathbf{a}_q}, u^{n-1}_{\mathbf{b}_p}\}_{S_n} = -\frac{4ik_{n-1}\mu_n^2}{\eta\mu_{n-1}^2}\int_{S_n} u^n_{\mathbf{a}_q}(\mathbf{r}) u^n_{\mathbf{b}_p}(\mathbf{r}) ds.$$

$$\{u^{n-1}_{\mathbf{a}_q}, u^{n-1}_{\mathbf{b}_p}\}_{S_n} = -\frac{4i\eta}{k_{n-1}\mu_{n-1}}\int_{S_n} \frac{\partial u^n_{\mathbf{a}_q}(\mathbf{r})}{\partial n}\frac{\partial u^n_{\mathbf{b}_p}(\mathbf{r})}{\partial n} ds.$$

Now, Eq. (13) follows by (15) combined with the last three relations. □

We also note the interesting *primary acoustic field general radiation relation*, involving the primary fields and the corresponding far-field patterns, proved in [11].

$$\frac{u^{pr}_{\mathbf{b}_q}(\mathbf{a}_q)}{h_0(k_q a_q)} + \frac{u^{pr}_{\mathbf{a}_q}(\mathbf{b}_q)}{h_0(k_q b_q)} = \frac{1}{2\pi}\int_{S^2} g^{pr}_{\mathbf{a}_q}(\hat{\mathbf{r}}) g^{pr}_{\mathbf{b}_q}(\hat{\mathbf{r}}) ds(\hat{\mathbf{r}}). \tag{16}$$

We prefer the term *radiation* instead of *scattering* in order to emphasize that (16) relates the primary fields radiated by two point-sources under the absence of the scatterer V.

Now, by combining Eqs. (2), (13) and (16), we obtain the following *secondary acoustic field general scattering relation.*

Corollary 3.1 *The secondary fields $u^{sec}_{\mathbf{a}_q}$ and $u^{sec}_{\mathbf{b}_q}$ in V_q and the total far-field patterns $g^0_{\mathbf{a}_q}$ and $g^0_{\mathbf{b}_q}$ satisfy*

$$\frac{k_0\rho_0}{k_q\rho_q}\frac{u^{sec}_{\mathbf{b}_q}(\mathbf{a}_q)}{h_0(k_q a_q)} + \frac{k_0\rho_0}{k_q\rho_q}\frac{u^{sec}_{\mathbf{a}_q}(\mathbf{b}_q)}{h_0(k_q b_q)} + \frac{k_0\rho_0}{k_q\rho_q}\frac{2\mathrm{sinc}(k_q|\mathbf{a}_q - \mathbf{b}_q|)}{h_0(k_q a_q) h_0(k_q b_q)} =$$
$$\frac{1}{2\pi}\int_{S^2} g^0_{\mathbf{a}_q}(\hat{\mathbf{r}}) g^0_{\mathbf{b}_q}(\hat{\mathbf{r}}) ds(\hat{\mathbf{r}}) + \mathcal{A}(\mathbf{a}_q, \mathbf{b}_q), \tag{17}$$

The influence of the boundary conditions on the surface S_n of the scatterer's core on the statements of theorem 3.1 and corollary 3.1 is actually displayed by the function $\mathcal{A}$.

In addition, we state the optical theorem and the mixed general scattering theorem.

The optical theorem is derived as a corollary of (17). First, we define the *q-excitation*, the *absorption*, and the *extinction cross-section*

$$\begin{aligned} \sigma^0_{\mathbf{a}_q} &= \frac{1}{k_0^2}\int_{S^2} |g^0_{\mathbf{a}_q}(\hat{\mathbf{r}})|^2 ds(\hat{\mathbf{r}}), \\ \sigma^{\mathrm{a}}_{\mathbf{a}_q} &= \frac{\rho_0}{\rho_{n-1}k_0}\mathrm{Im}\left[\int_{S_n} u^{n-1}_{\mathbf{a}_q}(\mathbf{r})\frac{\partial u^{n-1}_{\mathbf{a}_q}(\mathbf{r})}{\partial n}ds(\mathbf{r})\right], \\ \sigma^{\mathrm{e}}_{\mathbf{a}_q} &= \sigma^{\mathrm{a}}_{\mathbf{a}_q} + \sigma^0_{\mathbf{a}_q}. \end{aligned} \tag{18}$$

The absorption cross-section determines the amount of primary field's power, absorbed by the scatterer's core V_n. Clearly, $\sigma^{\mathrm{a}}_{\mathbf{a}_q} \geq 0$ for a resistive or conductive core.

The following *optical theorem* is derived by (17) for $\mathbf{a}_q = \mathbf{b}_q$ and taking into account the definition (14) of the function $\mathcal{A}$

Theorem 3.2 *The extinction cross-section $\sigma^{\mathrm{e}}_{\mathbf{a}_q}$ and the secondary field $u^{sec}_{\mathbf{a}_q}$, corresponding to the interior excitation of V by a point-source at $\mathbf{a}_q \in V_q$, are related by*

$$\sigma^{\mathrm{e}}_{\mathbf{a}_q} = (k_q\rho_0)/(k_0\rho_q)4\pi a_q^2\big(\mathrm{Re}[h_0(k_q a_q)u^{sec}_{\mathbf{a}_q}(\mathbf{a}_q)] + 1\big). \tag{19}$$

Apart from its theoretical value, the optical theorem also serves the efficient computation of the cross-section. More precisely, the cross-section can be determined by (19) with the use of a distinct value of the secondary field, without having to measure the samples of $g^0_{\mathbf{a}_q}$ in S^2 and perform numerical integration, as dictated by the definition (18).

