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Algebraic Solutions of Some Partial Differential Equations Arising in Financial Mathematics and Related Areas

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Abstract

The symmetry analysis of both linear and nonlinear evolution equations arising in Financial Mathematics and related areas can lead to rich results and provide a route to the solution of a variety of problems which are expressed as evolution partial differential equations with terminal conditions.

Keywords: Financial Mathematics, Lie Symmetries, Partial Differential Equations

1. Introduction

The use of Lie symmetries for the resolution of differential equations, be they ordinary or partial, is well-known. Perhaps it is surprising that the employment of symmetries is not universal. In Physics and Mechanics it is commonplace. In Biology and Economics not only are the methods largely unknown but even their introduction is resisted with an enthusiasm which by comparison makes the Luddites of nineteenth-century England at the cutting-edge of innovation.

In this paper we concentrate on some of the partial differential equations which arise in Financial Mathematics which is a field that becomes increasingly attractive to students dreaming of a career in Mathematics combined with an unmathematical style of living. So far the field has been almost completely successful in avoiding formal considerations of symmetry even though many writers have been successful in solving the evolution partial differential equations arising and so have been benefitting from the presence of underlying symmetry.

To our knowledge the first application of Lie's symmetry analysis in Financial Mathematics was by Gazizov and Ibragimov [5] to the famous Black-Scholes equation [2, 3]

$$\frac{\partial u}{\partial t} + \tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0 \tag{1.1}$$

subject to the initial condition

$$u(0,x) = \delta(x - x_0). \tag{1.2}$$

Guided by the number of Lie point symmetries possessed by (1.1) Gazizov and Ibragimov transformed it to the classical heat equation, *videlicet*

$$\frac{\partial \Phi}{\partial t} = \frac{\partial^2 \Phi}{\partial \zeta^2}, \quad \Phi(0,\zeta) = \delta(), \tag{1.3}$$

where the argument of the delta function was expressed in terms of the transformed variables, so that the problem solved was the standard one of the heat equation. More recently [1] we presented the solution to (1.1) with the more usual conditions, $u(0,x) = U_0$ and $u(T,x) = U$, the latter of which is generally described as a terminal condition.

A central theme of financial Mathematics is the purchase or sale of an option in the hope that its exercise at some future time will lead to a profit. The time (T) and price (U) at the exercise of the option are fixed. The value of the option varies with the fluctuations in the price of the stock, x, between the taking of the option ($t = 0$) and its exercise. The value of the stock is taken to vary according to some stochastic equation, such as

$$\mathrm{d}x = x\mu \mathrm{d}t + \sigma x \mathrm{d}B_t \tag{1.4}$$

for the Black-Scholes equation, where the first part is simply deterministic Malthusian growth and the second part the stochastic variation which is based on the Brownian version of a random walk. An application of Îto's Lemma leads to the Black-Scholes equation (1.1).

Equation (1.4) is an essential component of the modelling of the problem. Black and Scholes used Brownian motion and this leads to a Gaussian distribution for the probability function to describe the fluctuation. As a model for 'random walks' as they occur in reality Brownian motion has been demonstrated to have its failings. One such instance of failure is with distributions which have fat tails. Other models, such as the Levy process, have been developed to reflect these aspects and it is more than likely that several are not yet part of the public domain. Once one has the equivalent the development of the associated partial differential equation follows.

The algebraic study of the evolution partial differential equations arising in Financial Mathematics is motivated by the use of the symmetries to determine the solution. However, as is the nature of Mathematics, the very existence of the symmetries can lead one to mathematical explorations which doubtless will never earn one the obol.

One should emphasise that not all of the equations of Financial Mathematics are linear and indeed we use a nonlinear equation to illustrate the methodology of the solution of the partial differential equation plus terminal condition using Lie methods.

The paper is structured thusly. In §2 we recall some of the basic results of the existence of Lie point symmetries for evolution partial differential equations and indicate a certain incompleteness in our knowledge. Then we show how to solve a given problem using the techniques of symmetry analysis. In §3 we present the solution of the famous Black-Scholes equation by using the symmetries the equation possesses. In §4 we examine a succession of higher-dimensional equations for the modelling of the price of a commodity and are intrigued by the appearance of a hierarchy of algebras paralleling the hierarchy of dimensions. We present some final comments in §5.

