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The application of the D-adaptive model for the elastic bodies free vibration problem

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Abstract

An approach to solving the free vibration of elastic bodies with thin coating is proposed. The main idea of this approach is based on the formulation of the combined model, which permits the use of 3-D elasticity theory equations over the body domain and 2-D Timoshenko shell theory equations over the coating domain. The differential equations of the system are interconnected by special junction conditions.

1. Introduction

The purpose of this paper is to present an approach for the formulation of the D-adaptive [1] mathematical models (combined models) of the elastic deformation of structures with thin surface coating. In the papers [2, 3] the model of a thin coating is obtained from 3-D elasticity theory equations by passing to the limit with thin coating thickness tending to zero.

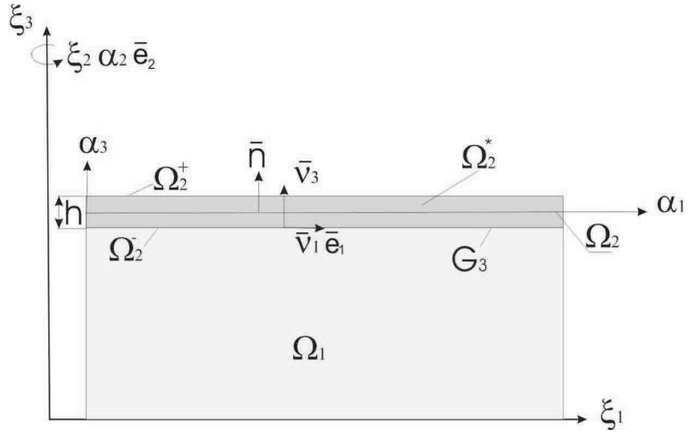
The main idea of our approach is based on the formulation of the combined model, which permits the use of 3-D elasticity theory model over one part of the domain, and 2-D Timoshenko's shell model over the other part. The differential equations of the system are interconnected by special boundary conditions - junction conditions. The numerical investigations of the problems which are described by combined mathematical models are performed by means of finite element method (FEM) [4, 5]. The numerical results obtaining by this method are checked with the available exact solutions reported by other authors.

2. Mathematical background and preliminary results

Let us consider an elastic continuum within a domain $\Omega \in R^3$, which consists of two parts $\Omega = \Omega_1 \cup \Omega_2^*$. We have regarded Ω_1 as an arbitrary 3-D domain with a

Lipschitzian boundary $G = G_1 \cup G_2 \cup G_3$. Let Ω_2^* be a 3-D domain limited by two surfaces Ω_2^-, Ω_2^+. We denote by h the distance between Ω_2^-, Ω_2^+ (thickness), which is small compared with the radius of curvature and other measurements of Ω_2^*, and by Ω_2 - the middle surface. The surface G_3 coincides with Ω_2^-.

Suppose that curvilinear orthogonal coordinated $\xi = (\xi_1, \xi_2, \xi_3)$ are determined in the domain Ω_1. Let us determine an orthogonal basis $\bar{v} = (\bar{v}_1, \bar{v}_2, \bar{v}_3)$ on the boundary G, here $\bar{v}_3$ is the unit normal to the surface. The radius-vector, which describes the points of the domain Ω_1, can be given in the form $\bar{R} = \bar{R}(\xi_1, \xi_2, \xi_3) \in \Omega_1$.



Let us put the Ω_2^* domain into curvilinear orthogonal coordinates $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. The radius vector, which describes the points of the domain Ω_2^*, can be given in the form:

$$\bar{r}(\alpha_1, \alpha_2, \alpha_3) = \bar{r}_0(\alpha_1, \alpha_2) + \alpha_3 \bar{n}(\alpha_1, \alpha_2) \in \Omega_2^*, \quad (\alpha_1, \alpha_2) \in \Omega_2, \quad \alpha_3 \in \left[-\frac{h}{2}, \frac{h}{2}\right],$$

where $\bar{r}_0$-radius-vector which determines surface Ω_2, $\bar{n}$-the unit normal to this surface. At each point of the Ω_2 the main orthogonal basis $(\bar{e}_1, \bar{e}_2, \bar{n})$ is determined. Here $\bar{e}_1$ and $\bar{e}_2$ are unit vectors which are directed along the α_1, α_2 directions respectively. The unit vectors $\bar{v}_1, \bar{v}_2$ coincide with vectors $\bar{e}_1, \bar{e}_2$. The normal $\bar{n}$ to the middle surface coincides with $\bar{v}_3$ and form α_3 axis. The Ω_2 domain has Lipschitzian boundary $\Gamma = \Gamma_1 \cup \Gamma_2$. Let us determine an orthogonal pair of the unit vectors $\bar{t} = (\bar{t}_1, \bar{t}_2)$ at the points of the middle surface boundary. Here $\bar{t}_1$-the outward normal to the boundary; $\bar{t}_2$-the tangent vector.

