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Robust stabilization of multivariable systems: Directionality and Super-optimisation

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Abstract

The paper reviews some new results by the authors in the area of super-optimal Nehari approximations with applications to robust stabilisation of multivariable systems. The super-optimal approximation problem is first introduced and a new matrix dilation method is developed for its solution. This method is entirely based on concrete state-space realisations, systems theory and linear-algebraic techniques and is thus directly implementable. In addition, all simplifying assumptions made in previous work (e.g. multiplicity of largest Hankel singular value, minimality of realisation, solvability of certain matrix Riccati equations, etc) are removed. Next, applications of super-optimization are considered for the "maximally robust stabilisation problem" (MRSP) in the case of unstructured additive uncertainty. The maximum robust stability radius $\epsilon^\star$ is derived and a "worst-case" direction is identified, along which all boundary uniformly-destabilizing perturbations are shown to lie, i.e. all perturbations of norm $\epsilon^\star$ which destabilize the closed-loop system for every optimal (maximally robust) controller. By imposing a parametric constraint on the projection of admissible perturbations along this direction (uniformly in frequency), it is shown that it is possible to extend the robust stability radius in every other direction, using a subset of all optimal (maximally-robust) controllers, by solving a super-optimal Nehari approximation problem. A closed-form expression is obtained for the constrained robust stability radius, $\mu^\star(\delta)$ which depends on the first two super-optimal levels of the closed-loop system, while the identified "worst-case" direction corresponds to the maximal Schmidt pair of a Hankel operator related to the problem. The results rely on the solution of a sequence of "distance-to-singularity" problems, which when applied to structured uncertainty models, produce a systematic method for breaching the convex upper bound of the "structured singular value" (μ), a problem which is at present the main bottleneck of effective synthesis in the area of robust control. It is shown that the proposed method is can be applied to a wide class of NP-hard problems where convex relaxations are used.

Keywords: Robust control, Additive uncertainty models, Hankel operators, Schmidt pairs, Super-optimisation, Structured Singular Value, Convex relaxations

1. Notation

All systems considered are assumed linear, time invariant and finite-dimensional. Let $\mathcal{R}^{p\times m}(s)$ denote the space of proper $p \times m$ rational matrix functions in s with real coefficients. Associated with $P(s) \in \mathcal{R}^{p\times m}(s)$ of McMillan degree n is a state-space realization $P(s) = C(sI - A)^{-1}B + D$ where $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{p\times n}$ and $D \in \mathcal{R}^{p\times m}$. For $P(s) \in \mathcal{R}(s)^{p\times m}$ let $P^{\sim}(s) := P'(-s)$ denote the *para-hermitian conjugate* of $P(s)$. Let $P(s)$ be partitioned as $P_{ij}(s)$, $i = 1, 2$, $j = 1, 2$. Then a state space realization of $P(s)$ can be written as:

$$P(s) := \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{ccc} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right]$$

and $P_{ij} = C_i(sI - A)^{-1}B_j + D_{ij}$ is a state-space realization of $P_{ij}(s)$. A lower linear fractional transformation of $P(s)$ and $K(s)$ is defined as $\mathcal{F}_l(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$ where $K(s)$ is of dimension $m \times p$ if $P_{22}(s)$ has dimension $p \times m$ and the indicated inverse exists. Similarly we define the upper linear fractional transformation of $P(s)$ and $K(s)$ as $\mathcal{F}_u(P, K) = P_{22} + P_{21}K(I - P_{11}K)^{-1}P_{12}$ for a compatible partitioning of $P(s)$ with $K(s)$ and provided that the indicated inverse exists.

The space $\mathcal{RL}_\infty$ consists of all proper real-rational transfer matrix functions which are analytic on the imaginary axis. $\mathcal{RH}_\infty^+$ and $\mathcal{RH}_\infty^-$ are the subspaces of $\mathcal{RL}_\infty$ consisting of all real-rational proper matrix functions which are analytic in the closed right-half plane and closed left-half plane, respectively. Thus $\mathcal{RL}_\infty = \mathcal{RH}_\infty^+ \oplus \mathcal{RH}_\infty^-$ where $\oplus$ denotes direct sum of subspaces. The norm $\|\cdot\|_\infty$ denotes either the $\mathcal{L}_\infty$-norm of a function in $\mathcal{L}_\infty$ or the $\mathcal{H}_\infty$-norm of a function in $\mathcal{H}_\infty$, depending on context.

The factorizations: $P(s) = N(s)M^{-1}(s)$, $P(s) = \tilde{M}^{-1}\tilde{N}(s)$ are said to be right and left co-prime factorisations of $P(s)$ (lcf and rcf), respectively, if $N, M, \tilde{N}, \tilde{M} \in \mathcal{RH}_\infty^+$ satisfy the following Diophantine matrix equation (also known as the "Bezout identity"):

$$\begin{pmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix} = I \tag{1}$$

for appropriate functions $U, V, \tilde{U}, \tilde{V} \in \mathcal{RH}_\infty^+$.

2. Background theory: Hankel operators, singular values and Schmidt vectors

In this section we first give some preliminary definitions related to Hankel operators, their singular values and their Schmidt vectors and outline some of their properties related to Nehari and super-optimal approximations. Given $G \in \mathcal{RL}_\infty^{p\times m}$, the Hankel operator with symbol G is defined as [Fra87], [Pel03], [ZDG96]:

$$\Gamma_G = P_+ M_G|_{\mathcal{H}_2^\perp} : \mathcal{H}_2^\perp \to \mathcal{H}_2, \quad \Gamma_G f := (P_+ M_G)f = P_+(Gf) \ \text{ for } \ f \in \mathcal{H}_2^\perp$$

Here M_G denotes the multiplication operator and P_+, P_- denote the orthogonal projections from $\mathcal{L}_2$ to $\mathcal{H}_2$ and $\mathcal{H}_2^\perp$, respectively. Since $G \in \mathcal{RL}_\infty$ is analytic on a vertical strip containing the imaginary axis, we can define its two-sided Laplace transform, $g(t) \in \mathcal{L}_2(-\infty,\infty)$, containing both causal and anti-causal parts. Here $\mathcal{L}_2(-\infty,\infty)$ denotes the space of all square-integrable functions with support on the real line. The equivalent definition of the Hankel operator in the time-domain is:

$$\Gamma_g : \mathcal{L}_2(-\infty,0] \rightarrow \mathcal{L}_2[0,\infty),\ \Gamma_g f = P_+(g * f),\ \text{ for }\ f \in \mathcal{L}_2(-\infty,0]$$

where $*$ denotes convolution. Thus

$$(\Gamma_g f)(t) = \begin{cases} \int_{-\infty}^{0} g(t-\tau)f(\tau)d\tau & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Clearly, the anti-causal part of the "impulse response" of $G(s)$ does not affect $(\Gamma_g f)(t)$, and hence it can be assumed without loss of generality that $G(s) \in \mathcal{RH}_\infty^+$ with $G(\infty) = 0$. Further, due to the isometric isomorphism between the $\mathcal{L}_2$ spaces in the time and frequency domains [ZDG96], $\|\Gamma_G\| = \|\Gamma_g\|$ and we can use the definitions of the Hankel operator in the two domains interchangeably. It further follows from the definition that the Hankel operator may be written as the composition of the controllability and observability operators, defined via a state-space realization of $G = (A,B,C)$, assumed minimal without loss of generality, as

$$\Psi_c : \mathcal{L}_2[-\infty,0] \rightarrow \mathcal{R}^n,\ \ \Psi_c u := \int_{-\infty}^{0} e^{-A\tau}Bu(\tau)d\tau$$

and

$$\Psi_o : \mathcal{R}^n \rightarrow \mathcal{L}_2[0,\infty),\ \ \Psi_o x_0 := Ce^{At}x_0,\ t \geq 0$$

where $n = \dim(A)$, i.e. $\Gamma_G = \Psi_o\Psi_c$. Thus Γ_g may be thought of as the operator mapping "past" inputs $u(t)$ to "future" outputs $y(t)$ via the initial state $x(0) = x_0$ [Glo84]. The adjoint operator of Γ_G can be shown to be [ZDG96]:

$$\Gamma_G^* : \mathcal{H}_2 \rightarrow \mathcal{H}_2^\perp,\ \Gamma_G^* = P_- M_{G^\sim}|_{\mathcal{H}_2}$$

and hence [ZDG96],

$$\Gamma_g^* = (\Psi_o\Psi_c)^* = \Psi_c^*\Psi_o^* : \mathcal{L}_2[0,\infty) \rightarrow \mathcal{L}_2(-\infty,0]$$

where Ψ_c^* and Ψ_o^* denote the adjoint operators of Ψ_c and Ψ_o, respectively:

$$\Psi_c^* : \mathcal{R} \rightarrow \mathcal{L}_2(-\infty,0],\ \ \Psi_c^* x_0 = B'e^{-A'\tau}x_0,\ \ \tau \leq 0$$

and

$$\Psi_o^* : \mathcal{L}_2[0,\infty) \rightarrow \mathcal{R}^n,\ \ \Psi_o^* y(t) = \int_0^\infty e^{A't}C'y(t)dt,\ \ t \geq 0$$