One of the features of the equations of Financial Mathematics is that they tend to be over-complicated when stated in terms of the original variables and parameters and it behoves one to seek the simplest form so that the essence of the problem is made manifest.

2. Some basic algebraic properties and an illustrative example

Many of the evolution partial differential equations of Financial Mathematics are linear $1+1$ evolution equations which have the basic form

$$u_t = u_{xx} + V(t,x)u, \tag{2.1}$$

where $V(t,x)$ is some function, which is the classical heat equation with a source term. The analysis of (2.1) for the existence of Lie point symmetries where a number of cases emerges is shown in Table 1.

Table 1

V	number of symmetries	algebra	
$V(t,x)$	$0+1+\infty$	$A_1 \oplus_s \infty A_1$	
$V(x)$	$1+1+\infty$	$2A_1 \oplus_s \infty A_1$	
$\omega^2 x^2 + \dfrac{\mu^2}{x^2}$	$3+1+\infty$	$\{A_1 \oplus sl(2,R)\} \oplus_s \infty A_1$	(2.2)
$\mu^2 + \dfrac{\mu^2}{x^2}$	$3+1+\infty$	$\{A_1 \oplus sl(2,R)\} \oplus_s \infty A_1$	
$\omega^2 x^2$	$5+1+\infty$	$\{sl(2,R) \oplus_s W\} \oplus_s \infty A_1$	
$\mu^2 x$	$5+1+\infty$	$\{sl(2,R) \oplus_s W\} \oplus_s \infty A_1$	
μ^2	$5+1+\infty$	$\{sl(2,R) \oplus_s W\} \oplus_s \infty A_1$,	

where W is the Heisenberg-Weyl algebra with

$$[\Sigma_1, \Sigma_2]_{LB} = 0, \ [\Sigma_1, \Sigma_3]_{LB} = 0, \ [\Sigma_2, \Sigma_3]_{LB} = \Sigma_1.$$

Subsequent transformation, as we have seen, brings us to the various forms of the Black-Scholes equation, not necessarily with $5+1+\infty$ Lie point symmetries which may well have been the attraction of the symmetry analysis of the Black-Scholes equation in the first place.

It is from the symmetries that the evolution partial differential equation possesses that one seeks to find the solution of the equation subject to its condition. It is really only the two cases, *ie* those of $5 + 1 + \infty$ and $3 + 1 + \infty$ Lie point symmetries, for which this procedure is possible. This is for a linear equation.

We illustrate the method of solution with the nonlinear equation

$$w_t + w_{xx} + ww_x - x = 0 \tag{2.3}$$

to emphasise the importance of the nongeneric symmetries. Equation (2.3) is a special case of a more general equation derived in the modelling of continuous-time stochastic saddlepoint systems. According to Miller and Weller [11] a number of classic models may be cast in this form. Examples are Blanchard's model relating stock market prices to the level of real activity in the economy [4] and the model of Krugman for target zones [8].

The Lie point symmetries of (2.3) are

$$\begin{aligned}
\Gamma_1 &= \mathrm{e}^{t}\left(\partial_x + \partial_w\right)\\
\Gamma_2 &= \mathrm{e}^{-t}\left(\partial_x - \partial_w\right)\\
\Gamma_3 &= \partial_t\\
\Gamma_4 &= \mathrm{e}^{2t}\left(\partial_t + x\partial_x + (2x - w)\partial_w\right)\\
\Gamma_5 &= \mathrm{e}^{-2t}\left(\partial_t - x\partial_x + (2x + w)\partial_w\right).
\end{aligned}$$

We consider the terminal problem

$$t = T \quad \text{and} \quad w(T, x) = W. \tag{2.4}$$

Take a general symmetry

$$\Gamma = \sum_{i=1}^{5} \alpha_i \Gamma_i.$$

Its application to the condition $t = T$ gives

$$\alpha_3 + \exp[2T]\alpha_4 + \exp[-2T]\alpha_5 = 0. \tag{2.5}$$

The application of Γ to $w = W$ gives

$$\exp[T]\alpha_1 - \exp[-T]\alpha_2 + (2x - W)\exp[2T]\alpha_4 + (2x + W)\exp[-2T]\alpha_5 = 0. \tag{2.6}$$

Since x is a free variable in (2.6), we have three conditions on the coefficients, *videlicet* (2.5) and

$$\begin{aligned}
\exp[T]\alpha_1 - \exp[-T]\alpha_2 - \exp[2T]\alpha_4 + W\exp[-2T]\alpha_5 &= 0\\
\exp[2T]\alpha_4 + \exp[-2T]\alpha_5 &= 0
\end{aligned}$$

from which it is evident that $\alpha_3 = 0$ and that there is a two-parameter solution for the nontrivial coefficients.