The free vibration problem of the continuum within the Ω_1 domain can be written in terms of the 3-D linear elasticity theory. Three equations are of the following form [6, 7]:

$$\frac{1}{H_1 H_2 H_3} \sum_{\beta=1}^{3} \frac{\partial}{\partial \xi_\beta} \left(\frac{H_1 H_2 H_3}{H_\beta} \sigma_{\beta i} \right) + \sum_{k=1}^{3} \frac{1}{H_i H_k} \frac{\partial H_i}{\partial \xi_k} \sigma_{ik} - \sum_{k=1}^{3} \frac{1}{H_i H_k} \frac{\partial H_k}{\partial \xi_i} \sigma_{kk} = \tilde{P}_i \quad (1)$$

$$i = 1, 2, 3; \quad k \neq i; \quad \xi \in \Omega_1$$
$$\tilde{P}_i = -\omega^2 \rho_1 U_i, \quad i = 1, 3.$$

Here H_i are the Lame's coefficients of the curvilinear coordinate system, σ_{ij} are the components of the stress tensor, the three components of displacement along the directions ξ_1, ξ_2, ξ_3 are respectively U_i. ω-circle frequency, ρ_1-density of the material.

The generalized Hook's law describes constitutive equations that related stress and strains $e_{ij}\left(i, j = 1, 3\right)$ for a linear elastic material:

$$\sigma_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} C_{ijkl} e_{kl}, \quad i, j = 1, 3; \tag{2}$$

here C_{ijkl} are elastic constants. For the important and frequently occurring case of isotropic homogeneous materials, nonzero components C_{ijkl} simplify to the following:

$$C_{iiii} = \frac{E_1(1 - v_1)}{(1 + v_1)(1 - 2v_1)}, \quad C_{iikk} = \frac{E_1(v_1)}{(1 + v_1)(1 - 2v_1)}, \quad C_{ikik} = \frac{E_1}{2(1 + v_1)},$$
$$i, k = 1, 3, \quad i \neq k,$$

where E_1, v_1 are Young's modulus and Poisson's ratio of the elastic body.

One can write down the strain- displacement relations:

$$e_{ii} = \frac{1}{H_i} \frac{\partial U_i}{\partial \xi_i} + \sum_{k=1}^{3} \frac{1}{H_i H_k} \frac{\partial H_i}{\partial \xi_k} U_k, \quad k \neq i, \quad \xi \in \Omega_1,$$
$$e_{ij} = \frac{H_j}{2H_i} \frac{\partial}{\partial \xi_i} \frac{U_j}{H_j} + \frac{H_i}{2H_j} \frac{\partial}{\partial \xi_j} \frac{U_i}{H_i}, \quad i, j = 1, 3. \tag{3}$$

$$C_{iiii} = \frac{E_1(1 - v_1)}{(1 + v_1)(1 - 2v_1)}, \quad C_{iikk} = \frac{E_1(v_1)}{(1 + v_1)(1 - 2v_1)}, \quad C_{ikik} = \frac{E_1}{2(1 + v_1)},$$
$$i, k = 1, 3, \quad i \neq k,$$

The free vibration problem of the continuum within the Ω_2 domain can be written in terms of the Timoshenko's shell theory. Five equations are of the following form [8]:

$$\frac{\partial(A_j T_i)}{A_i A_j \partial\alpha_j} - \frac{\partial(A_j) T_j}{A_i A_j \partial\alpha_i} + \frac{\partial(A_i S)}{A_i A_j \partial\alpha_j} + \frac{\partial(A_i) S}{A_i A_j \partial\alpha_j} + k_i Q_i + \frac{\partial(A_i k_i H)}{A_i A_j \partial\alpha_j} - \frac{k_j H \partial(A_i)}{A_i A_j \partial\alpha_j} = \tilde{p}_i;$$
$$k_1 T_1 + k_2 T_2 - \frac{\partial(A_2 Q_1)}{A_1 A_2 \partial\alpha_1} - \frac{\partial(A_1 Q_2)}{A_1 A_2 \partial\alpha_{21}} = \tilde{p}_3 \tag{4}$$
$$-Q_1 + \frac{\partial(A_j M_i)}{A_i A_j \partial\alpha_i} - \frac{\partial(A_j)}{A_i A_j \partial\alpha_i} M_j + \frac{\partial(A_i H)}{A_i A_j \partial\alpha_j} + \frac{H \partial(A_i)}{A_i A_j \partial\alpha_j} = \tilde{m}_i; \; i \neq j, \; i, j \neq 1, 2.$$
$$\tilde{p}_i = p_i^- - \omega^2 h \rho_2 u_i; \quad \tilde{m}_k = -\frac{h}{2} p_i^- - \omega^2 \frac{h^3}{12} \rho_2 \gamma_k; \; i = 1, 3; \; k = 1, 2.$$

Here A_1, A_2 are the Lame's coefficients, k_1, k_2 are the principal curvatures of the Ω_2 surface; T_k, S, Q_k, M_k, H $(k = 1, 2)$ denote the stresses and moments, p_i^- are the components of the surface loads on which are related to the α_1 coordinates, ρ_2-density of the material.

The stresses-moments and strains $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}, \chi_{11}, \chi_{22}, \chi_{12}$ relationship for the case of isotropic materials can be expressed in the terms of the Timoshenko's shells theory:

$$\begin{aligned} &T_k = B(\varepsilon_{kk} + v_2 \varepsilon_{ll}),\ S = B\frac{1 - v_2}{2}\varepsilon_{12},\ Q_k = G\varepsilon_{k3},\ M_k = D(\chi_{kk} + v_2\chi_{ll}),\\ &H = D\frac{1 - v_2}{2}\chi_{12};\ B = \frac{E_2 h}{1 - v_2^2},\ D = \frac{E_2 h^3}{1 - v_2^2},\ G = \frac{5}{12}E_2 h(1 + v_2),\\ &k \neq l; \quad k, l = 1, 2. \end{aligned} \tag{5}$$

Let u_1, u_2, w are the displacement components of the middle-surface points in the $\alpha_1, \alpha_2, \alpha_3$ directions and γ_1, γ_2 are rotation angular of a normal vector to the middle-surface in α_1, α_2 directions. The normal element, which is on the perpendicular to the middle surface will be not perpendicular to the middle surface after its deformation, it rotate on some angular do not bend and do not changing its length.

The strain components are expressed below in terms of the middle surface displacements:

$$\varepsilon_{ii} = \frac{\partial(u_i)}{A_i\partial\alpha_i} + \frac{\partial(A_i)}{A_iA_j\partial\alpha_j}u_j + k_i w; \quad \varepsilon_{12} = \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\frac{u_1}{A_1} + \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\frac{u_2}{A_2};$$

$$\varepsilon_{i3} = -k_i u_i + \frac{\partial(w)}{A_i\partial\alpha_i} + \gamma_i; \quad \chi_{ii} = \frac{\partial(\gamma_i)}{A_i\partial\alpha_i} + \frac{\partial(A_i)}{A_iA_j\partial\alpha_j}\gamma_j;$$

$$\begin{aligned} 2\chi_{12} = &\frac{k_1}{A_2}\frac{\partial u_1}{\partial\alpha_2} - \frac{k_2}{A_1A_2}\frac{\partial(A_1)}{\partial\alpha_2}u_1 + \frac{k_2}{A_1}\frac{\partial u_2}{\partial\alpha_1} - \frac{k_1}{A_1A_2}\frac{\partial(A_2)}{\partial\alpha_1}u_2\\ &+ \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\frac{\gamma_1}{A_1} + \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\frac{\gamma_2}{A_2}; \quad i, j = 1, 2, \quad i \neq j. \end{aligned} \tag{6}$$

On the boundaries of the Ω_1, Ω_2 domains we can write down the kinematics and static boundary conditions respectively:

$$U_{v_i} = 0; \quad i = 1, 3; \quad \xi \in G_1; \tag{7}$$

$$\sigma_{v_1 v_3} = 0; \quad i = 1, 3; \quad \xi \in G_2; \tag{8}$$

$$u_{t_k} = 0; \quad w = 0; \quad \gamma_{t_k} = 0; \quad k = 1, 2; \quad \alpha \in \Gamma_1; \tag{9}$$

$$T_{t_k} = 0; \quad Q = 0; \quad M_{t_k} = 0; \quad k = 1, 2; \quad \alpha \in \Gamma_2. \tag{10}$$