Now

$$\Psi_c\Psi_c^* x_0 = \left(\int_{-\infty}^{0} e^{-A\tau}BB'e^{-A'\tau}d\tau\right)x_0 = \left(\int_{0}^{\infty} e^{A\tau}BB'e^{A'\tau}d\tau\right)x_0 := Px_0$$

where P is the controllability gramian of the pair (A,B), which satisfies the Lyapunov equation $AP+PA'+BB'=0$. Thus P is the matrix representation of $\Psi_c\Psi_c^*$. Similarly,

$$\Psi_o^*\Psi_o x_0 = \left(e^{A't}C'Ce^{At}dt\right)x_0 := Qx_0$$

where Q is the observability gramian of the pair (A,C), which satisfies the Lyapunov equation $A'Q+QA+C'C=0$. Now the operators $\Gamma_g^*\Gamma_g$ and $\Gamma_g\Gamma_g^*$ have matrix representations $\Gamma_g^*\Gamma_g = \Psi_c^*\Psi_o^*\Psi_o\Psi_c$ and $\Gamma_g\Gamma_g^* = \Psi_o\Psi_c\Psi_c^*\Psi_o^*$, respectively. Thus their non-zero eigenvalues satisfy:

$$\lambda_i(\Gamma_g^*\Gamma_g) = \lambda_i(\Gamma_g\Gamma_g^*) = \lambda_i(\Psi_c^*\Psi_o^*\Psi_o\Psi_c) = \lambda_i(\Psi_c\Psi_c^*\Psi_o^*\Psi_o) = \lambda_i(PQ) := \sigma_i^2(\Gamma_G)$$

The $\sigma_i(\Gamma_G)$'s are the singular values of Γ_G (Hankel singular values of G). Let these be ordered as $\sigma_1 = \ldots = \sigma_r > \sigma_{r+1} \geq \ldots \geq \sigma_n > 0$ where n is the McMillan degree of G. Then, $\sigma_1 = \|\Gamma_G\|$ is the Hankel norm of G. Next, let $u_i(t) \in \mathcal{L}_2(-\infty,0]$, $u_i(t) \neq 0$, be an eigenvector of $\Gamma_g\Gamma_g^*$ corresponding to the eigenvalue σ_i^2. Then

$$\Gamma_g^*\Gamma_g u_i = \Psi_c^*\Psi_o^*\Psi_o\Psi_c u_i = \sigma_i^2 u_i$$

Pre-multiplying by Ψ_c and defining $x_i = \Psi_c u_i \in \mathcal{R}^n$ gives $PQx_i = \sigma_i^2 x_i$. Define $v_i = (1/\sigma_i)\Gamma_g u_i \in \mathcal{L}_2[0,\infty)$. Then the pair (u_i, v_i) satisfies $\Gamma_g u_i = \sigma_i v_i$ and $\Gamma_g^* v_i = \sigma_i u_i$ and is called a Schmidt pair of Γ_G. Thus

$$u_i(t) = \Psi_c^*\left(\frac{1}{\sigma_i}Qx_i\right) = \sigma_i^{-1}B'e^{-A't}Qx_i \in \mathcal{L}_2(-\infty,0]$$

and

$$v_i(t) = \Psi_o x_i = Ce^{At}x_i \in \mathcal{L}_2[0,\infty)$$

Let $\{u_1, u_2, \ldots, u_r\}$ and $\{v_1, v_2, \ldots, v_r\}$ be a collection of r linearly independent eigenvectors of $\Gamma_g^*\Gamma_g$ and $\Gamma_g\Gamma_g^*$, respectively, corresponding to the eigenvalue σ_1^2. Then

$$U(t) = \begin{bmatrix} u_1 & \ldots & u_r \end{bmatrix}(t) = \sigma_1^{-1}B'e^{-A't}Q\begin{bmatrix} x_1 & \ldots & x_r \end{bmatrix} \in \mathcal{L}_2^{m\times r}(-\infty,0]$$

and

$$V(t) = \begin{bmatrix} v_1 & \ldots & v_r \end{bmatrix}(t) = Ce^{At}\begin{bmatrix} x_1 & \ldots & x_r \end{bmatrix} \in \mathcal{L}_2^{p\times r}[0,\infty)$$

Taking the (bilateral) Laplace transform shows that

$$U(s) = -B'(sI+A')^{-1}\Xi \in \mathcal{RH}_2^{\perp,m\times r}, \;\; \Xi = \sigma_1^{-1}Q\begin{bmatrix} x_1 & x_2 & \ldots & x_r \end{bmatrix}$$

and

$$V(s) = C(sI-A)^{-1}\Theta \in \mathcal{H}_2^{p\times r}, \;\; \Theta = \begin{bmatrix} x_1 & x_2 & \ldots & x_r \end{bmatrix}$$

We can now invoke Nehari's theorem which states that [Fra87], [Glo84], [Pel03]

$$\inf_{Q \in \mathcal{H}_\infty^-} \|G + Q\| = \|\Gamma_G\| = \sigma_1 \tag{2}$$

It can be shown that the infimum in (2) is attained; further [ZDG96], [Glo84]:

$$\text{rank}_{\mathcal{R}(s)} U(s) = \text{rank}_{\mathcal{R}(s)} V(s) := l \leq \min(p, m, r) \tag{3}$$

and $(G + Q)U(s) = \sigma_1 V(s)$ for *every* (optimal) Q. This may be used to show that in the scalar case the optimal Nehari extension is unique and is given by $Q = G + \sigma_1 V(s)/U(s)$. In the matrix case the equation has motivated the derivation of the parametrization of all optimal solutions of (2) [Glo84], and also most methods used to solve the super-optimal distance problem, typically based on the construction of all-pass diagonalising transformations of $G + Q$ using $U(s)$ and $V(s)$ [HLG93], [LHG89].

3. Super-optimal Nehari extensions

A formal definition of the problem follows. Firstly, define

$$s_i^\infty(R) := \sup_{\omega \in \mathcal{R}} \sigma_i[R(j\omega)], \quad i = 1, 2, \ldots, \min(p, m).$$

If p and m are both greater than 1, then we define recursively the first and subsequent super-optimal levels of R as

$$s_i(R) := \inf_{Q \in \mathcal{S}_{i-1}(R)} s_i^\infty(R + Q) \quad i = 1, 2, \ldots, \min(p, m) \tag{4}$$

and the set of all i-th level super-optimal approximations of R as

$$\mathcal{S}_i(R) := \{Q \in \mathcal{S}_{i-1}(R) : s_i^\infty(R + Q) = s_i(R)\} \quad i = 1, 2, \ldots, \min(p, m)$$

In other words, we seek among all super-optimal approximations at the $(i-1)$-th level $\mathcal{S}_{i-1}(R)$ a set for which $s_i(R)$ is minimized (it turns out that the infimum in (4) is always attained). This set is not a singleton in general (apart from the case of $i = \min(p, m)$), but forms a subset of all $(i-1)$-th level super-optimal approximations of R, $\mathcal{S}_{i-1}(R)$. Due to the lexicographic nature of the problem, it is clear that every element of $\mathcal{S}_i(R)$ is also an element of $\mathcal{S}_{i-1}(R)$, i.e. that the super-optimal approximation sets nest as:

$$\mathcal{S}_0(R) \supseteq \mathcal{S}_1(R) \supseteq \ldots \supseteq \mathcal{S}_i(R) \supseteq \ldots \supseteq \mathcal{S}_{\min(p,m)}(R)$$

Note that for $i = 1$, (4) is taken to be a Nehari extension problem and hence we define $\mathcal{S}_0(R) := \mathcal{H}_\infty^{+, p \times m}$. The super-optimal approximation problem ([SODP]) considered in this paper can be formally defined as follows:

SODP Problem: Given a $G \in \mathcal{RH}_{\infty}^{-,\, p\times m}$, find the (unique) matrix-function $Q_{sup} \in \mathcal{H}_{\infty}^{+,\, p\times m}$ which minimizes the sequence

$$s^{\infty}(G+Q) = (s_1^{\infty}(G+Q), s_2^{\infty}(G+Q), \ldots, s_k^{\infty}(G+Q))$$

with respect to the lexicographic ordering, where $k = \min(p, m)$.

The approach followed here involves the reduction of the lexicographic minimization into a hierarchy of ordinary Nehari-extension problems of progressively reduced input-output dimensions. First the system to be approximated, $R(s)$, is embedded in an all-pass system $H(s)$ of higher dimensions. This acts as a "generator" of the optimal solution set of the Nehari extension problem, as all solutions can be obtained via a LFT of $H(s)$ with the ball of $\mathcal{H}_{\infty}$ of radius s_1^{-1} (i.e. the set of all stable s_1^{-1}-contractions) [Glo84]. Next, a sub-block of the optimal generator $H(s)$ is dilated to define a new square all-pass system $H(s)$, of lower dimensions compared to those of $H(s)$. Exploiting the all-pass nature of $H(s)$ and $H(s)$ and the fact that they share a common block, two diagonalising transformations of $H(s)$ can be defined from certain sub-blocks of $H(s)$ and $H(s)$.

First, the general solution of the optimal Nehari-extension problem is given under minimal assumptions:

Theorem 3.1 (Optimal Nehari approximation) *Consider* $R \in \mathcal{RH}_{\infty}^{-,\, p\times m}$ *with realization* $R \overset{s}{=} \left[\begin{array}{cc} A & B \\ C & 0 \end{array}\right]$ *where* $\lambda(A) \subseteq \mathcal{C}_+$.
Then there exists $Q_a \in \mathcal{RH}_{\infty}^{+,(p+m-r)\times(p+m-r)}$ *such that all* $Q \in \mathcal{H}_{\infty}^{+,\, p\times m}$ *such that* $\|R+Q\|_{\infty} = \|R^{\sim}\|_H = s_1$ *(Nehari optimal approximations of R) are given by*

$$Q = \mathcal{F}_l(Q_a, s_1^{-1}\, \mathcal{BH}_{\infty}^{(p-l)\times(m-l)})$$

where in which r *denotes the multiplicity of the largest Hankel singular value of* $R^{\sim}$*,* l *is defined in (3), and*

$$Q_a := \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \overset{s}{=} \left[\begin{array}{ccc} A_q & B_{q1} & B_{q2} \\ C_{q1} & D_{11} & D_{12} \\ C_{q2} & D_{21} & 0 \end{array}\right] \tag{5}$$

The corresponding "error" system is given by

$$H := \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} R+Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \overset{s}{=} \left[\begin{array}{cc} A_H & B_H \\ C_H & D_H \end{array}\right] \tag{6}$$

where $\|H_{22}\|_{\infty} < s_1$ *and* $Q_{ij} \in \mathcal{RH}_{\infty}^{+}$*, for* $i,j \in \{1,2\}$*. Further,* $HH^{\sim} = H^{\sim}H = s_1^2 I$ *and the following set of equations is satisfied*

$$\begin{aligned} P_H Q_H &= Q_H P_H = s_1^2 I \\ D_H D'_H &= D'_H D_H = s_1^2 I \\ A'_H Q_H + Q_H A_H + C'_H C_H &= 0 \\ A_H P_H + P_H A'_H + B_H B'_H &= 0 \\ D'_H C_H + B'_H Q_H &= 0 \\ D_H B'_H + C_H P_H &= 0 \end{aligned} \tag{7}$$

Here P_H and Q_H are the gramians of the realization of H given in (7).