After a little cosmetic manipulation we obtain the two symmetries

$$\begin{aligned}\Lambda_1 &= \cosh(T-t)\partial_x - \sinh(T-t)\partial_w \qquad (2.7)\\ \Lambda_2 &= \cosh 2(T-t)\partial_t - x\sinh 2(T-t)\partial_x + 2x\cosh 2(T-t)\partial_w \\ &\quad + w\sinh 2(T-t)\partial_w + w\exp[-(T-t)]\left(\partial_x + \partial_w\right). \qquad (2.8)\end{aligned}$$

We have two Lie point symmetries for (2.3) plus its terminal condition (2.4). In (2.3) we have three variables, one dependent and two independent. The existence of a symmetry implies a constraint and we can look to the symmetry to provide just one dependent and one independent variable so that an ordinary differential equation results and one can hope that this equation can be solved. We take as new variables the invariants of Λ_1 since it looks the easier to treat. An invariant, v, of Λ_1 satisfies $\Lambda_1 v = 0$. This requirement is

$$\cosh(T-t)\frac{\partial v}{\partial x} - \sinh(T-t)\frac{\partial v}{\partial w} = 0,$$

the characteristics of which are found from the solution of the associated Lagrange's system

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{\cosh(T-t)} = \frac{\mathrm{d}w}{-\sinh(T-t)}. \qquad (2.9)$$

From the first member of (2.9) it is evident that t is one of the invariants. We make use of this in the solution of the equation of obtained from the second and third members of (2.9), *videlicet*

$$0 = \mathrm{d}w + \tanh(T-t)\mathrm{d}x,$$

to obtain the second invariant $w + x\tanh(T-t)$.

The solution of the first-order linear partial differential equation is

$$\Phi(t, w + x\tanh(T-t)) = 0,$$

where Φ is an arbitrary function of its arguments. We 'solve' this equation to write

$$w = -x\tanh(T-t) + f(t) \qquad (2.10)$$

and to substitute this and the derivatives into (2.1). We obtain the linear first-order equation

$$\dot{f} - f\tanh(T-t) = 0$$

which is readily solved to give the solution

$$w(t,x) = K\mathrm{sech}(T-t) - x\tanh(T-t), \qquad (2.11)$$

where K is the constant of integration.

When we apply the terminal condition, $w(T,x) = W$, to (2.11), we see that $K = W$ and so the solution of (2.1) subject to the given terminal condition is

$$w(t,x) = W\operatorname{sech}(T-t) - x\tanh(T-t). \tag{2.12}$$

One may verify that the solution given in (2.12) is also an invariant of Λ_2, *ie* the second symmetry does not produce another solution. This is to be expected since the solution to such a problem is unique according to a theorem due to Feynman and Kac.

The advantage of this example has been that the process towards the determination of the solution has not been obscured by computational complications.

3. The Black-Scholes equation

The Black-Scholes equation, *videlicet*

$$\frac{\partial u}{\partial t} + \tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0, \tag{3.1}$$

subject to the terminal condition

$$u(T,x) = U, \tag{3.2}$$

is the most known of the evolution partial differential equations of the mathematics of finance.

The Lie point symmetries of (3.1) are

$$\begin{aligned}
\Gamma_1 &= x\partial_x \\
\Gamma_2 &= \sigma^2 tx\partial_x + [\log x - \lambda t]\, u\partial_u \\
\Gamma_3 &= \partial_t \\
\Gamma_4 &= 2t\partial_t + [\log x + \lambda t]\, x\partial_x + 2rtu\partial_u \\
\Gamma_5 &= 2\sigma^2 t^2\partial_t + 2\sigma^2 tx\log x\partial_x + \left[(\log x - \lambda t)^2 + 2r\sigma^2 t^2 - \sigma^2 t\right] u\partial_u \\
\Gamma_6 &= u\partial_u \\
\Gamma_7 &= f(t,x)\partial_u,
\end{aligned}$$

where $f(t,x)$ is any solution of (3.1).