Now, by considering one of the two point-sources receding to infinity ($a_0 \to \infty$) and the other still located in $\mathbf{b}_p \in V_p$, we assume that the scatterer is simultaneously excited by a plane and a spherical wave and obtain mixed scattering relations. Such relations play an important role in the point-source inverse scattering method of [19].

An incident plane acoustic wave propagating in the direction $\hat{\mathbf{d}}$ is given by

$$u^{inc}(\mathbf{r};\hat{\mathbf{d}}) = \exp(ik_0\hat{\mathbf{d}}\cdot\mathbf{r}).$$

By $u^p(\mathbf{r};\hat{\mathbf{d}})$, $u^{sc}(\mathbf{r};\hat{\mathbf{d}})$ and $g(\hat{\mathbf{r}};\hat{\mathbf{d}})$ we denote the total field in V_p, the scattered field in V_0 and the far-field pattern, all due to the above incident plane wave. Moreover, we have

$$u^p_{\mathbf{a}_0}(\mathbf{r}) \to u^p(\mathbf{r};-\hat{\mathbf{a}}_0), \quad g^{sec}_{\mathbf{a}_0}(\hat{\mathbf{r}}) \to g(\hat{\mathbf{r}};-\hat{\mathbf{a}}_0), \quad a_0 \to \infty$$

Now, we need the following relation (see [11], lemma 6.3)

$$\lim_{a_0\to\infty}\int_{S^2} g^{pr}_{\mathbf{a}_0}(\hat{\mathbf{r}})g^0_{\mathbf{b}_p}(\hat{\mathbf{r}})ds(\hat{\mathbf{r}}) = 2\pi[g^0_{\mathbf{b}_p}(-\hat{\mathbf{a}}_0) - g^0_{\mathbf{b}_p}(\hat{\mathbf{a}}_0)e^{2ik_0a_0}]$$

The following *mixed total acoustic field general scattering theorem* is derived by the general scattering theorem 3.1 for $a_0 \to \infty$ and taking into account the last relation.

Theorem 3.3 *A total far-field pattern $g^0_{\mathbf{b}_p}$, corresponding to interior excitation of V by a point-source at $\mathbf{b}_p \in V_p$, a total field u^p in V_p and a far-field pattern g, both due to an incident on V plane wave u^{inc}, satisfy*

$$\frac{k_0\rho_0}{k_p\rho_p}\frac{u^p(\mathbf{b}_p;-\hat{\mathbf{a}}_0)}{h_0(k_pb_p)} = g^0_{\mathbf{b}_p}(-\hat{\mathbf{a}}_0) + \frac{1}{2\pi}\int_{S^2} g(\hat{\boldsymbol{r}};-\hat{\mathbf{a}}_0)g^0_{\mathbf{b}_p}(\hat{\mathbf{r}})ds(\hat{\mathbf{r}}) + \lim_{a_0\to\infty}\mathcal{A}(\mathbf{a}_0,\mathbf{b}_p).$$

Finally, we note that the formulations of the reciprocity and mixed reciprocity theorems are independent of the boundary conditions on the scatterer's core. Thus, for this type of relations we refer to theorems 3.2 and 6.1 of [11].

4. Mathematical Formulation: Electromagnetic Waves

The geometrical characteristics of the layered scatterer V considered hereafter are those described in Section 2. Concerning the physical characteristics we suppose that the layers V_j (j=1,...,$n-1$), are homogeneous with real dielectric permittivity ϵ_j and magnetic permeability μ_j. The scatterer's core V_n is resistive or conductive. The exterior V_0 of V is homogeneous with real constants ϵ_0 and μ_0.

The scatterer V is excited by a time-harmonic spherical electromagnetic wave, generated by a dipole located at $\mathbf{a}_\ell$ of layer V_ℓ (ℓ=0,...,n). The radiated by the dipole electric field $\mathbf{E}^{pr}_{\mathbf{a}_\ell}$ constitutes the *primary field* of the Sommerfeld's method and is expressed by

$$\mathbf{E}^{pr}_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell) = \frac{a_\ell\exp(-ik_\ell a_\ell)}{ik_\ell}\nabla\times\left(\frac{\exp(ik_\ell|\mathbf{r}-\mathbf{a}_\ell|)}{|\mathbf{r}-\mathbf{a}_\ell|}\hat{\mathbf{p}}_\ell\right), \qquad \mathbf{r}\in\mathbb{R}^3\backslash\{\mathbf{a}_\ell\} \tag{20}$$

where $\hat{\mathbf{p}}_\ell$ is a constant unit vector with $\hat{\mathbf{p}}_\ell\cdot\hat{\mathbf{a}}_\ell = 0$. Relation (20) represents the electric field radiated by a magnetic dipole with moment $\hat{\mathbf{p}}_\ell$ [18], [21].

The spherical wave (20) reduces to a plane wave with direction of propagation $-\hat{\mathbf{a}}_\ell$ and polarization $\hat{\mathbf{p}}_\ell\times\hat{\mathbf{a}}_\ell$, when the dipole recedes to infinity [23]

$$\lim_{a_\ell\to\infty}\mathbf{E}^{pr}_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell) = (\hat{\mathbf{p}}_\ell\times\hat{\mathbf{a}}_\ell)\exp(-ik_\ell\hat{\mathbf{a}}_\ell\cdot\mathbf{r}) \equiv \mathbf{E}^{inc}(\mathbf{r};-\hat{\mathbf{a}}_\ell,\hat{\mathbf{p}}_\ell\times\hat{\mathbf{a}}_\ell).$$

The secondary electric fields in V_j ($j{\neq}\ell$) and V_ℓ are denoted by $\mathbf{E}^j_{\mathbf{a}_\ell}$ and $\mathbf{E}^{sec}_{\mathbf{a}_\ell}$. Applying Sommerfeld's method, the total electric field $\mathbf{E}^\ell_{\mathbf{a}_\ell}$ in V_ℓ is defined as the superposition of the primary and the secondary field

$$\mathbf{E}^\ell_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell) = \mathbf{E}^{pr}_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell) + \mathbf{E}^{sec}_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell), \quad \mathbf{r}\in V_\ell\setminus\{\mathbf{a}_\ell\}. \tag{21}$$

The total field in V_j $(j \neq \ell)$ coincides with the secondary field $\mathbf{E}^j_{\mathbf{a}_\ell}$.