Proof. *See [Glo84]; see also [JL93] and [HJ98] for a more general setting.*

Remark 3.1 The realization of R need not be assumed minimal. However, we require that $\lambda(A) \subseteq \mathcal{C}_+$. If R has McMillan degree n, it can be shown [Glo86] that Q_a given in (5) has degree $n-r$; in addition, $\sigma_i(Q_a) = \sigma_{i+r}(R^\sim)$, $i = 1, 2, \ldots, n-r$ [Glo86], [Glo84].

Next, using $H_{22} = Q_{22} \in \mathcal{RH}_\infty^{+,(m-l)\times(p-l)}$ (with $\|Q_{22}\| < s_1$ from Theorem 3.1), we construct an s_1-allpass matrix function H, corresponding to a new system $\widehat{R} \in \mathcal{RH}_\infty^{-,\,(p-l)\times(m-l)}$ defined from its $(1,1)$ block. It is shown that H acts as a s_1-suboptimal Nehari generator of $\widehat{R}$, i.e. that the LFT of H with the s_1^2-ball of $\mathcal{H}_\infty$ generates the set

$$\mathcal{S}(\hat{R}, s_1) = \{\Psi \in \mathcal{H}_\infty : \|\hat{R} + \Psi\| \leq s_1\}$$

Using this structure, it is possible to construct all level-two super-optimal approximations of R, which lie inside the set of all optimal approximations, Q, of R. By choosing all Q inside the subset, the corresponding "error" systems $R + Q$ will now minimize the first as well as the second singular values of R (for $l = 1$), i.e. this subset defines the super-optimal approximations of R with respect to the first two levels. The method can be repeated using a recursive procedure until all degrees of freedom have been exhausted.

The construction of H is based on the following proposition, first stated at a transfer function level. A state-space construction of H follows, proving that it acts as an s_1-suboptimal Nehari generator of the anti-stable projection of its $(1,1)$ block.

Proposition 3.1 *Let H_{22} be defined in 3.1 with $\|H_{22}\|_\infty < s_1$. Then,*

1. *There exists a square transfer matrix $H_{21} \in \mathcal{RH}_\infty$ such that $H_{21}H_{21}^\sim = s_1^2 I - H_{22}H_{22}^\sim$ and $H_{21}^{-1} \in \mathcal{RH}_\infty$.*
2. *There exists a square transfer matrix $H_{12} \in \mathcal{RH}_\infty$ such that $H_{12}^\sim H_{12} = s_1^2 I - H_{22}^\sim H_{22}$ and $H_{12}^{-1} \in \mathcal{RH}_\infty$.*

3 The system

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} := \begin{pmatrix} -H_{12}H_{22}^{\sim}H_{21}^{\sim} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

is in $\mathcal{RL}_\infty$ *and is* s_1*-allpass. Further, let* $-H_{12}H_{22}^{\sim}H_{21}^{\sim} = \hat{R} + Q_{11}$ *where* $\hat{R} \in \mathcal{RH}_\infty^-$ *and* $Q_{11} \in \mathcal{RH}_\infty^+$*. Then* $\|\hat{R}^\sim\|_H < s_1$*.*

Proof. *Follows from parts (1) and (2) of [ZDG96], Corollary 13.22 and [Glo86]. The proof follows from a detailed construction involving elements from the theory of algebraic Riccati equations and spectral factorization; details are omitted.*

Remark 3.2 Since $s_1 = \sigma_1(R^\sim)$ the inequality of part (3) says that $\sigma_1(\hat{R}) < \sigma_1(R^\sim)$. This can actually be strengthened to $\sigma_1(\hat{R}) < \sigma_{r+1}(R^\sim)$, where r is the multiplicity of the largest Hankel singular value of $R^\sim$.

A detailed state-space construction of H and its properties are given in Theorem 3.2 below.

Theorem 3.2 *Consider*

$$H_{22} = Q_{22} \stackrel{s}{=} \left[\begin{array}{cc} A_q & B_{q2} \\ C_{q2} & 0 \end{array} \right] \in \mathcal{RH}_\infty^{+,(m-l)\times(p-l)}, \quad \|Q_{22}\|_\infty < s_1$$

defined in Theorem 3.1. Then there exist unique stabilizing solutions P_2 *and* Q_2 *to the following algebraic Riccati equations:*

$$\begin{array}{c} A_q P_2 + P_2 A_q' + B_{q2}B_{q2}' + s_1^{-2} P_2 C_{q2}' C_{q2} P_2 = 0 \\ A_q' Q_2 + Q_2 A_q + C_{q2}' C_{q2} + s_1^{-2} Q_2 B_{q2} B_{q2}' Q_2 = 0 \end{array} \tag{8}$$

respectively. Define:

$$R := Q_2 P_2 - s_1^2 I \tag{9}$$

Then R *is non-singular. Further, there exists a* $Q_a \in \mathcal{H}_\infty^{+,(p+m-2l)\times(p+m-2l)}$ *with realization*

$$Q_a := \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{ccc} A_q & B_{q1} & B_{q2} \\ C_{q1} & 0 & s_1 I \\ C_{q2} & s_1 I & 0 \end{array} \right] \tag{10}$$

where

$$\begin{array}{l} C_{q1} = -s_1^{-1} B_{q2}' Q_2 \\ B_{q1} = -s_1^{-1} P_2 C_{q2}' \end{array} \tag{11}$$

so that $Q = \mathcal{F}_l(Q_a, s_1^{-1}\, \mathcal{BH}_\infty^{(p-l)\times(m-l)})$ *is the set of all* s_1-*suboptimal Nehari extensions of a system* $\widehat{R} \in \mathcal{RH}_\infty^{-,(p-l)\times(m-l)}$ *defined as:*

$$\widehat{R} \stackrel{s}{=} \left[\begin{array}{cc} \widehat{A} & \widehat{B} \\ \widehat{C} & 0 \end{array} \right] \tag{12}$$

in which

$$\begin{aligned}\widehat{A} &= -(A_q + s_1^{-2} P_2 C_{q2}' C_{q2})' = -A_q' - s_1^{-2} C_{q2}' C_{q2} P_2 \\ \widehat{B} &= -s_1^{-1} C_{q2}' \\ \widehat{C} &= s_1^{-1} B_{q2}' R\end{aligned} \tag{13}$$

The corresponding "error system"

$$H = \widehat{R}_a + Q_a = \begin{pmatrix} \widehat{R} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \tag{14}$$

is s_1-allpass and has a realization

$$H := \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} \widehat{R} + Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \stackrel{s}{=} \left[\begin{array}{cccc} \widehat{A} & 0 & \widehat{B} & 0 \\ 0 & A_q & B_{q1} & B_{q2} \\ \widehat{C} & C_{q1} & 0 & s_1 I \\ 0 & C_{q2} & s_1 I & 0 \end{array} \right] \tag{15}$$

which satisfies the following set of all-pass equations:

$$\begin{aligned} &A_H' Q_H + Q_H A_H + C_H' C_H = 0 \\ &A_H P_H + P_H A_H' + B_H B_H' = 0 \\ &D_H' C_H + B_H' Q_H = 0 \\ &D_H B_H' + C_H P_H' = 0 \\ &D_H D_H' = D_H' D_H = s_1^2 I \\ &P_H Q_H = Q_H P_H = s_1^2 I \end{aligned} \tag{16}$$

in which Q_H and P_H are the gramians of the realization of H given in (15).

Proof. *The proof is based on [Glo84]; see also [JL93] and [HJ98] for a more general setting. Details are omitted.*

The following theorem constructs a diagonalising transformation of H and solves the level-two SODP.

Theorem 3.3 *Let H and H be as defined in Theorems 3.1 and 3.2, respectively. Then: (i) There exist square inner matrix functions V and $W^{\sim}$, such that:*

$$R + \mathcal{S}_1(R) = V \begin{pmatrix} s_1 \alpha & 0 \\ 0 & \hat{R} + \mathcal{S}(\hat{R}, s_1) \end{pmatrix} W^{\sim} \tag{17}$$

where $\alpha(s) \in \mathcal{R}^{l \times l}$ is anti-inner. (ii) Also,

$$\|R^{\sim}\|_H = s_1(R) = s_2(R) = \ldots = s_l(R) > s_{l+1}(R) = \|\hat{R}^{\sim}\|_H$$

Further,

$$\mathcal{S}_1(R) = \mathcal{S}_2(R) = \ldots = \mathcal{S}_l(R) = \mathcal{F}_l(Q_a, s_1^2\, \mathcal{BH}_\infty^{(p-l)\times(m-l)})$$

and

$$\mathcal{S}_{l+1}(R) = \mathcal{F}_l[Q_a, \mathcal{F}_u(\underline{Q}_a^{-1}, \mathcal{S}_1(\widehat{R}))] \subseteq \mathcal{S}_1(R)$$

where Q_a *and* $\underline{Q}_a$ *are defined in Theorems 3.1 and 3.2.*

Proof. The proof follows via a rather long argument using the all-pass character of the optimal and sub-optimal generators H *and* $\bar{H}$*.*

Remark 3.3 Part (i) of the Theorem establishes a connection between $\mathcal{S}_1(R)$, the set of all optimal Nehari approximations of R with $\mathcal{S}(\hat{R}, s_1)$, the set of all s_1-suboptimal approximations of $\hat{R}$. Since, (i) V and $W^\sim$ are square-inner; (ii) $\alpha \in \mathcal{R}^{l\times l}(s)$ is anti-inner, and (iii) $\|\hat{R}\|_H < s_1$, it follows that the first l super-optimal levels of R are equal to s_1 and that the $(l+1)$-th super-optimal level of R is equal to $\|\hat{R}^\sim\|_H$ (since the set of all optimal Nehari approximations of $\hat{R}$ is contained in the set of all s_1-suboptimal approximations of $\hat{R}$). The Theorem also shows the recursive character of the problem which is made explicit in part (ii).