Again we take a general symmetry

$$\Gamma = \sum_{i=1}^{6} \alpha_i \Gamma_i.$$

and its application to the condition $t = T$ and $u = U$ gives the single one-parameter Lie point symmetry

$$\Lambda = 2(T-t)\partial_t - [\log x + \lambda t]\, x\partial_x + 2r(T-t)u\partial_u. \tag{3.3}$$

The invariants of Λ are

$$\begin{aligned} \eta &= u\exp[-rt] \\ \zeta &= (T-t)^{-\frac{1}{2}}\left\{\log x+\Lambda(2T-t)\right\}. \end{aligned}$$

We write

$$u=\exp[rt]g\left\{(T-t)^{-\frac{1}{2}}\left[\log x+\Lambda(2T-t)\right]\right\} \tag{3.4}$$

and obtain the linear second-order ordinary differential equation

$$\frac{g''}{g'}=-\frac{\zeta}{\sigma^2}.$$

the solution of which we perform with some care.

We integrate this once to obtain

$$\begin{aligned} \log g'(\zeta)-\log g'(\zeta_0) &= -\left(\frac{\zeta^2}{2\sigma^2}-\frac{\zeta_0^2}{2\sigma^2}\right) \\ g'(\zeta) &= g'(\zeta_0)\exp\left(\frac{\zeta_0^2}{2\sigma^2}\right)\exp\left(-\frac{\zeta^2}{2\sigma^2}\right). \end{aligned}$$

We now perform the second quadrature and obtain

$$g(\zeta)-g(\zeta_0)=g'(\zeta_0)\exp\left(\frac{\zeta_0^2}{2\sigma^2}\right)\int_{\zeta_0}^{\zeta}\exp\left(-\frac{\zeta^2}{2\sigma^2}\right).$$

We now have

$$u(t,x)=\exp[rt]\left\{g(\zeta_0)+g'(\zeta_0)\exp\left(\frac{\zeta_0^2}{2\sigma^2}\right)\int_{\zeta_0}^{\zeta}\exp\left(-\frac{\zeta^2}{2\sigma^2}\right)\mathrm{d}\zeta\right\}.$$

At $t=0$ the integral is zero and so $g(\zeta_0)=U_0$.

At $t=T$ we have

$$\begin{aligned} U &= \exp[rT]\left\{U_0+g'(\zeta_0)\exp\left(\frac{\zeta_0^2}{2\sigma^2}\right)\int_{\zeta_0}^{\infty}\exp\left(-\frac{\zeta^2}{2\sigma^2}\right)\mathrm{d}\zeta\right\} \\ &= \exp[rT]\left\{U_0+g'(\zeta_0)\exp\left(\frac{\zeta_0^2}{2\sigma^2}\right)\sqrt{2}\sigma\int_{\zeta_0/\sigma\sqrt{2}}^{\infty}\exp\left(-s^2\right)\mathrm{d}s\right\} \end{aligned}$$

from which it is evident that

$$g'(\zeta_0)=\frac{U\exp[-rT]-U_0}{\exp\left[\frac{\zeta_0^2}{2\sigma^2}\right]\sigma\sqrt{2}\,\frac{\sqrt{\pi}}{2}\,\mathrm{erfc}\left(\frac{\zeta_0}{\sigma\sqrt{2}}\right)}.$$

Thus we have the solution to the Black-Scholes equation with the given terminal condition to be

$$u(t,x) = \exp[rt]\left\{U_0 + \frac{U\exp[-rT] - U_0}{\frac{\sqrt{\pi}}{2}\operatorname{erfc}\left(\frac{\zeta_0}{\sigma\sqrt{2}}\right)}\int_{\zeta_0}^{\zeta}\exp\left(-\frac{\zeta^2}{2\sigma^2}\right)\mathrm{d}\zeta\right\}.$$

4. Effects on the Pricing of Commodities

The stochastic behaviour of commodity prices plays a central role in the models for the evaluation of financial contingent claims on the commodity and in the procedures for the evaluation of investments to extract or produce the commodity. In his presidential speech to the American Finance Association Eduardo Schwartz [12] addressed the question of the relationship between the prices of various commodities. As the published version of this speech runs to some fifty pages, one can imagine that the address was lengthy. Much of the paper is concerned with comparisons of the models with data concerned with the prices of oil, copper and gold.

