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Well posedness of the stochastic Drude-Born-Fedorov model in electromagnetics

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Abstract

In this work we present some results on the stochastic Drude-Born-Fedorov (DBF) model for the evolution of electromagnetic fields in chiral media, in the time domain. The problem reduces to a Cauchy problem for a Sobolev type stochastic evolution equation with additive noise. We show that the problem is strongly well posed.

Keywords: Stochastic differential equations, Maxwell's equations, Sobolev type equations, DBF model, chiral media, electromagnetic media, time domain analysis

1. Introduction

The propagation of electromagnetic waves in chiral media is the subject of many studies, and numerous references are available in the literature. Chiral media find a wide range of applications from medicine to thin film technology. The mathematical modeling of such media is done through the modification of the constitutive relations for the well known Maxwell's equations in a region $\Omega \subset \mathbb{R}^3$, $t > 0$:

$$\left.\begin{aligned} \frac{\partial D}{\partial t} - curlH &= -J_e \\ \frac{\partial B}{\partial t} + curlE &= -J_m \\ divD &= 0 \\ divB &= 0 \end{aligned}\right\} \qquad (1)$$

where E is the electric field, H is the magnetic field, D is the electric displacement, B is the magnetic induction and J_e, J_m are the densities of the electric and magnetic current, respectively. Here, as we can see, we have assumed the divergence-free property for the fields D and B. The complete constitutive relations for chiral media are nonlocal in time and have the form:

$$D{=}\varepsilon E + c_e \star E + c_m \star H, \qquad B{=}\mu H + \sigma_e \star E + \sigma_m \star H$$

where by $\star$ we denote the convolution $\alpha \star U = \int_0^t \alpha(t-s,x)U(s,x)ds$.

Maxwell's equations (1) under these constitutive relations constitute the full nonlocal problem for chiral media. A time-domain analysis for chiral media under the full constitutive relations can be found in [15]. Related work can be found in [1], [2], [7], [9], [10]. Though the mathematical treatment of the full problem is feasible, in a number of important applications this may be cumbersome to handle. Thus, local approximations to the full constitutive relations have been proposed, that are capable of keeping the general features of chiral media, without the mathematical complications introduced by the non locality of the integral terms. In practice, a very common approximation scheme to the full constitutive relations is the so-called Drude-Born-Fedorov (DBF) approximation which leads to the constitutive relations:

$$D{=}\varepsilon(I+\beta curl)E, \qquad B{=}\mu(I+\beta curl)H$$

where $\varepsilon > 0$ is the electric permittivity, $\mu > 0$ is the magnetic permeability and $\beta \neq 0$ is the chirality measure. This approximation has been extensively used in the modeling of chiral media, especially in the time harmonic case (i.e. when we consider special solutions of the form $E(t,x) = E(x)e^{-i\omega t}$, $H(t,x) = H(x)e^{-i\omega t}$), and is formally justified when the $c_e, c_m, \sigma_e, \sigma_m$ are localized functions of their arguments and we focus on certain frequency ranges. However, recently the DBF approximation has been used for the study of electromagnetic fields in chiral media in the time domain, as we can see in [4], [7], [15].

It is the aim of the present paper to study the well posedness of the DBF model for chiral media in the time domain when the source terms driving the fields are considered as random fields. This necessitates the study of a stochastic Sobolev type evolution equation. The structure of the paper is as follows: In section 2 we review some results concerning the Cauchy problem of Sobolev type associated with the time domain analysis for deterministic chiral media, using the DBF constitutive relations. In section 3, we set up the stochastic model of this problem. We assume that the stochastic external fields are modeled by an infinite dimensional Wiener process, thus turning the model to a stochastic Sobolev type evolution equation with additive noise. Using semigroup theory and properties of the stochastic convolution we prove that the stochastic problem is strongly well posed and we make some remarks concerning the behaviour as the chirality parameter β becomes small, which clarify certain issues concerning the use of the DBF model, under this functional background, as a modeling tool for chiral media in the time domain.

2. Formulation of the deterministic model

In this section, we formulate the deterministic model. In order to set up an initial-boundary value problem for Maxwell's equations (1) under the DBF constitutive relations, so that a well posed Cauchy problem can be formulated, we need some initial-boundary conditions and some extra assumptions. In both the deterministic and the stochastic model we will assume that Maxwell's equations (1) hold in Ω, for $t > 0$, where Ω is a bounded and simply connected domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$.

2.1. *The initial-boundary value problem*

We present here the initial and boundary conditions and some extra assumptions, in order to understand the important role that the spaces[1]

$$H(div0;\Omega) := \left\{U \in L^2(\Omega) : \ divU = 0\right\}^{2}$$

$$H_0(div;\Omega) := \{U \in H(div;\Omega) : \ U \cdot n = 0 \ \text{ in } \ \partial\Omega\}$$

$$\mathbf{W} = H_0(div;\Omega) \cap H(div0;\Omega)$$

will play and why the space $\mathbf{W}_s = \{U \in \mathbf{W} \ : curlU \in \mathbf{W}\}$ will be chosen as the domain of definition of the operator $curl$.