The following Theorem establishes bounds on the super-optimal levels. The proof is similar to a parallel result in [LHG89], but the assumption involving the multiplicity of the largest Hankel singular value of $R^\sim$ is removed.

Theorem 3.4 (Super-optimal level bounds) *The* $(l+1)$*-th super-optimal level is bounded above by the* $(r+1)$*-th Hankel singular value of* $R^\sim$*, i.e.*

$$\sigma_1(\hat{R}^\sim) = s_{l+1}(R) \leq \sigma_{r+1}(R^\sim) < s_1(R) = s_2(R) = \ldots = s_l(R) = \sigma_1(R^\sim)$$

Proof. Follows by generalizing a parallel result in [LHG89]. The assumption involving the multiplicity of the largest Hankel singular value of $R^\sim$ *here is removed.*

Remark 3.4 The result of Theorem 3.4 may be propagated to establish upper bounds for the subsequent super-optimal levels $s_i(R)$, $i > l+1$.

4. Robust control of multivariable systems

In this section we consider a maximally robust stabilisation problem under unstructured additive uncertainty. We investigate the set of optimal solutions and show that, in the multivariable case, super-optimisation can be used to guarantee stabilisation of a larger class of perturbations relative to an arbitrary optimal controller. This observation has also been made by Nyman for the co-prime uncertainty case [Nym99], who identified the extended set of perturbations stabilised by the (unique) super-optimal controller. His description of this set, however, is rather implicit (as it is formulated in the form of a weighted-norm) and thus not really suitable for the further investigation of its structure, or for control design purposes. Here, we attempt to identify the stronger robust-stability properties arising by using the super-optimal solution to the problem in terms of *directionality*. This arises naturally from the observation that every boundary (maximum-norm) perturbation which is uniformly destabilising (i.e. which destabilises the closed-loop system for every maximal robust controller) lies in a certain direction which is identified. Our approach leads to a simple closed-form expression for the extended robust-stability radius (for every direction other than the worst-case direction) and can be easily applied to control design problems involving both structured and unstructured uncertainty models.

The feedback structure considered is shown in Figure 1. In this diagram G represents the nominal plant, K the feedback compensator which must be designed and Δ an additive perturbation of G.

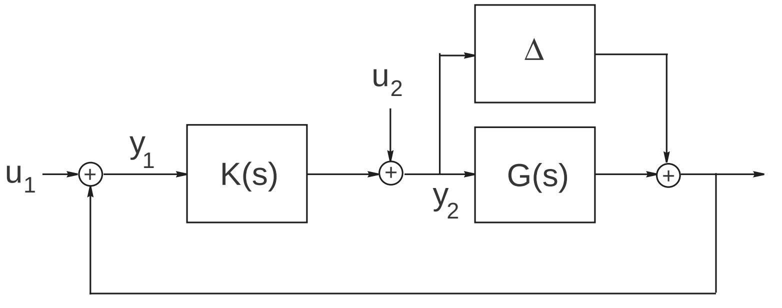

Figure 1: Feedback system with additive uncertainty

Remark 4.1 To simplify the presentation we make the following assumptions: (i) $G \in \mathcal{RH}_\infty^-$, $G(\infty) = 0$; (ii) $\Delta \in \mathcal{B}_\epsilon\mathcal{H}_\infty^+ = \{\Delta \in \mathcal{RH}_\infty : \|\Delta\|_\infty < \epsilon\}$. Both assumptions can be easily relaxed at the expense of increased notational complexity and involve no real loss of generality. The standard assumptions normally made for problems of this type are: (i)$'$ $G \in \mathcal{RL}_\infty$, i.e. that $G(s)$ is real-rational, proper and free of poles on the imaginary axis, and (ii)$'$, $\Delta \in \mathcal{RL}_\infty$, $\eta(G + \Delta) = \eta(G)$ where $\eta(\cdot)$ denotes number of poles in the open right-half-plane (counted in a McMillan degree sense). Regarding (i)$'$, it can be shown [Glo86] that introducing a suitable preliminary feedback transformation cancels completely the stable part of the nominal plant without changing the nature of the problem. Assumption (ii)$'$ implies that the nominal and perturbed plant are constrained to have the same number of poles in the open right-half plane (although not necessarily at the same locations). Clearly condition (ii)$'$ is satisfied if (i) and (ii) hold. All results presented in this paper still apply under the more general assumption (ii)$'$, provided that condition $\eta(G + \Delta) = \eta(G)$ is introduced as an additional qualification in the definition of several sets.

Definition 4.1 When $\Delta = 0$ we will say that the feedback system in Figure 1 is *internally stable* if and only if the four transfer functions $(u_1\ u_2) \to (y_1\ y_2)$ are *well-posed* and lie in $\mathcal{RH}_\infty^+$. This is denoted by writing $(G, K) \in \mathcal{S}$, or equivalently $K \in \mathcal{K}$, where $\mathcal{K}$ denotes the set of all stabilising compensators.

We can now define the sub-optimal and optimal robust stabilisation problems:

Definition 4.2 (ϵ-suboptimal robust stabilisation problem): Given $\epsilon \geq 0$ determine if there exists a non-empty subset of $\mathcal{K}$, such that for each $K \in \mathcal{K}$, $(G + \Delta, K) \in \mathcal{S}$ for every $\Delta \in \mathcal{B}_\epsilon\mathcal{H}_\infty^+$. If this is the case, we say that G has a robust-stability radius of at least ϵ.

Definition 4.3 Maximum Robust Stabilisation problem (MRSP): Find $\epsilon^* = \sup \epsilon$ such that there exists $K \in \mathcal{K}$ for which $(G + \Delta, K) \in \mathcal{S}$ for every $\Delta \in \mathcal{B}_\epsilon\mathcal{H}_\infty^+$ and the corresponding set of all such K, $\mathcal{K}_1 \subseteq \mathcal{K}$.

The corresponding result can be used to solve both the optimal and ϵ-sub-optimal robust stabilisation problem:

Theorem 4.1 *Suppose that $(G, K) \in \mathcal{S}$. Then $(G + \Delta, K) \in \mathcal{S}$ for every $\Delta \in \mathcal{B}_\epsilon\mathcal{H}_\infty^+$ if and only if $\|T\|_\infty < \epsilon^{-1}$ where $T = K(I - GK)^{-1}$. Hence $\epsilon^* = (\inf_{K\in\mathcal{K}} \|T(K)\|)^{-1}$.*

Proof. *See [Vid85], [Glo86].*

The optimization given in Theorem 4.1 can be simplified using the Youla parametrisation of all stabilizing controllers [Fra87], [Vid85], [ZDG96]. First bring in left and right coprime factorisations of G with inner denominators, i.e. $G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ such that $M^{\sim}M = MM^{\sim} = I$ and $\tilde{M}\tilde{M}^{\sim} = \tilde{M}^{\sim}\tilde{M} = I$. Let U, V, $\tilde{U}$, $\tilde{V}$ satisfy $\tilde{V}M - \tilde{U}N = I$ and $-\tilde{N}U + \tilde{M}V = I$. Then, the set of all stabilising controllers $\mathcal{K}$ and the corresponding set of all control-sensitivity functions can be written as:

$$\mathcal{K} = \{(U + MQ)(V + NQ)^{-1}\ :\ Q \in \mathcal{H}_\infty^+\}$$

and

$$\mathcal{T} = \{(U + MQ)\tilde{M}\ :\ Q \in \mathcal{H}_\infty^+\}$$

respectively. Using the fact that M and $\tilde{M}$ are inner, we get

$$(\epsilon^*)^{-1} = \inf\{\|M^{\sim}U + Q\|\ ;\ Q \in \mathcal{H}_\infty^+\}$$

which is a Nehari approximation problem of the form discussed in section 3 with $R = M^{\sim}U \in \mathcal{RH}_\infty^-$. Further, it may be shown [Glo86] that $\|\Gamma_{R^\sim}\|_H = \sigma_n(\Gamma_{G^\sim})$ and hence from Nehari's theorem, the maximum robust stability radius is $\epsilon^* = \sigma_n(\Gamma_{G^\sim})$. From Theorem 3.3 of the last section we conclude that the set of all optimal (maximally robust) controllers and the corresponding set of all optimal control-sensitivity functions can be written as

$$\mathcal{K}_1 = \{(U + MQ)(V + NQ)^{-1}\ :\ Q \in \mathcal{S}_1(R)\}$$

and

$$\mathcal{T}_1 = \{(U + MQ)\tilde{M} \; : \; Q \in \mathcal{S}_1(R)\} = Y \begin{pmatrix} s_1\alpha & 0 \\ 0 & \hat{R} + \mathcal{S}(\hat{R}, s_1) \end{pmatrix} X$$

respectively, where we have defined $Y = MV$ and $X = W^{\sim}\tilde{M}$. Note that Y and X lie in $\mathcal{RL}_\infty$ and that they are square all-pass.