Here we consider the mathematical analysis of the equations constructed to model the stochastic processes. Schwartz presented three models. The first model considers merely the stochastic behaviour of the commodity. The second includes a variable known as the convenience yield. The third adds the possibility of stochastic interest rates.

4.1. *The Equations*

— **The one-factor model** has the Hamilton-Jacobi-Bellman equation

$$\tfrac{1}{2}\sigma^2 S^2 F_{SS} + \kappa(\mu - \lambda - \log S)SF_S - F_T = 0. \tag{4.1}$$

The terminal condition is

$$F(0,S) = S. \tag{4.2}$$

— **The two-factor model** adds to the spot price, S, of (4.1) the instantaneous convenience yield, δ, which may be interpreted as the flow of services accruing to the holder of the spot commodity but not to the owner of a futures contract. The evolution partial differential equation for this model is [6]

$$\tfrac{1}{2}\sigma_1^2 S^2 F_{SS} + \rho\sigma_1\sigma_2 SF_{S\delta} + \tfrac{1}{2}\sigma_2^2 F_{\delta\delta} + (r-\delta)SF_S + (\kappa(\alpha-\delta) - \lambda)F_\delta - F_T = 0 \tag{4.3}$$

for which the terminal condition is now $F(0,S,\delta) = S$.

— **The three-factor model** has as state variables the spot price of the commodity, the instantaneous convenience yield and the instantaneous interest rate.

Essentially this is an extension of Model 2 with the previous constant, r, now becoming a variable. The equation is

$$\begin{aligned}&\sigma_1^2 S^2 F_{SS} + \sigma_2^2 F_{\delta\delta} + \sigma_3^2 F_{rr} + 2\rho_1\sigma_1\sigma_2 S F_{S\delta} + 2\rho_2\sigma_2\sigma_3 F_{\delta r} + 2\rho_3\sigma_3\sigma_1 S F_{rS}\\ &+2(r-\delta)SF_S + 2\kappa(\alpha-\delta)F_\delta + 2a(m-r)F_r - 2F_T = 0 \end{aligned} \tag{4.4}$$

for which the terminal condition is now $F(0,S,\delta,r) = S$.

Equations (4.1), (4.3) and (4.4) are a bit of a mess. Perhaps the high salaries paid to persons working in the financial sector are partly as compensation for having to look at such terrible objects. Under a sequence of elementary point transformations they may be reduced to the simpler forms

—

$$\tfrac{1}{2}\Phi_{xx} - x\Phi_x - \Phi_t = 0 \tag{4.5}$$

for (4.1), where

$$\begin{aligned} t &= T\\ x &= \log S + \lambda - \mu + \tfrac{1}{2}\sigma^2\\ \Phi(t,x) &= F(T,S). \end{aligned}$$

—

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + 2(M-Ny)\frac{\partial\Phi}{\partial x} - 2y\frac{\partial\Phi}{\partial y} - 2\frac{\partial\Phi}{\partial t} = 0, \tag{4.6}$$

where the sequence of transformations is somewhat lengthier being in turn

$$\begin{aligned} S &= \exp[s] \quad\text{and}\quad F(T,S,\delta) = G(T,s,\delta)\\ u &= \frac{s}{\sigma_1\sqrt{1-\rho^2}} - \frac{\rho\sigma_1\delta}{\sigma_2}\\ v &= \frac{\delta}{\sigma_2}\\ \zeta(T,u,v) &= G(T,s,\delta) \text{ and}\\ \zeta &= \exp[at+bu+cv]\Phi \end{aligned}$$

for suitable values of the constants a, b and c. One of the remaining constants can be removed by rescaling of the independent variables

—

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} - 2(ay+bz+f)\frac{\partial\Phi}{\partial x} - 2(cy+dz)\frac{\partial\Phi}{\partial y} - 2z\frac{\partial\Phi}{\partial z} - 2\frac{\partial\Phi}{\partial t} = 0 \tag{4.7}$$

after a similar set of point transformations. An essential feature is that the second derivatives may be diagonalised to the standard form of the three-dimensional Laplacian in cartesian coordinates.