The total electric fields $\mathbf{E}^j_{\mathbf{a}_\ell}$ satisfy the vector equation

$$\nabla \times \nabla \times \mathbf{E}^j_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell) - k_j^2 \mathbf{E}^j_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell) = \mathbf{0}, \tag{22}$$

for $\mathbf{r} \in V_j$ if $j \neq \ell$ and $\mathbf{r} \in V_\ell \backslash \{\mathbf{a}_\ell\}$ if $j = \ell$, where k_j is the wavenumber in V_j.

On the surfaces S_j $(j \neq n)$ the total fields satisfy the transmission boundary conditions

$$\begin{aligned} &\hat{\mathbf{n}} \times \mathbf{E}^{j-1}_{\mathbf{a}_\ell}(\mathbf{r}) - \hat{\mathbf{n}} \times \mathbf{E}^j_{\mathbf{a}_\ell}(\mathbf{r}) = \mathbf{0}, \quad \mathbf{r} \in S_j \\ &(\mu_j/\mu_{j-1})\hat{\mathbf{n}} \times \nabla \times \mathbf{E}^{j-1}_{\mathbf{a}_\ell}(\mathbf{r}) - \hat{\mathbf{n}} \times \nabla \times \mathbf{E}^j_{\mathbf{a}_\ell}(\mathbf{r}) = \mathbf{0}, \quad \mathbf{r} \in S_j \end{aligned} \tag{23}$$

The total field on the surface S_n of the core satisfies the conductive boundary condition

$$\begin{aligned} &\hat{\mathbf{n}} \times \mathbf{E}^{n-1}_{\mathbf{a}_\ell}(\mathbf{r}) - \hat{\mathbf{n}} \times \mathbf{E}^n_{\mathbf{a}_\ell}(\mathbf{r}) = \mathbf{0}, \quad \mathbf{r} \in S_n \\ &\hat{\mathbf{n}} \times \left[\hat{\mathbf{n}} \times \nabla \times \mathbf{E}^{n-1}_{\mathbf{a}_\ell}(\mathbf{r})\right] - \hat{\mathbf{n}} \times \left[\hat{\mathbf{n}} \times \nabla \times \mathbf{E}^n_{\mathbf{a}_\ell}(\mathbf{r})\right] = i\omega\mu_0\tau(\mathbf{r})\hat{\mathbf{n}} \times \mathbf{E}^{n-1}_{\mathbf{a}_\ell}(\mathbf{r}), \quad \mathbf{r} \in S_n \end{aligned} \tag{24}$$

where the function $\tau(\mathbf{r})$ is the integrated conductivity [14], [15]. Note that when $\tau \to \infty$, the boundary conditions (24) reduce to that of a perfect electric conducting core. Detailed analysis of the conductive boundary condition is contained in [14].

The secondary field $\mathbf{E}^0_{\mathbf{a}_\ell}$ in V_0 satisfies the Silver-Müller radiation condition [21]

$$\lim_{r\to\infty} \left[\hat{\mathbf{r}} \times \nabla \times \mathbf{E}^0_{\mathbf{a}_\ell}(\mathbf{r}) + ik_0 r \mathbf{E}^0_{\mathbf{a}_\ell}(\mathbf{r})\right] = \mathbf{0},$$

uniformly for all directions $\hat{\mathbf{r}}$.

Combination of the techniques of [14] and [24] shows that the above problem is well posed.

Besides, the fields $\mathbf{E}^0_{\mathbf{a}_\ell}$ and $\mathbf{E}^{pr}_{\mathbf{a}_\ell}$ have the asymptotic expressions in the radiation zone

$$\begin{aligned} \mathbf{E}^0_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell) &= \mathbf{g}^0_{\mathbf{a}_\ell}(\hat{\mathbf{r}};\hat{\mathbf{p}}_\ell) h_0(k_0 r) + \mathcal{O}(r^{-2}), \quad r \to \infty. \\ \mathbf{E}^{pr}_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell) &= \mathbf{g}^{pr}_{\mathbf{a}_\ell}(\hat{\mathbf{r}};\hat{\mathbf{p}}_\ell) h_0(k_\ell r) + \mathcal{O}(r^{-2}), \quad r \to \infty. \end{aligned} \tag{25}$$

The functions $\mathbf{g}^0_{\mathbf{a}_\ell}$ and $\mathbf{g}^{pr}_{\mathbf{a}_\ell}$ will be referred to as the *total* and *primary electric far-field pattern*. By [12] we have

$$\mathbf{g}^{pr}_{\mathbf{a}_\ell}(\hat{\mathbf{r}};\hat{\mathbf{p}}_\ell) = \frac{\exp(-ik_\ell \mathbf{a}_\ell \cdot \hat{\mathbf{r}})}{h_0(k_\ell a_\ell)} (\hat{\mathbf{r}} \times \hat{\mathbf{p}}_\ell),$$