Remark 4.2 It may be shown that the first row of X and the first row of Y are associated with the Schmidt pair of the Hankel operator with symbol $R^{\sim}$; in fact earlier solutions to the SODP derive a parallel result to that of the diagonalisation of Theorem 3.3 by following a sequence of frequency-depending scalings of the Schmidt vectors of $R^{\sim}$.

All optimal compensators in the set $\mathcal{K}_1$ maximize the robust stability radius for the class of additive unstructured perturbations, i.e. guarantee that all $\Delta \in \mathcal{B}_{\epsilon^*}\mathcal{H}_\infty^+$ do not destabilize the feedback system. Pick any $K \in \mathcal{K}_1$; then there must exist $\Delta \in \partial\mathcal{B}_{\epsilon^*}\mathcal{H}_\infty^+ = \{\Delta \in \mathcal{H}_\infty^+ : \|\Delta\|_\infty = \epsilon^\star\}$ such that $(G_\Delta, K) \notin \mathcal{S}$ (for otherwise $\epsilon^\star$ would not be optimal). Such (real-rational) Δ's are constructed in [Vid85]. In the sequel we identify controllers within $\mathcal{K}_1$ that guarantee improved robust stability properties (apart from stabilising all $\Delta \in \mathcal{B}_{\epsilon^*}\mathcal{H}_\infty^+$. We first give the following definition:

Definition 4.4 A $\Delta \in \partial\mathcal{B}_{\epsilon^*}\mathcal{H}_\infty^+$ is called *uniformly destabilising* if $(G + \Delta, K) \notin \mathcal{S}$ for every $K \in \mathcal{K}_1$.

The next Lemma is ensures that (real-rational) uniformly destabilising perturbations exist on the boundary of $\mathcal{B}_{\epsilon^*}\mathcal{H}_\infty^+$ The proof of the Lemma (which is omitted) relies on a direct construction of such perturbations using the techniques of [Vid85]. The construction reveals that all frequencies are "equally critical", in the sense that such perturbations can be constructed so that the generalised Nyquist stability criterion of the open-loop perturbed system is be violated at an arbitrary frequency (including zero and infinity). For simplicity we assume from this point that the first two superoptimal levels of $R^{\sim}$ are distinct, i.e. that $l = 1$.

Lemma 4.1 *There exist (real-rational) uniformly destabilising perturbations on* $\partial\mathcal{B}_{\epsilon^*}\mathcal{H}_\infty^+$.

Proof. The proof follows by direct construction similar to that given in [Vid85]. See [GHJ00] for details.

The next Lemma shows that a necessary condition for a $\Delta \in \partial\mathcal{B}_{\epsilon^*}\mathcal{H}_\infty^+$ to be uniformly destabilising is that it is aligned with a particular direction at an arbitrary frequency.

Lemma 4.2 *If* $\Delta \in \partial\mathcal{B}_{\epsilon^*}\mathcal{H}_\infty^+$ *is uniformly destabilising, then* $\|x'\Delta y\|_\infty = \epsilon^*$, *where* x' *and* y *denote the first row and column of* X *and* Y, *respectively.*

Proof. See [GHJ00].

Remark 4.3 The constraint $\|x'\Delta y\|_\infty = \epsilon^*$ can be interpreted as a projection (uniform in frequency) of $\Delta(j\omega)$ in a direction defined by vectors x and y: For two matrices $A, B \in \mathcal{C}^{p\times m}$, define the inner product: $\langle A, B\rangle = \text{trace}(B^*A)$, with corresponding norm $\|A\|_F^2 = \langle A, A\rangle = \sum_{i=1}^{p}\sum_{j=1}^{m}|a_{ij}|^2$. Then $\|x'\Delta y\|_\infty = \epsilon^*$ can be written as $|x'(j\omega)\Delta(j\omega)y(j\omega)| = \epsilon^*$ for all $\omega \in \mathcal{R}$, or equivalently $|\langle \Delta(j\omega), x(-j\omega)y'(-j\omega)\rangle| = \epsilon^*$ for all $\omega \in \mathcal{R}$.

Remark 4.4 Lemma 4.2 shows that all uniformly destabilising perpurbations Δ are constrained to have a projection (uniformly in frequency) equal to $\epsilon^\star$ along the *fixed* direction defined by vectors x and y. This means that it is impossible to extend the robust stability radius along this direction, using a subset of all maximally robust controllers $\mathcal{K}_1$ (assume that we still want to stabilize all $\Delta \in \mathcal{B}_{\epsilon^*}\mathcal{H}_\infty^+$. Moreover, all frequencies are equally critical, in the sense that we can construct uniformly destabilising perturbations such that the generalised Nyquist criterion is violated at an arbitrary frequency.

To motivate the formulation of an optimization problem which allows us to extend the robust stability radius in all directions (other than the "most critical" direction), consider the following "distance to singularity" problem:

Let A be a $n\times n$ complex non-singular matrix with singular value decomposition $A = U\Sigma V^* = \sum_{i=1}^{n}\sigma_i u_i v_i^*$ with $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$, $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_{n-2} \geq \sigma_{n-1} > \sigma_n > 0$. What is the minimum norm perturbation $\|E\|$ such that $A - E$ is singular? It is well known that the unique solution is given by the rank 1 matrix $E_o = \sigma_n u_n v_n^*$ so that $\|E_o\| = \sigma_n$. Thus in this case $u_n^\star E_o v_n = \sigma_n$ or $\langle u_n v_n^*, E_o\rangle = \sigma_n$. Thus E_o has a projection σ_n in the most critical direction $\langle u_n v_n^*, \cdot\rangle$. Suppose now that we constrain the magnitude of the projection of allowable perturbations in this direction, i.e. impose the restriction that $|\langle u_n v_n^*, E_o\rangle| \leq \phi$ for some non-negative constant $\phi \leq \sigma_n$. Since now the new minimum-norm singularizing perturbation cannot have a projection of magnitude σ_n in the most-critical direction, we expect the constrained optimal distance to singularity $\gamma(\phi)$ to be larger than σ_n; further, the tighter the constraint (ϕ decreases), the more $\gamma(\phi)$ should deviate from σ_n. The full solution to the problem is provided by the following Lemma.

Lemma 4.3 *Let A be a square non-singular complex matrix which has a singular value decomposition $A = U\Sigma V^* = \sum_{i=1}^{n}\sigma_i u_i v_i^*$, where $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$, $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_{n-2} \geq \sigma_{n-1} > \sigma_n > 0$. Then all E which minimize*

$$\gamma(\phi) = \min\{\|E\| : det(A - E) = 0, |\langle u_n v_n^*, E\rangle| \leq \phi \leq \sigma_n\}$$

are given by:

$$E = U\begin{bmatrix} \phi & \nu & 0 \\ \nu^* & -\phi & 0 \\ 0 & 0 & P_s \end{bmatrix} V^*$$

where P_s is arbitrary except for the constraint $\|P_s\| \leq \sqrt{\sigma_n\sigma_{n-1} + \phi(\sigma_n - \sigma_{n-1})}$ and ν is given by $\nu = \sqrt{(\phi + \sigma_{n-1})(\sigma_n - \phi)}e^{j\theta}$, $\theta \in [0, 2\pi)$. The minimum value of $\gamma(\phi)$ is given by the right-hand side of the equation involving the constraint on $\|P_s\|$.

Proof. *See [LCLS+ 84]. For a number of generalizations to the problem see [JHMG06].*

Remark 4.5 In the above formulation of the problem σ_n and σ_{n-1} are fixed and so the constrained distance to singularity $\gamma(\phi)$ is a function only of ϕ. Suppose that somehow we could influence the level of σ_{n-1}, assuming that σ_n and ϕ are fixed. Then, in order to maximize $\gamma(\phi)$, we would have to maximize σ_{n-1}, i.e. make the gap $\sigma_{n-1} - \sigma_n$ as large as possible, an observation which motivates super-optimization.

Motivated by the above result we proceed as follows: Suppose we impose a structure on the permissible uncertainty set, by defining the set:

$$\mathcal{E}(\delta, \mu) = \left\{\Delta \in \mathcal{B}_\mu\mathcal{H}_\infty^+ : \|x'\Delta y\|_\infty \leq (1-\delta)\epsilon^\star\right\}$$

Then we formulate the following optimization problem:

Constrained maximum robust stabilization (CMRS): For a fixed δ, $0 \leq \delta \leq 1$, find all $K \in \mathcal{K}$ that solve: $\sup\{\mu : (G+\Delta, K) \in \mathcal{S}$ for all $\Delta \in \mathcal{E}(\delta,\mu) \cup \mathcal{B}_{\epsilon^*}\mathcal{H}_\infty^+\}$ and the corresponding value of the supremum $\mu = \mu^\star(\delta)$.

Remark 4.6 (i) Note that since we still require that all $\Delta \in \mathcal{B}_{\epsilon^*}\mathcal{H}_\infty^+$ are stabilised, the set of optimal controllers which solve CMRS must be a subset of $\mathcal{K}_1$. (ii) When $\delta = 0$ the constraint $\|x'\Delta y\|_\infty \leq (1-\delta)\epsilon^\star$ is redundant (i.e. no structure is imposed) and thus $\mathcal{E}(0,\mu) = \mathcal{B}_{\epsilon^*}\mathcal{H}_\infty^+$; hence in this case the solution to the CMRS problem is trivial and is given by $\mathcal{K}_1$ and $\mu^\star(0) = \epsilon^\star$.