4.2. *The Symmetries of the Simplified Equations*

The application of Program LIE [7, 13] to the three equations produces the sets of Lie point symmetries listed below.
Equation (4.5):

$$\begin{aligned}
\Gamma_1 &= \exp[\kappa T] S\partial_S \\
\Gamma_2 &= \exp[\kappa T]\left\{\sigma^2 S\partial_S + \left[2\kappa(\log S + \lambda - \mu) + \sigma^2\right]\partial_F\right\} \\
\Gamma_3 &= \partial_T \\
\Gamma_4 &= \exp[2\kappa T]\left\{\partial_T + \left[\tfrac{1}{2}\sigma^2 + \kappa(\log S + \lambda - \mu)\right]\partial_S\right\} \\
\Gamma_5 &= \exp[-2\kappa T]\left\{\tfrac{1}{2}\sigma^2\partial_T - \left[\tfrac{1}{4}\sigma^4 + \tfrac{1}{2}\kappa\sigma^2(\log S + \lambda - \mu)\right] S\partial_S\right. \\
&\quad \left. - \left[(\log S + \lambda - \mu)^2 + \frac{\sigma^4}{4\kappa^2} - \frac{\sigma^2}{2\kappa} + \frac{\sigma^2}{\kappa}(\log S + \lambda - \mu)\right] F\partial_F\right\} \\
\Gamma_6 &= F\partial_F \\
\Gamma_7 &= f(T,S)\partial_F,
\end{aligned}$$

where $f(T,S)$ is a solution of (4.5).
Equation (4.6):

$$\begin{aligned}
\Gamma_1 &= f(t,x,y)\partial_\Phi \\
\Gamma_2 &= \Phi\partial_\Phi \\
\Gamma_3 &= \partial_x \\
\Gamma_4 &= t\left(N^2+1\right)\partial_x + N\partial_y - (Mt + x - Ny)\Phi\partial_\Phi \\
\Gamma_5 &= \exp[t]\left[N\partial_x + \partial_y\right] \\
\Gamma_6 &= \exp[-t]\left[N\partial_x - \partial_y + y\Phi\partial_\Phi\right] \\
\Gamma_7 &= \partial_t,
\end{aligned}$$

where $f(t,x,y)$ is a solution of (4.6).

We note that (4.6) is less well-endowed with symmetry than (4.5).
Equation (4.7):

$$\begin{aligned}
\Gamma_1 &= f(t,x,y,z)\partial_\Phi \\
\Gamma_2 &= \Phi\partial_\Phi \\
\Gamma_3 &= \partial_x \\
\Gamma_4 &= [(ad-bc)^2 + a^2 + c^2]t\partial_x + [a + (ad-bc)d]\partial_y - c[ad-bc]\partial_z + \\
&\quad c[-cx + ay - (ad-bc)z + cft]\Phi\partial_\Phi \\
\Gamma_5 &= \exp[t]\{[b + (ad-bc)]\partial_x + d\partial_y + (1-c)\partial_z\} \\
\Gamma_6 &= \exp[-t]\{[b - ad + bc]\partial_x + d\partial_y - (1+c)\partial_z - 2(1+c)z\Phi\partial_\Phi\} \\
\Gamma_7 &= \exp[ct][a\partial_x + c\partial_y]
\end{aligned}$$

$$\begin{aligned}\Gamma_8 &= \exp[-ct]\left\{[a(1-c^2+d^2)-2bcd]\partial_x + c(c^2-d^2-1)\partial_y + 2c^2 d\partial_z \right.\\ &\quad \left. +2c^2[(c^2-1)y + d(c+1)z]\Phi\partial_\Phi\right\}\\ \Gamma_9 &= \partial_t,\end{aligned}$$

where f is a solution of (4.7) as specified. We note that the number of Lie point symmetries is increasing as the number of factors increases.

4.3. *The Solutions in the Literature*

Equation (4.1) is essentially the same as the Black-Scholes equation. With such a supply of symmetry one is not surprised that the solution which satisfies the terminal condition has been found to be

$$F(T,S) = \exp\left[\log S\exp[-\kappa T] + (1-\exp[-\kappa T])\,\alpha * + \frac{\sigma^2}{4\kappa}\,(1-\exp[-2\kappa T])\right],$$

where $\alpha* = \mu - \lambda - \frac{1}{2}\sigma^2/\kappa$.