The secondary field $\mathbf{E}^{sec}_{\mathbf{a}_0}$, due to a dipole at $\mathbf{a}_0 \in V_0$, has the asymptotic expression

$$\mathbf{E}^{sec}_{\mathbf{a}_0}(\mathbf{r};\hat{\mathbf{p}}_\ell) = \mathbf{g}^{sec}_{\mathbf{a}_0}(\hat{\mathbf{r}};\hat{\mathbf{p}}_\ell) h_0(k_0 r) + \mathcal{O}(r^{-2}), \quad r \to \infty.$$

Function $\mathbf{g}^{sec}_{\mathbf{a}_0}$ is named the *secondary electric far-field pattern*. The above relations imply

$$\mathbf{g}^0_{\mathbf{a}_0}(\hat{\mathbf{r}};\hat{\mathbf{p}}_\ell) = \mathbf{g}^{pr}_{\mathbf{a}_0}(\hat{\mathbf{r}};\hat{\mathbf{p}}_\ell) + \mathbf{g}^{sec}_{\mathbf{a}_0}(\hat{\mathbf{r}};\hat{\mathbf{p}}_\ell).$$

5. Electromagnetic Scattering Relations

The layered scatterer V is excited by two magnetic dipoles located at $\mathbf{a}_\ell \in V_\ell$ and $\mathbf{b}_m \in V_m$ (ℓ,m=0,...,n) with moments $\hat{\mathbf{p}}_\ell$ and $\hat{\mathbf{q}}_m$ ($m \geq \ell$), generating in V_ℓ, V_m primary $\mathbf{E}^{pr}_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell)$, $\mathbf{E}^{pr}_{\mathbf{b}_m}(\mathbf{r};\hat{\mathbf{q}}_m)$, secondary $\mathbf{E}^{sec}_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell)$, $\mathbf{E}^{sec}_{\mathbf{b}_m}(\mathbf{r};\hat{\mathbf{q}}_m)$, and total fields $\mathbf{E}^{\ell}_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell)$, $\mathbf{E}^{m}_{\mathbf{b}_m}(\mathbf{r};\hat{\mathbf{q}}_m)$. We will use the notation [2]

$$\left\{\mathbf{E}_1,\mathbf{E}_2\right\}_{S_j} = \int_{S_j} \Big[(\hat{\mathbf{n}}\times\mathbf{E}_1)\cdot(\nabla\times\mathbf{E}_2) - (\hat{\mathbf{n}}\times\mathbf{E}_2)\cdot(\nabla\times\mathbf{E}_1)\Big] ds.$$

First, we state the following useful

Lemma 5.1 *The primary* $\mathbf{E}^{pr}_{\mathbf{a}_\ell}$ *and total field* $\mathbf{E}^{\ell}_{\mathbf{b}_m}$ *in* V_ℓ*, the total fields* $\mathbf{E}^{0}_{\mathbf{a}_\ell}$ *and* $\mathbf{E}^{0}_{\mathbf{b}_m}$ *in* V_0*, and the corresponding far-field patterns* $\mathbf{g}^0_{\mathbf{a}_\ell}$, $\mathbf{g}^0_{\mathbf{b}_m}$ *satisfy*

$$\lim_{\varepsilon\to 0}\left\{\mathbf{E}^{pr}_{\mathbf{a}_\ell}(\cdot;\hat{\mathbf{p}}_\ell),\mathbf{E}^{\ell}_{\mathbf{b}_m}(\cdot;\hat{\mathbf{q}}_m)\right\}_{S_\varepsilon(\mathbf{a}_\ell)} = -4\pi i(a_\ell/k_\ell)\exp(-ik_\ell a_\ell)\big(\nabla\times\mathbf{E}^{\ell}_{\mathbf{b}_m}(\mathbf{a}_\ell;\hat{\mathbf{q}}_m)\big)\cdot\hat{\mathbf{p}}_\ell. \tag{26}$$

$$\lim_{R\to\infty}\left\{\mathbf{E}^{0}_{\mathbf{a}_\ell}(\cdot;\hat{\mathbf{p}}_\ell),\mathbf{E}^{0}_{\mathbf{b}_m}(\cdot;\hat{\mathbf{q}}_m)\right\}_{S_R(\mathbf{0})} = \frac{2i}{k_0}\int_{S^2}\mathbf{g}^0_{\mathbf{a}_\ell}(\hat{\mathbf{r}};\hat{\mathbf{p}}_\ell)\cdot\mathbf{g}^0_{\mathbf{b}_m}(\hat{\mathbf{r}};\hat{\mathbf{q}}_m)ds(\hat{\mathbf{r}}). \tag{27}$$

Proof. After lengthy calculations and using (20), we find

$$\begin{aligned}
&\left\{\mathbf{E}^{pr}_{\mathbf{a}_\ell}(\cdot;\hat{\mathbf{p}}_\ell),\mathbf{E}^{\ell}_{\mathbf{b}_m}(\cdot;\hat{\mathbf{q}}_m)\right\}_{S_\varepsilon(\mathbf{a}_\ell)} = \\
&\quad a_\ell\exp(-ik_\ell a_\ell)\Bigg\{\int_{S_\varepsilon(\mathbf{a}_\ell)}\hat{\mathbf{r}}\cdot\nabla\times\Big[\big(\hat{\mathbf{p}}_\ell\cdot\nabla h_0(k_\ell|\mathbf{r}-\mathbf{a}_\ell|)\big)\mathbf{E}^{\ell}_{\mathbf{b}_m}(\mathbf{r};\hat{\mathbf{q}}_m)\Big]ds \\
&\quad -\int_{S_\varepsilon(\mathbf{a}_\ell)}\hat{\mathbf{r}}\cdot\nabla h_0(k_\ell|\mathbf{r}-\mathbf{a}_\ell|)\Big[\hat{\mathbf{p}}_\ell\cdot\big(\nabla\times\mathbf{E}^{\ell}_{\mathbf{b}_m}(\mathbf{r};\hat{\mathbf{q}}_m)\big)\Big]ds \\
&\quad -k_\ell^2\int_{S_\varepsilon(\mathbf{a}_\ell)}\hat{\mathbf{r}}\cdot\Big[\mathbf{E}^{\ell}_{\mathbf{b}_m}(\mathbf{r};\hat{\mathbf{q}}_m)\times\hat{\mathbf{p}}_\ell\Big]h_0(k_\ell|\mathbf{r}-\mathbf{a}_\ell|)ds\Bigg\}
\end{aligned}$$

The first integral on the right-hand side vanishes by Stokes theorem. Hence, by applying the mean value theorem for surface integrals and letting $\varepsilon\to 0$, we obtain (26).