The solution of the CMRS problem is summarized in the following Theorem which is the main result of the paper. The Theorem is stated without a proof due to lack of space. Note that (s_1, s_2) denote the first two super-optimal levels of R and we assume that $s_1 > s_2$. Further, $\mathcal{K}_1$ denotes the set of all optimal (maximally robust) controllers and $\mathcal{K}_2$ the set of all super-optimal controllers with respect to the first two levels, so that $\mathcal{K}_2 \subseteq \mathcal{K}_1$.

Theorem 4.2 *In previously defined notation:*

1 For each $\delta \in [0,1]$,

$$\mu^*(\delta) = \sqrt{\frac{1}{s_1}\left(\frac{\delta}{s_2} + \frac{1-\delta}{s_1}\right)} \geq \epsilon^\star$$

with equality only in the case $\delta = 0$. Here s_1 and s_2 are the first two (distinct) super-optimal levels of R with $s_1 = (\epsilon^*)^{-1}$.

2 For each $0 < \delta \leq 1$ *the following two statements are equivalent:*

(a) $(G + \Delta, K) \in \mathcal{S}$ *for every* $\Delta \in \mathcal{E}(\delta, \mu^*(\delta)) \cup \mathcal{B}_{\epsilon^*}\mathcal{H}_\infty^+$

(b) $K \in \mathcal{K}_2$ *(the set of all super-optimal controllers with respect to the first two levels).*

3 (a) $\mathcal{E}(0, \mu^*(0)) = \mathcal{B}_{\epsilon^*}\mathcal{H}_\infty^+$.

(b) *For each* $K \in \mathcal{K}_2$, $(G + \Delta, K) \in \mathcal{S}$ *for every* $\Delta \in \bigcup_{\delta\in[0,1]} \mathcal{E}(\delta, \mu^*(\delta))$.

4 If $\sigma_n < \sigma_{n-1}$ *are the two smallest Hankel singular value of* G*, then*

$$\mu^*(\delta) \geq \sqrt{\delta\sigma_n\sigma_{n-1} + (1-\delta)\sigma_n^2}$$

Proof. *See [GHJ00].*

Remark 4.7 (i) As expected the constrained robust stability radius $\mu^\star(\delta)$ is a strictly increasing function of δ with $\mu^\star(0) = \epsilon^\star$. Moreover, for a fixed $\delta \neq 0$ and s_1, $\mu^\star(\delta)$ is a decreasing function of s_2. Thus the structured robust stability radius $\mu^\star(\delta)$ increases with an increasing gap between the first two super-optimal levels. (ii) For each $\delta \neq 0$ the set of optimal controllers is the same, namely $\mathcal{K}_2$. Thus each super-optimal controller guarantees the stability of all perturbations in the union of the sets $\bigcup_{\delta\in[0,1]} \mathcal{E}(\delta, \mu^*(\delta))$ which contains the the ball of radius $\epsilon^\star$ as a subset.

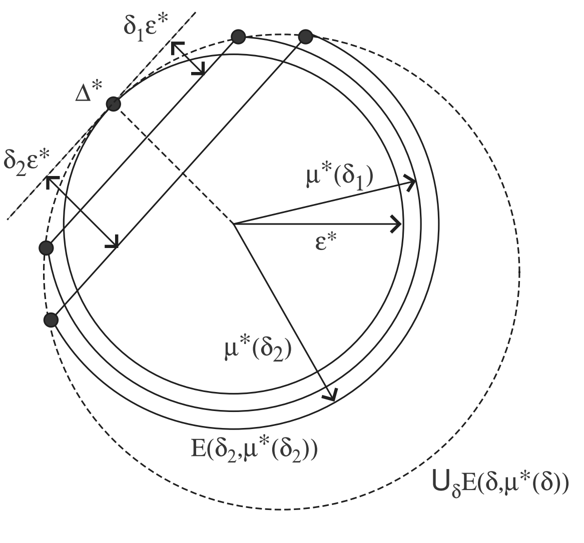

Figure 2: Constrained robust stability radii

Remark 4.8 When the model uncertainty set is unstructured, the above theorem shows that by using a superoptimal controller (with respect to the first two levels) guarantees robust stabilization for a larger class of uncertainties ($\bigcup_{\delta \in [0,1]} \mathcal{E}(\delta, \mu^*(\delta))$) compared to the class guaranteed to be stabilised by using an arbitrary optimal controller ($\mathcal{B}_{\epsilon^*}\mathcal{H}_\infty^+$). In the case when the uncertainty set is structured, we can give the following interpretation of the theorem: Consider a normalised structured uncertainty set $\mathcal{B}\Delta_\mathcal{S} = \{\Delta \in \mathcal{S} : \|\Delta\| \le 1\} \subseteq \mathcal{B}\mathcal{H}_\infty$, where $\mathcal{S}$ is an arbitrary structure. Suppose that we can determine the maximum value of δ in the interval $[0,1]$, say $\delta^\star$, such that $\|x'\Delta y\|_\infty \le 1 - \delta^\star$ for all $\Delta \in \mathcal{B}\Delta_\mathcal{S}$. Since there exists a controller $K \in \mathcal{K}_2$ which stabilises all Δ such that (i) $\|\Delta\|_\infty < \mu^\star(\delta^\star)$ and (ii) $\|x'\Delta y\|_\infty \le \epsilon^\star(1-\delta^\star)$, then $\mu^\star(\delta^\star)$ is a lower bound of the robust stability radius relative to structure $\mathcal{S}$, which is tighter than the "unstructured" bound, i.e. $\mu^\star(\delta^\star) > \epsilon^\star$, provided $\delta^\star \neq 0$. This approach can be used to breach the convex upper bound of the structured singular value under complex block-structured uncertainties (see [JHMG06] for details).

Example 4.1 Consider the unit-ball of the diagonal structure $\mathcal{D}\mathcal{H}_\infty^+$,

$$\mathcal{B}\mathcal{D}\mathcal{H}_\infty^+ = \{\Delta = \mathrm{diag}(\delta_1(s), \delta_2(s), \ldots, \delta_n(s)) : \delta_i(s) \in \mathcal{R}\mathcal{H}_\infty^+, \|\delta_i(s)\|_\infty \le 1\}$$

Assume that $s_1(R) > s_2(R) > 0$ and let $x_i(s)$ and $y_i(s)$ denote the i-th element od x and y, respectively. Let also $a_i(s) = x_i(s)y_i(s)$, $i = 1, 2, \ldots, n$. Then:

$$\begin{aligned}\max_{\Delta \in s_1^{-1}\mathcal{B}\mathcal{D}\mathcal{H}_\infty^+} \|x'\Delta y\|_\infty &= \max_{|\delta_i| < s_i^{-1}} \max_{\omega \in \mathcal{R}} \frac{1}{s_1} \left|\sum_{i=1}^n \delta_i(j\omega) a_i(j\omega)\right| \\ &= \frac{1}{s_1} \max_{\omega \in \mathcal{R}} \max_{\phi_i \in [0, 2\pi)} \left|\sum_{i=1}^n e^{j\phi_i} a_i(j\omega)\right| \\ &= \frac{1}{s_1} \max_{\omega \in \mathcal{R}} \left|\sum_{i=1}^n a_i(j\omega)\right| := \frac{\gamma_{\max}}{s_1}\end{aligned}$$

Note that the Cauchy-Schwartz inequality implies that:

$$\left|\sum_{i=1}^n a_i(j\omega)\right|^2 = \left|\sum_{i=1}^n x_i(j\omega) y_i(j\omega)\right|^2 \le \left(\sum_{i=1}^n |x_i(j\omega)|^2\right)\left(\sum_{i=1}^n |y_i(j\omega)|^2\right) = 1$$

and hence $\gamma_{\max} \le 1$. Setting $\delta^* = 1 - \gamma_{\max}$, and using Theorem 4.2 shows that

$$\mu^*(\delta^*) = \sqrt{\frac{1}{s_1}\left(\frac{\delta^*}{s_2} + \frac{1-\delta^*}{s_1}\right)} = \sqrt{\frac{1}{s_1}\left(\frac{1-\gamma_{\max}}{s_2} + \frac{\gamma_{\max}}{s_1}\right)}$$

is a lower bound of the structured robust stability radius relative to $\mathcal{D}\mathcal{H}_\infty^+$.

5. Breaching the convex bound of the structured singular value

In this section we use the results developed in the last two sections to derive a method for breaching the convex upper bound of the structured singular value in the case of block-diagonal complex uncertainty [Sm97], [PD93], [FT88]. The efficient calculation of the structured singular value is currently one of the main bottlenecks of robust control.

Let $M \in \mathcal{C}^{n \times n}$, denote by $\mathbf{\Delta}$ a complex block-diagonal (including repeated scalar blocks) structured uncertainty set and define:

$$\mathcal{B}\mathbf{\Delta} = \{\Delta \in \mathbf{\Delta} : \|\Delta\| \leq 1\}$$

The complex structure singular value of M is defined as:

$$\mu_{\mathbf{\Delta}}^{-1}(M) = r_{\mathbf{\Delta}}(M) = \min\{\Delta \in \mathbf{\Delta} : \det(I - \Delta M) = 0\}$$

where $r_{\mathbf{\Delta}}(M)$ is the structured-distance to singularity relative to structure $\mathbf{\Delta}$. It is well-known that the computation of $\mu_{\mathbf{\Delta}}$ is an NP-hard problem. To get (a convex) upper bound define complementary structure commuting with $\mathbf{\Delta}$:

$$\mathbf{D} = \{D \in \mathcal{C}^{n \times n} : D = D' > 0, D\Delta = \Delta D \text{ for all } \Delta \in \mathbf{\Delta}\}$$

Then:

$$\mu_{\mathbf{\Delta}}(M) \leq \inf_{D \in \mathbf{D}} \|D^{1/2} M D^{-1/2}\| := \gamma_0$$

Since

$$\|D^{1/2} M D^{-1/2}\| \leq \gamma \Leftrightarrow \gamma^2 D - M' D M \geq 0$$

calculation of γ_0 is a convex problem (e.g. solved via Linear Matrix Inequalities). Suppose the infimising $D = D_0 > 0$ (the case when the infimising D is singular can also be taken into account). Redefine: $M \leftarrow \gamma_0^{-1} D_0^{1/2} M D_0^{-1/2}$. Then if the largest singular value of M in non-repeated, $\mu_{\mathbf{\Delta}}(M) = 1$ [?]. Hence we assume multiplicities in the singular values of M. We treat simultaneously the cases when:

(i) M is redefined at output of D-iteration (useful for breaching convex bound).