Here

$$U_{v_i} = \sum_{k=1}^{3} U_k v_{ik}; \sigma_{v_i v_j} = \sum_{k=1}^{3}\sum_{\beta=1}^{3} \sigma_{k\beta} v_{ik} v_{j\beta};$$

$$u_{t_k} = \sum_{\beta=1}^{2} u_\beta t_{k\beta}; \quad \gamma_{t_k} = \sum_{\beta=1}^{2} \gamma_\beta t_{k\beta};$$

$$T_{t_1} = \sum_{k=1}^{2} t_{1k}^2 T_k - 2t_{11}t_{12}S - t_{11}t_{12}(k_1+k_2)H;$$

$$T_{t_2} = \sum_{k=1}^{2} t_{1k}t_{2k}T_k + (t_{11}t_{22}+t_{12}t_{21})S + (k_1 t_{12}t_{21} + k_2 t_{12}t_{21})H;$$

$$Q = t_{11}Q_1 + t_{12}Q_2; \; M_{t_1} = \sum_{k=1}^{2} t_{1k}^2 M_k - 2t_{11}t_{12}H;$$

$$M_{t_2} = \sum_{k=1}^{2} t_{1k}t_{2k}M_k + (t_{11}t_{22}+t_{12}t_{21})H;$$

where the components $v_{ij} = \cos(v_i, \xi_j)$; $t_{kj} = \cos(t_k, \alpha_j)$ are direction cosines of the vectors v_i.

Let us introduce into consideration a junction conditions on the surface G_3. These correlations express the continuity of the displacements of the medium and static equilibrium conditions:

$$U_{v_i} = u_i - \frac{h}{2}\gamma_i, \quad U_3 = w, \quad i = 1,2, \quad G_3 = \Omega_2^-; \tag{11}$$

$$\sigma_{v_j v_3}(U_1, U_2, U_3) = -p_j^-; \quad j = 1,3, \quad G_3 = \Omega_2^-. \tag{12}$$

Lets bring together all those relations (1)-(6), boundary conditions (7)-(10) and junction conditions (11)-(12) that will be needed for a D-adaptive (combined) model problem.

3. The variation statement of the problem

Let us denote and define:

$$V = (V_1(\xi), V_2(\xi), V_3(\xi)); \quad v = (v_1(\alpha), v_2(\alpha), v_3(\alpha), v_4(\alpha), v_5(\alpha)); \quad \bar{V} = (V, v)$$

$$D_1 = \left\{V : V \in \left[W_2^{(1)}(\Omega_2)\right]^3, \; V_i(\xi) = 0, \; i = 1,3, \; \xi \in G_1\right\},$$

$$D_2 = \left\{v : v \in \left[W_2^{(1)}(\Omega_2)\right]^5, \; v_i(\alpha) = 0, \; i = 1,5, \; \alpha \in \Gamma_1\right\},$$

$$D = \left\{(V,v) : V \in D_1,\ v \in D_2,\ V_{v_j} = v_j - \frac{h}{2} v_{j+3},\ V_3 = v_3;\ j = 1,2;\ \xi \in G_3; \alpha \in \Omega_2\right\}.$$

For solving variation problem we need to find $(\omega, \bar{U})$, here ω - scalier, and $\bar{U} \in D$ (week solution [5]), which satisfied relations (13).

$$A(\bar{U}, \bar{V}) = \omega^2 M(\bar{U}, \bar{V}) + B(\bar{U}, \bar{V}). \tag{13}$$

Here

$$A(\bar{U}, \bar{V}) = \sum_{i=1}^{3} A_i(U, V_i) + \sum_{j=1}^{5} a_j(u, v_j);$$

$$M(\bar{U}, \bar{V}) = \sum_{i=1}^{3} M_i(U, V_i) + \sum_{j=1}^{5} m_j(u, v_j);$$

$$B(\bar{U}, \bar{V}) = \sum_{i=1}^{3} B_i(U, V_i) + \sum_{j=1}^{5} b_j(u, v_j).$$