Eq. (27) is verified by letting $R\to\infty$ and using for the total fields $\mathbf{E}^{0}_{\mathbf{a}_\ell}$ and $\mathbf{E}^{0}_{\mathbf{b}_m}$ the asymptotic form (25). □

Now, we formulate and prove the *total electric field general scattering theorem.*

Theorem 5.1 *The total fields* $\mathbf{E}^{\ell}_{\mathbf{b}_m}$, $\mathbf{E}^{m}_{\mathbf{a}_\ell}$ *and far-field patterns* $\mathbf{g}^0_{\mathbf{b}_m}$, $\mathbf{g}^0_{\mathbf{a}_\ell}$*, corresponding to the interior excitation of* V *by dipoles at* $\mathbf{a}_\ell\in V_\ell$ *and* $\mathbf{b}_m\in V_m$*, are related by*

$$\begin{aligned}
&\frac{k_0\mu_0}{ik_\ell^2\mu_\ell}\,\frac{\nabla\times\mathbf{E}^{\ell}_{\mathbf{b}_m}(\mathbf{a}_\ell;\hat{\mathbf{q}}_m)}{h_0(k_\ell a_\ell)}\cdot\hat{\mathbf{p}}_\ell - \frac{k_0\mu_0}{ik_m^2\mu_m}\,\frac{\nabla\times\mathbf{E}^{m}_{\mathbf{a}_\ell}(\mathbf{b}_m;\hat{\mathbf{p}}_\ell)}{h_0(k_m b_m)}\cdot\hat{\mathbf{q}}_m \\
&+\frac{1}{2\pi}\int_{S^2}\mathbf{g}^0_{\mathbf{a}_\ell}(\hat{\mathbf{r}};\hat{\mathbf{p}}_\ell)\cdot\mathbf{g}^0_{\mathbf{b}_m}(\hat{\mathbf{r}};\hat{\mathbf{q}}_m)ds(\hat{\mathbf{r}}) = \mathcal{E}(\mathbf{a}_\ell,\mathbf{b}_m;\hat{\mathbf{p}}_\ell,\hat{\mathbf{q}}_m),
\end{aligned} \tag{28}$$

where

$$\mathcal{E}(\mathbf{a}_\ell,\mathbf{b}_m;\hat{\mathbf{p}}_\ell,\hat{\mathbf{q}}_m) = -\frac{\mu_0}{\mu_{n-1}}\frac{ik_0}{4\pi}\left\{\mathbf{E}^{n-1}_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell),\mathbf{E}^{n-1}_{\mathbf{b}_m}(\mathbf{r};\hat{\mathbf{q}}_m)\right\}_{S_n}. \quad (29)$$

For a conductive core holds

$$\mathcal{E}(\mathbf{a}_\ell,\mathbf{b}_m;\hat{\mathbf{p}}_\ell,\hat{\mathbf{q}}_m) = -\frac{\mu_0}{\mu_{n-1}}\frac{k_0}{2\pi}\Big[\mathrm{Im}(k_n^2)\int_{V_n}\mathbf{E}^{n}_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell)\cdot\mathbf{E}^{n}_{\mathbf{b}_m}(\mathbf{r};\hat{\mathbf{q}}_m)dv(\mathbf{r})$$
$$+\omega\mu_0\int_{S_n}\mathrm{Re}(\tau(\mathbf{r}))\big(\hat{\mathbf{n}}\times\mathbf{E}^{n}_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell)\big)\cdot(\hat{\mathbf{n}}\times\mathbf{E}^{n}_{\mathbf{b}_m}(\mathbf{r};\hat{\mathbf{q}}_m))ds(\mathbf{r})\Big] \quad (30)$$

Proof. The main idea lies on two alternative computations of $\left\{\mathbf{E}^m_{\mathbf{a}_\ell},\mathbf{E}^m_{\mathbf{b}_m}\right\}_{S_m}$. First, the total field superposition (21) in layer V_ℓ implies

$$\left\{\mathbf{E}^\ell_{\mathbf{a}_\ell},\mathbf{E}^\ell_{\mathbf{b}_m}\right\}_{\partial V_\ell} = \left\{\mathbf{E}^{pr}_{\mathbf{a}_\ell},\mathbf{E}^\ell_{\mathbf{b}_m}\right\}_{\partial V_\ell} + \left\{\mathbf{E}^{sec}_{\mathbf{a}_\ell},\mathbf{E}^\ell_{\mathbf{b}_m}\right\}_{\partial V_\ell}.$$