(ii) General M's (scaled as $M \leftarrow \|M\|^{-1} M$).

Let $M \in \mathcal{C}^{n \times n}$ have an ordered singular value decomposition:

$$M = U \Sigma V' = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} I_m & 0 \\ 0 & \Sigma_2 \end{pmatrix} \begin{pmatrix} V_1' \\ V_2' \end{pmatrix}$$

with $\Sigma_2 = \mathrm{diag}(\sigma_{m+1}, \ldots, \sigma_n)$ where $\sigma_{m+1} < 1$. Define $A = \Sigma^{-1} = \mathrm{diag}(I_m, A_2)$ where $A_2 = \mathrm{diag}(a_{m+1}, \ldots, a_n)$ with $1 < a_{m+1} \leq \cdots \leq a_n$. Suppose we can establish (non-trivial) bounds:

$$\phi_1 := \max\{\rho(V_1' \Delta U_1) : \Delta \in \mathcal{B}\mathbf{\Delta}\} \leq \bar{\phi}_1 \leq 1$$
$$\phi_2 := \max\{\|V_1' \Delta U_1\| : \Delta \in \mathcal{B}\mathbf{\Delta}\} \leq \bar{\phi}_2 \leq 1$$

Then,

$$\begin{aligned} r_{\mathbf{\Delta}}(M) &= \min\{\|\Delta\| : \Delta \in \mathbf{\Delta}, \det(I - \Delta M) = 0\} \\ &= \min\{\|\Delta\| : \Delta \in \mathbf{\Delta}, \det(A - V'\Delta U) = 0\} \\ &\geq \min\{\|\Delta\| : \rho(E_1'\Delta E_1) \leq \bar{\phi}_1\|\Delta\|, \|E_1'\Delta E_1\| \leq \bar{\phi}_2\|\Delta\|, \det(A - \Delta) = 0\} \\ &:= r_{\mathbf{\Delta}}(M) \end{aligned}$$

Here E_1 is the matrix formed my the first m columns of I_n. Clearly, the tightest possible bound is obtained if we use ϕ_1 and ϕ_2 in place of $\bar{\phi}_1$ and $\bar{\phi}_2$, respectively. Hence, to establish the bound $r_{\mathbf{\Delta}}(M)$ we need (i) to calculate ϕ_1 and ϕ_2 (or at least $\bar{\phi}_1$ and $\bar{\phi}_2$ less than or equal to 1), and (ii) solve the optimisation problem:

$$\min\{\|\Delta\| : \rho(E_1'\Delta E_1) \leq \bar{\phi}_1\|\Delta\|, \|E_1'\Delta E_1\| \leq \bar{\phi}_2\|\Delta\|, \det(A - \Delta) = 0\} \tag{18}$$

To calculate ϕ_1 (or $\bar{\phi}_1$) note that:

$$\max_{\Delta \in \mathcal{B}\mathbf{\Delta}} \rho(V_1'\Delta U_1) = \max_{\Delta \in \mathcal{B}\mathbf{\Delta}} \rho(\Delta U_1 V_1') := \max_{\Delta \in \mathcal{B}\mathbf{\Delta}} \rho(\Delta M_0) = \mu_{\mathbf{\Delta}}(M_0)$$

and hence calculation of ϕ_1 is a reduced-rank μ-problem [PD93], [FT88]. Some progress for solving certain classes of reduced-rank μ problems is reported in [Br94].

The calculation of ϕ_2 can be performed using the following result:

Lemma 5.1 *In previous notation,*

$$\phi_2^2 = \max_{\Delta \in \mathcal{B}\mathbf{\Delta}} \|V_1'\Delta U_1\|^2 = \min_{D \in \mathcal{D},\ D - V_1 V_1' \geq 0,\ \gamma I - U_1' D U_1 \geq 0} \gamma$$

Furthermore,

$$\inf_{D \in \mathcal{D}} \|D^{1/2} M D^{-1/2}\| = 1 \quad \Rightarrow \quad \phi_2 = 1$$

Proof. See [JHMG06].

Remark 5.1 The first part of the Lemma shows that the calculation of ϕ_2 reduces to a convex optimization problem which can be solved via efficient computation techniques (e.g. Linear Matrix Inequalities). The second part shows that if M results from the output of the D-iteration then $\phi_2 = 1$.

The calculation of $r_{\mathbf{\Delta}}$ via the solution of optimization problem (18) is a challenging problem which is addressed in [JHMG06]. This is achieved via a lengthy procedure by solving a sequence of distance-to-singularity problems of increased complexity of the form:

$$\gamma_{\mathbf{\Delta_{11}}} = \min\{\|\Delta\|, \det(A - \Delta) = 0, E_1'\Delta E_1 \in \mathbf{\Delta_{11}}\}$$

The sequence of problems solved in [JHMG06] include:

(i) $\mathbf{\Delta_{11}} = \mathcal{C}^{n\times n}$ (unconstrained distance to singularity)

(ii) $\mathbf{\Delta_{11}} = \{\delta \in \mathcal{C} : |\delta| \leq \phi\}, 0 \leq \phi \leq 1$ [LCLS+84]

(iii) $\mathbf{\Delta_{11}} = \mathcal{C}^{m\times m}$

(iv) $\mathbf{\Delta_{11}} = \{0_{m\times m}\}$

(v) $\mathbf{\Delta_{11}} = \{\Delta_{11}\}, \|\Delta_{11}\| \leq 1, \det(I_m - \Delta_{11}) \neq 0$

(vi) $\mathbf{\Delta_{11}} = \{\Delta_{11} : \|\Delta_{11}\| \leq 1, 1 \notin \lambda(\Delta_{11})\}$

(vii) $\mathbf{\Delta_{11}} = \{\Delta_{11} : \rho(\Delta_{11}) \leq \phi_1, \|\Delta_{11}\| \leq \phi_2\}, \phi_1 \leq \phi_2$

The main result obtained by solving this sequence of problems is stated next:

Theorem 5.1 *Assume that* $0\leq\bar{\phi}_1\leq\bar{\phi}_2\leq 1$. *Then*

$$\begin{aligned} r_{\mathbf{\Delta}} &= \min\{\|\Delta\| : \det(A-\Delta)=0, \rho({E_1}'\Delta E_1) \leq \bar{\phi}_1\|\Delta\|, \|{E_1}'\Delta E_1\| \leq \bar{\phi}_2\|\Delta\|\} \\ &= \min\{\gamma > 1 : \|(\gamma I - \Delta^m_{\bar{\phi}_1,\bar{\phi}_2})(\gamma^{-1} I - \Delta^m_{\bar{\phi}_1,\bar{\phi}_2})^{-1}\| = a_{m+1}\} \end{aligned}$$

and is increasing in α_{m+1}, *where* $\Delta^m_{\bar{\phi}_1,\bar{\phi}_2}$ *is a Toeplitz matrix defined as:*

$$\Delta^m_{\phi_1,\phi_2}(i,j) = \left\{ \begin{array}{cc} 0, & j<i \\ \bar{\phi}_1, & j=i \\ (-\frac{\bar{\phi}_1}{\bar{\phi}_2})^{j-i-1}\frac{\bar{\phi}_2{}^2-\bar{\phi}_1{}^2}{\bar{\phi}_2}, & j>i \end{array} \right\}$$

e.g. for $m=4$,

$$\Delta^m_{\phi_1,\phi_2} = \begin{bmatrix} \phi_1 & \alpha & \beta & \gamma \\ 0 & \phi_1 & \alpha & \beta \\ 0 & 0 & \phi_1 & \alpha \\ 0 & 0 & 0 & \phi_1 \end{bmatrix}$$

with

$$\alpha = \frac{\phi_2^2-\phi_1^2}{\phi_2}, \beta = -\frac{\phi_1}{\phi_2}\frac{\phi_2^2-\phi_1^2}{\phi_2}, \gamma = (\frac{\phi_1}{\phi_2})^2\frac{\phi_2^2-\phi_1^2}{\phi_2}$$

Furthermore,

$$\bar{\phi}_1 < 1 \Rightarrow r_{\mathbf{\Delta}}(M) > 1 \quad \textit{and} \quad r_{\mathbf{\Delta}}(M) > 1.$$

Finally, $r_{\mathbf{\Delta}}(M) = r_{\mathbf{\Delta}}(M)$ *if and only if there exists* $\Delta \in \mathbf{\Delta}$ *such that*

$$V_1'\Delta U_1 = [r_{\mathbf{\Delta}}(M)]^{-1} W'\Delta^m_{\bar{\phi}_1,\bar{\phi}_2} W,$$

for some unitary W.

Proof. See [JHMG06]

Remark 5.2 The evaluation of $r_{\mathbf{\Delta}}(M)$ is an eigenvalue problem of dimension $4m \times 4m$. Let $a = a_{m+1}$, $\Psi = \Delta^m_{\bar{\phi}_1,\bar{\phi}_2}$ and $\Psi_s = \Psi + \Psi'$. Then

$$a = \|(\gamma I - \Psi)(\gamma^{-1} I - \Psi)^{-1}\|$$
$$\Rightarrow \det\left(\gamma I - \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ a^2 I & -a^2\Psi_s & (a^2-1)\Psi'\Psi & \Psi_s \end{bmatrix}\right) = 0$$

Thus γ is the smallest real eigenvalue larger than 1 of a $4m \times 4m$ matrix.