In the case of (4.3) the solution subject to the terminal condition $F(0,S,\delta) = S$ is

$$F(T,S,\delta) = S\exp\left[-\delta\frac{1-\exp[-\kappa T]}{\kappa} + A(T)\right],$$

where

$$\begin{aligned}A(T) &= \left(r - \alpha * + \frac{1}{2}\frac{\sigma_2^2}{\kappa^2} - \frac{\rho\sigma_1\sigma_2}{\kappa}\right)T + \sigma_2^2\frac{1-\exp[-2\kappa T]}{4\kappa^3}\\ &\quad + \left(\alpha * \kappa + \rho\sigma_1\sigma_2 - \frac{\sigma_2^2}{\kappa}\right)\frac{1-\exp[-\kappa T]}{\kappa^2}\end{aligned}$$

and $\alpha* = \alpha - \lambda/\kappa$.

Finally Schwartz gives the solution of (4.4) as

$$F(T,S,\delta,r) = S\exp\left[-\delta\frac{1-\exp[-\kappa T]}{\kappa} + r\frac{1-\exp[-aT]}{a} + C(T)\right], \tag{4.8}$$

where

$$\begin{aligned}C(T) &= \left[(\kappa\hat{\alpha} + \sigma_1\sigma_2\rho_1)(1-\exp[-\kappa T]) - \kappa T)\right]/\kappa^2\\ &\quad - \left[\sigma_2^2\left(4(1-\exp[-\kappa T]) - (1-\exp[-2\kappa T]) - 2\kappa T\right)\right]/\left(4\kappa^3\right)\\ &\quad - \left[(am^* + \sigma_1\sigma_3\rho_3)(1-\exp[-aT]) - aT)\right]/a^2\\ &\quad - \left[\sigma_3^2\left(4(1-\exp[-aT]) - (1-\exp[-2aT]) - 2aT\right)\right]/\left(4a^3\right)\\ &\quad + \sigma_1\sigma_3\rho_3\left[\frac{(1-\exp[-\kappa T]) + (1-\exp[-aT]) - (1-\exp[-(\kappa+a)T])}{\kappa a(\kappa+a)}\right.\\ &\quad \left. + \frac{\kappa^2(1-\exp[-aT]) + a^2(1-\exp[-\kappa T]) - \kappa a^2 T - a\kappa^2 T}{\kappa^2 a^2(\kappa+a)}\right].\end{aligned}$$

4.4. *Algebraic Observations*

The evolution partial differential equation for the one-factor model has the maximal number of Lie point symmetries and is simply the classical heat equation in disguise. The algebraically interesting part of our results comes when one examines the equations for the two- and three-factor models. Although there is a diminution in the number of Lie point symmetries by comparison with the maximum to be found in $1+2$ and $1+3$ equations, there is still a considerable amount of symmetry remaining. The method for the construction of the solutions is still available. We note that the number of symmetries increases with the number of 'space' variables. Although it may be difficult to envisage additional factors entering into the modelling of the pricing of a commodity, a deeper study of the general problem would be of mathematical interest. The symmetries as we have listed them are for the worst case scenario. The possibility of an increased number of symmetries if some of the parameters in the evolution partial differential equations become zero is real. One should bear in mind that the constants as they are written in (4.6) and (4.7) are combinations of the constants of the model. Although the constants are necessarily positive, their combinations have the potential to be zero.

5. Conclusions

We have presented some algebraic features of evolution partial differential equations arising in Financial Mathematics. This has been more by way of illustrative example and even then has been far from complete (see for example [14, 15, 10]). We have concentrated on the endpoint of the modelling process. We did mention that the basis of the modelling of the stochastic processes underlying the evolution partial differential equations presented here is and has been 'under review'. One can of course investigate the symmetries of the stochastic differential equations and compare them with those of the evolution partial differential equations resulting after the application of the standard theorems. A fundamental need is a full mathematical treatment of the models used for 'random walks' in a manner accessible to the symmetrically inclined workers in Financial Mathematics so that they can develop and have the tools necessary in the new setting which may well be in terms of coupled integral equations[1].

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[1] One recalls that Einstein relied on the narrowness of the distribution to go from the integral equation describing Brownian motion to the standard diffusion equation.

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