Applying a similar procedure to theorem 3.1 and utilizing (27), we get

$$\left\{\mathbf{E}^\ell_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell),\mathbf{E}^\ell_{\mathbf{b}_m}(\mathbf{r};\hat{\mathbf{q}}_m)\right\}_{S_\ell} = \frac{2i}{k_0}\frac{\mu_\ell}{\mu_0}\int_{S^2}\mathbf{g}^0_{\mathbf{a}_\ell}(\hat{\mathbf{r}};\hat{\mathbf{p}}_\ell)\cdot\mathbf{g}^0_{\mathbf{b}_m}(\hat{\mathbf{r}};\hat{\mathbf{q}}_m)ds(\hat{\mathbf{r}}).$$

$$\left\{\mathbf{E}^\ell_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell),\mathbf{E}^\ell_{\mathbf{b}_m}(\mathbf{r};\hat{\mathbf{q}}_m)\right\}_{S_{\ell+1}} = (\mu_\ell/\mu_m)\left\{\mathbf{E}^m_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell),\mathbf{E}^m_{\mathbf{b}_m}(\mathbf{r};\hat{\mathbf{q}}_m)\right\}_{S_m}$$

By Green's theorem $\left\{\mathbf{E}^{sec}_{\mathbf{a}_\ell},\mathbf{E}^\ell_{\mathbf{b}_m}\right\}_{\partial V_\ell}=0$ and $\left\{\mathbf{E}^{pr}_{\mathbf{a}_\ell},\mathbf{E}^\ell_{\mathbf{b}_m}\right\}_{\partial V_\ell}$ coincides with $\left\{\mathbf{E}^{pr}_{\mathbf{a}_\ell},\mathbf{E}^\ell_{\mathbf{b}_m}\right\}_{S_{\varepsilon_1}(\mathbf{a}_\ell)}$. Now, (26) for $\varepsilon_1\to 0$ and the last three relations give

$$\left\{\mathbf{E}^m_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell),\mathbf{E}^m_{\mathbf{b}_m}(\mathbf{r};\hat{\mathbf{q}}_m)\right\}_{S_m} =$$
$$\frac{2i}{k_0}\frac{\mu_m}{\mu_0}\int_{S^2}\mathbf{g}^0_{\mathbf{a}_\ell}(\hat{\mathbf{r}};\hat{\mathbf{p}}_\ell)\cdot\mathbf{g}^0_{\mathbf{b}_m}(\hat{\mathbf{r}};\hat{\mathbf{q}}_m)ds(\hat{\mathbf{r}}) - 4\pi i\frac{a_\ell\mu_m}{k_\ell\mu_\ell}\exp(ik_\ell a_\ell)\big(\nabla\times\mathbf{E}^\ell_{\mathbf{b}_m}(\mathbf{a}_\ell;\hat{\mathbf{q}}_m)\big)\cdot\hat{\mathbf{p}}_\ell.$$

Second, the total field superposition (21) in V_m gives

$$\left\{\mathbf{E}^m_{\mathbf{a}_\ell},\mathbf{E}^m_{\mathbf{b}_m}\right\}_{\partial V_m} = \left\{\mathbf{E}^m_{\mathbf{a}_\ell},\mathbf{E}^{pr}_{\mathbf{b}_m}\right\}_{\partial V_m} + \left\{\mathbf{E}^m_{\mathbf{a}_\ell},\mathbf{E}^{sec}_{\mathbf{b}_m}\right\}_{\partial V_m}.$$

By the above arguments, the last relation implies

$$\left\{\mathbf{E}^m_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell),\mathbf{E}^m_{\mathbf{b}_m}(\mathbf{r};\hat{\mathbf{q}}_m)\right\}_{S_m} =$$
$$\frac{\mu_m}{\mu_{n-1}}\left\{\mathbf{E}^{n-1}_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell),\mathbf{E}^{n-1}_{\mathbf{b}_m}(\mathbf{r};\hat{\mathbf{q}}_m)\right\}_{S_n} + 4\pi i\frac{b_m}{k_m}\exp(-ik_m b_m)\big(\nabla\times\mathbf{E}^m_{\mathbf{a}_\ell}(\mathbf{b}_m;\hat{\mathbf{p}}_\ell)\big)\cdot\hat{\mathbf{q}}_m,$$

and (28) follows. Moreover, by imposing on the conductive core's surface S_n the boundary conditions (24) and applying the Gauss divergence theorem in V_n, we get the expression (30) of function $\mathcal{E}$. □

Now, we note the *primary electric field general radiation relation*, proved in [12].

$$\frac{\nabla\times\mathbf{E}^{pr}_{\mathbf{a}_\ell}(\mathbf{b}_\ell;\hat{\mathbf{p}}_\ell)}{ik_\ell h_0(k_\ell b_\ell)}\cdot\hat{\mathbf{q}}_\ell - \frac{\nabla\times\mathbf{E}^{pr}_{\mathbf{b}_\ell}(\mathbf{a}_\ell;\hat{\mathbf{q}}_\ell)}{ik_\ell h_0(k_\ell a_\ell)}\cdot\hat{\mathbf{p}}_\ell = \frac{1}{2\pi}\int_{S^2}\mathbf{g}^{pr}_{\mathbf{a}_\ell}(\hat{\mathbf{r}};\hat{\mathbf{p}}_\ell)\cdot\mathbf{g}^{pr}_{\mathbf{b}_\ell}(\hat{\mathbf{r}};\hat{\mathbf{q}}_\ell)ds(\hat{\mathbf{r}}). \quad (31)$$

By (21), (28) and (31) we get the *secondary electric field general scattering relation.*