The results of the section can be summarized in the following algorithm which may be used to breach the convex upper bound of the structured singular value:

Algorithm 1:

(i) "D-iteration": $\gamma_0 := \min_{D\in\mathbf{D}} \|D^{1/2} M D^{-1/2}\|$ (LMI).

(ii) Re-define: $M \leftarrow \gamma_0^{-1} D_0^{1/2} M D_0^{-1/2}$.

(iii) Perform SVD of $M = U_1 V_1' + U_2 \Sigma_2 V_2'$. Set m=multiplicity of largest singular value, $A = \text{diag}(I_m, \Sigma_2^{-1})$.

(iv) If $m = 1$ there is no gap, i.e. $\mu_{\mathbf{\Delta}}(M) = \gamma_0$ - Exit.

(v) Find tightest possible bound $\bar{\phi}_1$ of $\mu_{\mathbf{\Delta}}(U_1 V_1')$ (m-rank). If $\bar{\phi}_1 = 1$ no improvement possible - Exit.

(vi) Set $\bar{\phi}_2 = 1$.

(vii) Form $\Delta^m_{\bar{\phi}_1,\bar{\phi}_2}$ and solve corresponding $4m \times 4m$ eigenvalue problem to find: $r_{\mathbf{\Delta}}(M) = \bar{\mu}_{\mathbf{\Delta}}^{-1}(M)$.

(viii) Reverse scaling: $\bar{\mu}_{\mathbf{\Delta}}(M) \leftarrow \gamma_0 \bar{\mu}_{\mathbf{\Delta}}(M)$.

The general method followed in this section for breaching the convex upper bound resembles similar methods developed by the authors for other problems where convex relaxations are used. To emphasize these similarities, consider the quadratic integer programming (QIP) problem and its semi-definite relaxation:

$$\gamma := \max_{x\in\{-1,1\}^n} x'Qx \leq \min_{D-Q\geq 0, D=\text{diag}(D)} \text{trace}(D) := \bar{\gamma}$$

where $Q = Q' \in \mathcal{R}^{n\times n}$. We refer to the maximization on the LHS of this inequality as the *primal problem* and to the minimisation on the RHS as the *dual*. Clearly, the computational complexity of the primal problem grows exponentially with n since it requires 2^{n-1} evaluations. It can be shown that (see [Mal06], [HJM]):

(i) The optimal solution $D = D_0$ of the dual problem is unique.

(ii) $\dim \mathrm{Ker}(D_0 - Q) = 1 \Rightarrow \gamma = \bar{\gamma}$

(iii) Let $V \in \mathcal{R}^{n \times m}$, $V'V = I_m$, whose columns span $\mathrm{Ker}(D_0 - Q)$ (potentially $m \ll n$). Then:

$$\gamma = \bar{\gamma} \Leftrightarrow \gamma_m := \frac{1}{n} \max_{x \in \{-1,1\}^n} x'VV'x = 1$$

(iv) Introduce an appropriate row perturbation matrix P so that $PV = [V_1' \ V_2']'$ with $\det(V_1) \neq 0$. Then:

$$\gamma = \bar{\gamma} \Leftrightarrow \exists\, z \in \{-1,1\}^m \,:\, V_2 V_1^{-1} z \in \{-1,1\}^{n-m}$$

("certificate" of zero duality gap requiring 2^m function evaluations).

(v) Let λ_+ smallest positive e-value of $D_0 - Q$. Then:

$$\gamma \leq \bar{\gamma} - n(1 - \gamma_m)\lambda_+ < \bar{\gamma}$$

(vi) For fixed m, solution of

$$n\gamma_n = \max_{x \in \{-1,1\}^n} x'VV'x$$

is of complexity $O(n^m)$ and can be solved efficiently (for low m) using properties of zonotopes and hyperplane arrangements [AF96].

6. Conclusions

The paper has proposed a concrete linear-algebraic approach for solving the super-optimal Nehari approximation problem without unnecessary assumptions and has considered some of its applications in the area of robust multivariable control. It was shown that the maximum robust stabilisation problem subject to additive unstructured perturbations can be reduced to a Nehari approximation, the solution of which gives the maximum robust stability radius and a complete parametrisation of all optimal (maximally robust) controllers. By analysing the properties of the optimal solution, a "worst-case" direction was identified in the uncertainty space, along which all boundary uniformly-destabilizing perturbations were shown to lie. By imposing a parametric constraint on the projection of admissible perturbations along this direction (uniformly in frequency), it was shown that it is possible to extend the robust stability radius in every other direction, using a subset of all optimal (maximally-robust) controllers. This involves the solution of a two-level super-optimal Nehari approximation and leads to a closed-form expression of the extended (direction-dependent) robust stability radius involving the first two super-optimal levels of the system which is approximated. An alternative interpretation of the main results has allowed us to develop a systematic method for breaching the convex upper bound of the structured singular value in the case of complex block-diagonal uncertainty structures. This relies on a novel convex relaxation technique which is potentially applicable to a wide class of non-convex optimisation problems.

References

[AF96] D. Avis and K. Fukuda *Reverse search for enumeration.* Discrete Apl. Math., 65:21–46, 1996.

[Br94] R.D. Braatz, P.M. Young, J.C. Doyle, and M. Morari. *Computational complexity of μ calculation.* IEEE Trans. Automatic Control, 39(5):1000–1002, May 1994.

[FT88] M.K.H. Fan and A.L. Tits *m-form numerical range and the computation of the structured singular value.* IEEE Trans. Automatic Control, 33(3):284–289, March 1988.

[Fra87] B. A. Francis *A Course in $\mathcal{H}_\infty$ Control Theory.* Lecture Notes in Control and Information Sciences, Vol. 88, Springer-Verlag, New York, 1987.

[GHJ00] S. K. Gungah, G. D. Halikias, and I. M. Jaimoukha *Maximally robust controllers for multivariable systems.* SIAM journal of Control and Optimization, 38:6(2000), 1805–1829.

[Glo84] K. Glover *All optimal Hankel-norm approximations of linear multivariable systems and their $\mathcal{L}_\infty$ error bounds.* International Journal of Control, 39(1984), 1115–1193.

[Glo86] K. Glover *Robust stabilization of linear multivariable systems: Relations to approximation.* International Journal of Control, 43(1986), 741–766.

[GL95] M. Green and David J. N. Limebeer *Linear robust control.* Information and System Sciences Series, Prentice-Hall, Englewood, 1995.

[HLG93] G.D. Halikias, D.J.N. Limebeer and K. Glover *A State-Space Algorithm for the Super-Optimal Hankel-Norm Approximation Problem.* SIAM journal of Control and Optimization, 31:4(1993), 960-982.

[HJ98] G. D. Halikias and I.M. Jaimoukha *The two-block superoptimal AAK problem. Mathematics of Control, Signals and Systems*, 11(1998), 244–264.

[HJM] G.D. Halikias, I.M. Jaimoukha and U. Malik *New bounds on the unconstrained integer programming problem.* Journal of Global Optimization (submitted), 2006.

[JHMG06] I. M. Jaimoukha, G.D. Halikias, U. Malik, and S.K. Gungah *On the gap between the complex structured singular value and its convex upper bound.* SIAM journal of Control and Optimization, 45:4(2006), 1251–1278.

[JL93] I. M. Jaimoukha and D. J. N. Limebeer *A state-space algorithm for the solution of the two-block superoptimal problem.* SIAM Journal of Control and Optimization 31:5(1993), 1115–1134.

[KHJ] J. Kiskiras, G.D. Halikias and I.M. Jaimoukha *Robust stabilization of multivariable systems under coprime factor perturbations.* European Control Conference ECC07 (submitted), 2007.

[LCLS+84] N.A. Lechtomaki, D.A. Castanon, B.C. Levy, G. Stein, N.R. Sandell, and M. Athans *Robustness and modelling error characterization.* IEEE Transactions on Automatic Control, AC-29:3(1984), 212–220.

[LHG89] D. J. N. Limebeer, G.D. Halikias and K. Glover, *A state-space algorithm for the computation of super-optimal matrix interpolating functions.* International Journal of Control, 50:6(1989), 2431–2466.

[Mal06] U. Malik, I.M. Jaimoukha, G.D. Halikias, and S.K. Gungah. *On the gap between the quadratic integer programming problem and its semidefinite relaxation.* Mathematical Programming, Series A, 107:3(2006),505–515.

[Nym99] P.O. Nyman *Multidirectional optimal robustness under gap and coprime factor uncertainties.* Proceedings of the 38th Conf. on Decision & Control, (1999), 3617–3620.

[PD93] A. Packard and J.C. Doyle *The complex structured singular value.* Automatica, 21(1):71–109, 1993.

[Pel03] V.V. Peller *Hankel Operators and their Applications.* Springer Monographs in Mathematics, Springer-Verlag, New York (2003).

[Sm97] R. Smith. *A study of the gap between the structured singular value and its convex upper bound for low–rank matrices.* IEEE Trans. Automatic Control, AC-42:8(1997), 1176–1179.

[Vid85] M. Vidyasagar *Control system synthesis: A factorization appoach.* Series in Signal Processing, Optimization and Control, vol. 7, MIT Press, Cambridge, MA, 1985.

[You86] N.J. Young *The Nevanlinna-Pick problem for matrix-valued functions.* Journal of Operator Theory, 15(1986), 239–265.

[ZDG96] K. Zhou, J. C. Doyle, and K. Glover *Robust and Optimal control* Prentice-Hall, 1996.

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