$$A_i(U, V_i) = \int_{\Omega_1} \left(\sum_{\beta=1}^{3} \frac{\partial V_i}{H_\beta \partial \xi_\beta} \sigma_{\beta i} - \sum_{k=1}^{3} \frac{V_i}{H_i H_k} \frac{\partial H_i}{\partial \xi_k} \sigma_{ik} + \sum_{k=1}^{3} \frac{V_i}{H_i H_k} \frac{\partial H_k}{\partial \xi_i} \sigma_{kk} \right) d\Omega,$$

$$M_i(U, V_i) = \rho_1 \int_{\Omega_1} V_i U_i d\Omega,$$

$$B_i(U, V) = \int_{G_3} V_{v_i} \sigma_{v_1 v_3} dG_3,$$

$$\text{for} \quad i = 1,3, \quad k \neq i;$$

$$a_i(u, v_i) = \int_{\Omega_2} \left(\frac{T_i \partial(v_i)}{A_i \partial \alpha_j} + \frac{v_j T_j \partial(A_j)}{A_i A_j \partial \alpha_i} + \frac{S A_i \partial}{A_j \partial \alpha_j} \left(\frac{v_i}{A_i} \right) - v_i k_i Q_i + \frac{k_i H \partial(v_i)}{A_j \partial \alpha_j} - \frac{k_j H v_i \partial(A_i)}{A_i A_j \partial \alpha_j} \right) d\Omega,$$

$$\alpha_{i+3}(u, v_{i+3} \int_{\Omega_2} \left(v_{i+3} Q_i + \frac{M_i \partial(v_{i+3})}{A_i \partial \alpha_i} + \frac{v_{i+3} M_j \partial(A_j)}{A_i A_j \partial \alpha_i} + \frac{A_i H \partial}{A_j \partial \alpha_j} \left(\frac{v_{i+3}}{A_i} \right) \right) d\Omega,$$

$$a_3(u, v_3) = \int_{\Omega_2} \left(\sum_{n=1}^{2} \left(k_n T_n v_3 + \frac{Q_n \partial(v_3)}{A_n \partial \alpha_n} \right) \right) d\Omega,$$

$$m_i(u, v_i) = \rho_2 h \int_{\Omega_2} v_i u_i d\Omega, \quad m_3(w, v_3) = \rho_2 h \int_{\Omega_2} v_3 w d\Omega,$$

$$m_{i+3}(u, v_{i+3}) = \frac{\rho_2 h^3}{12} \int_{\Omega_2} v_{i+3} \gamma_i d\Omega,$$

$$b_k(u, v_k) = \int_{\Omega_2} v_k f_k^- \, d\Omega,$$

$$f_i^- = -\sigma_{v_i v_3}, \quad f_3^- = -\sigma_{v_3 v_3}, \quad f_{i+3}^- = \frac{h}{2}\sigma_{v_i v_3},$$

$$k = 1, 5, \quad i, j = 1, 2; \quad i \neq j; \quad \forall\, \bar{V} = (V, v) \in D.$$

for each function $\bar{V} = (V, v) \in D$.

Let us add all the equations (13) and consider now the sum and show that it is equal zero.

$$\sum_{i=1}^{3} B_i(U, V_i) + \sum_{j=1}^{5} b_j(u, v_j) = 0, \quad \forall\, \bar{V} \in D. \tag{14}$$

Taking into account the fact that $\bar{V} \in D$ and (11)-(12), we can now easily show that this sum equals zero

$$\int_{G_3} \sum_{i=1}^{3} V_{v_i} \sigma_{v_i v_3} dG_3 - \int_{\Omega_2} \left(\sum_{j=1}^{3} v_j \sigma_{v_j v_3} - \sum_{k=1}^{2} v_{3+k} \frac{h}{2} \sigma_{v_k v_3} \right) d\Omega_2 = \int_{G_3} \sum_{i=1}^{3} V_{v_i} \sigma_{v_i v_3} dG_3 - \int_{G_3} \sum_{i=1}^{3} V_{v_i} \sigma_{v_i v_3} dG_3 = 0.$$