Corollary 5.1 *The secondary fields* $\mathbf{E}^{sec}_{\mathbf{a}_\ell}$, $\mathbf{E}^{sec}_{\mathbf{b}_\ell}$ *and far-field patterns* $\mathbf{g}^0_{\mathbf{a}_\ell}$, $\mathbf{g}^0_{\mathbf{b}_\ell}$ *satisfy*

$$\frac{\nabla\times\mathbf{E}^{sec}_{\mathbf{b}_\ell}(\mathbf{a}_\ell;\hat{\mathbf{q}}_\ell)}{ik_\ell h_0(k_\ell a_\ell)}\cdot\hat{\mathbf{p}}_\ell-\frac{\nabla\times\mathbf{E}^{sec}_{\mathbf{a}_\ell}(\mathbf{b}_\ell;\hat{\mathbf{p}}_\ell)}{ik_\ell h_0(k_\ell b_\ell)}\cdot\hat{\mathbf{q}}_\ell-\frac{1}{2\pi}\int_{S^2}\mathbf{g}^{pr}_{\mathbf{a}_\ell}(\hat{\mathbf{r}};\hat{\mathbf{p}}_\ell)\cdot\mathbf{g}^{pr}_{\mathbf{b}_\ell}(\hat{\mathbf{r}};\hat{\mathbf{q}}_\ell)ds(\hat{\mathbf{r}})$$
$$+\frac{1}{2\pi}\frac{k_\ell\mu_\ell}{k_0\mu_0}\int_{S^2}\mathbf{g}^0_{\mathbf{a}_\ell}(\hat{\mathbf{r}};\hat{\mathbf{p}}_\ell)\cdot\mathbf{g}^0_{\mathbf{b}_\ell}(\hat{\mathbf{r}};\hat{\mathbf{q}}_\ell)ds(\hat{\mathbf{r}})=\frac{k_\ell\mu_\ell}{k_0\mu_0}\mathcal{E}(\mathbf{a}_\ell,\mathbf{b}_\ell;\hat{\mathbf{p}}_\ell,\hat{\mathbf{q}}_\ell),\tag{32}$$

Next, we state the optical theorem for interior dipole excitation of a layered scatterer. First, we define the *ℓ-excitation*, the *absorption*, and the *extinction cross-section*

$$\sigma^0_{\mathbf{a}_\ell}=\frac{1}{k_0^2}\int_{S^2}|\mathbf{g}^0_{\mathbf{a}_\ell}(\hat{\mathbf{r}};\hat{\mathbf{p}}_\ell)|^2ds(\hat{\mathbf{r}}),$$
$$\sigma^{\mathrm{a}}_{\mathbf{a}_\ell}=\frac{\mu_0}{\mu_{n-1}k_0}\mathrm{Im}\left[\int_{S_n}\hat{\mathbf{n}}\cdot\big(\mathbf{E}^{n-1}_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell)\times\nabla\times\mathbf{E}^{n-1}_{\mathbf{a}_\ell}(\mathbf{r};\hat{\mathbf{p}}_\ell)\big)ds(\mathbf{r})\right],$$
$$\sigma^{\mathrm{e}}_{\mathbf{a}_\ell}=\sigma^{\mathrm{a}}_{\mathbf{a}_\ell}+\sigma^0_{\mathbf{a}_\ell}.$$

The following *optical theorem* is derived by (32) for $\mathbf{a}_\ell=\mathbf{b}_\ell$ and $\hat{\mathbf{p}}_\ell=\hat{\mathbf{q}}_\ell$ and using the definition (29) of function $\mathcal{E}$.

Theorem 5.2 *The extinction cross-section* $\sigma^{\mathrm{e}}_{\mathbf{a}_\ell}$ *and the secondary field* $\mathbf{E}^{sec}_{\mathbf{a}_\ell}$, *corresponding to the interior excitation of* V *by a dipole at* $\mathbf{a}_\ell\in V_\ell$, *are related by*

$$\sigma^{\mathrm{e}}_{\mathbf{a}_\ell}=\frac{k_\ell\mu_0}{k_0\mu_\ell}4\pi a_\ell^2\left[\mathrm{Re}\Big(ik_\ell^{-1}h_0(k_\ell a_\ell)\big(\nabla\times\mathbf{E}^{sec}_{\mathbf{a}_\ell}(\mathbf{a}_\ell;\hat{\mathbf{p}}_\ell)\big)\cdot\hat{\mathbf{p}}_\ell\Big)+\frac{2}{3}\right].$$

Finally, we consider one dipole receding to infinity ($a_0\to\infty$) and the other still located in $\mathbf{b}_m\in V_m$; hence the scatterer is simultaneously excited by a plane and a spherical wave.

An incident field with polarization $\hat{\mathbf{p}}$ and direction of propagation $\hat{\mathbf{d}}$ is given by

$$\mathbf{E}^{inc}(\mathbf{r};\hat{\mathbf{d}},\hat{\mathbf{p}})=\hat{\mathbf{p}}\exp(ik_0\hat{\mathbf{d}}\cdot\mathbf{r}).$$

By $\mathbf{E}^m(\mathbf{r};\hat{\mathbf{d}},\hat{\mathbf{p}})$, $\mathbf{E}^{sc}(\mathbf{r};\hat{\mathbf{d}},\hat{\mathbf{p}})$ and $\mathbf{g}(\hat{\mathbf{r}};\hat{\mathbf{d}},\hat{\mathbf{p}})$ we denote the total field in V_m, the scattered field in V_0 and the far-field pattern, all due to the above incident plane wave. We have

$$\mathbf{E}^m_{\mathbf{a}_0}(\mathbf{r};\hat{\mathbf{p}}_0)\to\mathbf{E}^m(\mathbf{r};-\hat{\mathbf{a}}_0,\hat{\mathbf{p}}_0\times\hat{\mathbf{a}}_0),\quad\mathbf{g}^{sec}_{\mathbf{a}_0}(\hat{\mathbf{r}};\hat{\mathbf{p}}_0)\to\mathbf{g}(\hat{\mathbf{r}};-\hat{\mathbf{a}}_0,\hat{\mathbf{p}}_0\times\hat{\mathbf{a}}_0),\quad a_0\to\infty$$

Now, we need the relation (see [12], (6.4)).