Hence, taking (14) into consideration we get

$$A(\bar{U}, \bar{V}) - \omega^2 M(\bar{V}, \bar{V}) = 0. \tag{15}$$

4. The application of the finite element method to the combined model

Let us consider the free vibration combined problem, which deals with rotation bodies with thin coating. Cylindrical coordinates (r, φ, z) are naturally suited to such problems, with the z-axis being the axis of rotational symmetry and φ-axis being circular axis. We shall put the shell (coating) domain into (α_1, φ) coordinates. For the combined problem the displacement, strain and stress vectors can be written in the form

$$U = (U_r, U_z, U_\varphi, u_1, u_2, w, \gamma_1, \gamma_2);$$
$$\varepsilon = (e_{rr}, e_{\varphi\varphi}, e_{zz}, e_{r\varphi}, e_{rz}, e_{\varphi z}, \varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}, \kappa_{11}, \kappa_{22}, \kappa_{12});$$
$$\sigma = (\sigma_{rr}, \sigma_{\varphi\varphi}, \sigma_{zz}, \sigma_{r\varphi}, \sigma_{rz}, \sigma_{\varphi z}, \bar{\sigma}_{11}, \bar{\sigma}_{22}, \bar{\sigma}_{12}, \bar{\sigma}_{13}, \bar{\sigma}_{23}).$$

We shall seek a solution of the problem by means of a semianalitical FEM. According to this technique the trigonometric functions $\sin m\varphi, \cos m\varphi$ defined at the $[0, 2\pi]$, are selected as basic functions along the circular coordinate φ. These functions constitute an orthogonal system in the energy metric of the operators of the elasticity theory and Timoshenko's shell theory. As to other variables, quadratic approximations of the finite element method are used. Due to the orthogonally of the basic functions at the $[0, 2\pi]$, the problem decomposes into L problems ($m = 0, 1, 2, \dots L$), and we shall call m a harmonic number. For solving such kind of problem, we will use the method of the iteration in the subspace [4].

5. Conclusions

Let us consider cylinder with coating, which occupies the $\Omega \in R^3$ domain where $\Omega = \Omega_1 \cup \Omega_2$ (fig.1). On the boundary $\Gamma_1^{(1)}$ stiff fastening conditions are given. We can write down the free vibration problem within the in terms of the 3-D linear elasticity theory and the Timoshenko's shell theory equations within the Ω_2 domain, which is a middle surface of the Ω_2^* domain, as it shown in Fig.2 Ω_2 ($h = 0.02m$).

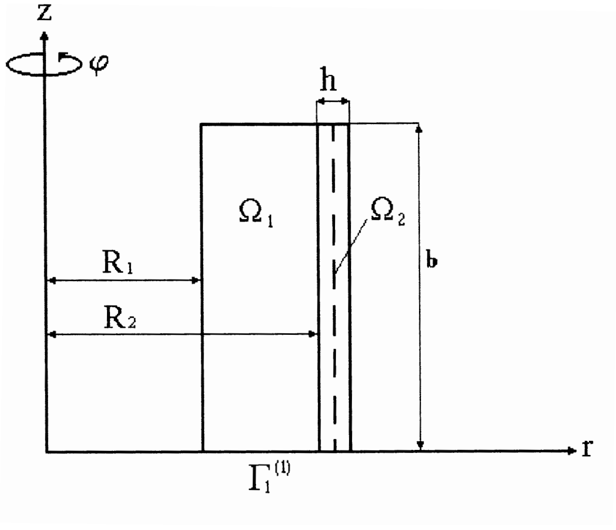


Fig. 2

Values of geometric parameters of the body are selected as:

$$b = 10m; \quad R_1 = 4m; \quad R_2 = 6m.$$

The material properties data of the cylinder and coating are given below:

$$E_1 = 3.6 \times 10^6 \ N/m^2, \quad v_1 = 0.475, \quad \rho_1 = 1150 \ N/m^2.$$

$$E_2 = a2 \times 10^{12} N/m^2, \quad v_2 = 0.3, \quad p_3 = 32000 N/m^2.$$

We will investigate vibration characteristics $\lambda = \omega/2\varphi$ for $m = 0.6$. From the Table 1 one can see that the lowest frequency is for $m = 4$. Table 2 show the depending of the free vibration frequencies on thickness (h) for $m = 4$.

Table 1

$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
4.70048	4.70291	4.88357	3.82197	3.52410	4.04589	4.87468
5.15320	5.43938	5.31822	5.48262	6.15703	6.96269	7.46961
5.70793	6.15503	6.52176	6.50162	6.99359	7.59788	7.87409

Table 2

$b_2=0.05$	$b_2=0.03$	$b_2=0.02$	$b_2=0.01$	$b_2=0.005$	$b_2=0.001$
4.137652	3.705769	3.524100	3.314059	3.202108	3.094666
6.115694	6.112317	6.157032	6.099259	5.990738	4.342801
6.929431	6.930078	6.993585	6.858809	6.107997	5.941217

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