$$\lim_{a_0\to\infty}\int_{S^2}\mathbf{g}^{pr}_{\mathbf{a}_0}(\hat{\mathbf{r}};\hat{\mathbf{p}}_0)\cdot\mathbf{g}^0_{\mathbf{b}_m}(\hat{\mathbf{r}};\hat{\mathbf{q}}_m)ds(\hat{\mathbf{r}})=2\pi(\hat{\mathbf{p}}_0\times\hat{\mathbf{a}}_0)\cdot\left[\mathbf{g}^0_{\mathbf{b}_m}(-\hat{\mathbf{a}}_0;\hat{\mathbf{q}}_m)+\mathbf{g}^0_{\mathbf{b}_m}(\hat{\mathbf{a}}_0;\hat{\mathbf{q}}_m)e^{2ik_0a_0}\right]$$

The following *mixed total electric field general scattering theorem* is derived by the general scattering theorem 5.1 for $a_0\to\infty$ and using the last equation.

Theorem 5.3 *A total far-field pattern* $\mathbf{g}^0_{\mathbf{b}_m}$*, due to a dipole at* $\mathbf{b}_m \in V_m$*, a total field* $\mathbf{E}^m$ *in* V_m *and a far-field pattern* $\mathbf{g}$*, both due to an incident on* V *plane wave* $\mathbf{E}^{inc}$ *satisfy*

$$(\hat{\mathbf{p}}_0 \times \hat{\mathbf{a}}_0) \cdot \mathbf{g}^0_{\mathbf{b}_m}(-\hat{\mathbf{a}}_0; \hat{\mathbf{q}}_m) - \frac{k_0\mu_0}{k_m\mu_m} b_m \exp(-ik_m b_m)\big(\nabla \times \mathbf{E}^m(\mathbf{b}_m; -\hat{\mathbf{a}}_0, \hat{\mathbf{p}}_0 \times \hat{\mathbf{a}}_0)\big) \cdot \hat{\mathbf{q}}_m$$
$$+ \frac{1}{2\pi} \int_{S^2} \mathbf{g}(\hat{\mathbf{r}}; -\hat{\mathbf{a}}_0, \hat{\mathbf{p}}_0 \times \hat{\mathbf{a}}_0) \cdot \mathbf{g}^0_{\mathbf{b}_m}(\hat{\mathbf{r}}; \hat{\mathbf{q}}_m) ds(\hat{\mathbf{r}}) = \lim_{a_0 \to \infty} \mathcal{E}(\mathbf{a}_0, \mathbf{b}_m; \hat{\mathbf{p}}_0, \hat{\mathbf{q}}_m).$$

For electromagnetic reciprocity and mixed reciprocity relations we refer to theorems 3.2 and 6.1 of [12]. Immediate consequence of theorems 6.1 of [12] and 5.3 constitutes

Corollary 5.2 *The fields and far-field patterns of theorem 5.3 are related by*

$$(\hat{\mathbf{p}}_0 \times \hat{\mathbf{a}}_0) \cdot \left[\mathbf{g}^0_{\mathbf{b}_m}(-\hat{\mathbf{a}}_0; \hat{\mathbf{q}}_m) - \exp(-2ik_m b_m)\mathbf{g}^0_{\mathbf{b}_m}(\hat{\mathbf{a}}_0; \hat{\mathbf{q}}_m)\right]$$
$$+ \frac{1}{2\pi} \int_{S^2} \mathbf{g}(\hat{\mathbf{r}}; -\hat{\mathbf{a}}_0, \hat{\mathbf{p}}_0 \times \hat{\mathbf{a}}_0) \cdot \mathbf{g}^0_{\mathbf{b}_m}(\hat{\mathbf{r}}; \hat{\mathbf{q}}_m) ds(\hat{\mathbf{r}}) = \lim_{a_0 \to \infty} \mathcal{E}(\mathbf{a}_0, \mathbf{b}_m; \hat{\mathbf{p}}_0, \hat{\mathbf{q}}_m).$$

$$\frac{k_0\mu_0}{k_m\mu_m} b_m \exp(-ik_m b_m)\hat{\mathbf{q}}_m \cdot \left[\nabla \times \big(\mathbf{E}^m(\mathbf{b}_m; \hat{\mathbf{a}}_0, \hat{\mathbf{p}}_0 \times \hat{\mathbf{a}}_0) - \mathbf{E}^m(\mathbf{b}_m; -\hat{\mathbf{a}}_0, \hat{\mathbf{p}}_0 \times \hat{\mathbf{a}}_0)\big)\right]$$
$$+ \frac{1}{2\pi} \int_{S^2} \mathbf{g}(\hat{\mathbf{r}}; -\hat{\mathbf{a}}_0, \hat{\mathbf{p}}_0 \times \hat{\mathbf{a}}_0) \cdot \mathbf{g}^0_{\mathbf{b}_m}(\hat{\mathbf{r}}; \hat{\mathbf{q}}_m) ds(\hat{\mathbf{r}}) = \lim_{a_0 \to \infty} \mathcal{E}(\mathbf{a}_0, \mathbf{b}_m; \hat{\mathbf{p}}_0, \hat{\mathbf{q}}_m).$$

The first of these equations actually represents a non-homogeneous Fredholm integral equation of the second kind with respect to $\mathbf{g}^0_{\mathbf{b}_m}$ and integral kernel $\mathbf{g}$.

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◇ Nikolaos L. Tsitsas
Department of Mathematics
University of Athens
GR-15784 Athens, GREECE
`ntsitsas@math.uoa.gr`

◇ Christodoulos Athanasiadis
Department of Mathematics
University of Athens
GR-15784 Athens, GREECE
`cathan@math.uoa.gr`