First, we can check that $divJ_e = divJ_m = 0$. We also assume for the fields $J_e, \ J_m$ that

$$J_e \cdot n = J_m \cdot n = 0 \ \text{ in } \ \partial\Omega, \ t > 0.$$

For the initial data $E(0,x) = E_0, \quad H(0,x) = H_0,$ we assume the divergence free property and that

$$E_0 \cdot n = H_0 \cdot n = 0 \ \text{ in } \ \partial\Omega,$$

and as boundary conditions we choose these of a conductor satisfying

$$E \times n = J_E, \qquad H \times n = J_H, \quad \text{ in } \ \partial\Omega,$$

where n is the unit outward normal vector to $\partial\Omega$. We also need the space

$$H(div;\partial\Omega) = \left\{U \in L^2(\partial\Omega) : \ divU \in L^2(\partial\Omega)\right\}.$$

We assume that $J_E, J_H \in H(div;\partial\Omega)$, satisfy $div_\tau J_E = div_\tau J_H = 0$, where by div_τ, we denote the boundary divergence operator $div_\tau : H(div;\partial\Omega) \to L^2(\partial\Omega)$.

One can prove now that $E(\cdot,t) \cdot n = H(\cdot,t) \cdot n = 0 \ \text{ in } \ \partial\Omega, \quad t > 0$. Indeed, using the identity $curlU \cdot n = div_\tau(U \times n)$ in $\partial\Omega$, we compute

$$\frac{\partial}{\partial t}(E \cdot n) = \frac{1}{\varepsilon}\frac{\partial}{\partial t}(D \cdot n) - \beta\frac{\partial}{\partial t}(curlE \cdot n) = \frac{1}{\varepsilon}(curlH - J_e) \cdot n - \beta\frac{\partial}{\partial t}(curlE \cdot n)$$

$$= \frac{1}{\varepsilon}(div_\tau J_H - J_e \cdot n) - \beta\frac{\partial}{\partial t}\ (div_\tau J_E) = 0.$$

[1]For the properties of the functional spaces introduced in electromagnetic theory, we refer to [6], [14].

[2]For simplicity we write $L^2(\Omega)$ instead of $L^2(\Omega)^3$, $H^1(\Omega)$ instead of $H^1(\Omega)^3$ etc...

In a similar way it can be proved that $\dfrac{\partial}{\partial t}(H \cdot n) = 0$.

Hence, $E(\cdot,t) \cdot n = E_0 \cdot n = 0$ and $H(\cdot,t) \cdot n = H_0 \cdot n = 0$, in $\partial\Omega$, $t > 0$.

One can easily see that the assumptions made for $E, H, E_0, H_0, J_e, J_m, J_E, J_H$ are compatible. Note that in the frequency domain $U \cdot n = 0 \Leftrightarrow curlU \cdot n = 0$. Therefore the initial conditions and the functional set up in the time domain chosen in this paper, transports this crucial assertion from the frequency domain to the time domain.

Maxwell's equations (1) supplemented with the above initial-boundary conditions and extra assumptions, under the DBF constitutive relations, lead to the following initial-boundary value problem for E, H:

$$\left.\begin{array}{rl} \varepsilon\frac{\partial}{\partial t}(I+\beta curl)E = curlH - J_e & \text{in } \Omega, \quad t>0, \\ \mu\frac{\partial}{\partial t}(I+\beta curl)H = -curlE - J_m & \text{in } \Omega, \quad t>0, \\ divE = 0 & \text{in } \Omega, \quad t>0, \\ divH = 0 & \text{in } \Omega, \quad t>0, \\ E\cdot n = H\cdot n = curlE\cdot n = curlH\cdot n = 0 & \text{in } \partial\Omega, \quad t>0, \\ E(\cdot,0)=E_0, \quad H(\cdot,0)=H_0 & \text{in } \Omega. \end{array}\right\} \tag{2}$$

As we can see from the set of equations (2), a suitable functional space in which a corresponding Cauchy problem which deals with both E and H can be posed, is the closed space $\mathbf{H} = \mathbf{W} \times \mathbf{W}$, where $\mathbf{W} = H(div0;\Omega) \cap H_0(div;\Omega)$ (see [4], [18]). In fact $\mathbf{H}$, as a closed subspace of the Hilbert space $L^2(\Omega) \times L^2(\Omega)$, when equipped with the inner product

$$\left(\begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix} \cdot \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix}\right)_{\mathbf{H}} = \int_\Omega (\varepsilon\phi_1 \cdot \phi_2 + \mu\psi_1 \cdot \psi_2) \ dx$$

is also a Hilbert space. For the operator $\ell = I + \beta curl$, since $divcurlE = divcurlH = 0$ and $curlE \cdot n = curlH \cdot n = 0$, we choose $D(\ell) = \mathbf{W}_S = \{U \in \mathbf{W} \ : curlU \in \mathbf{W}\}$, which by [18] is a dense subset of $\mathbf{W}$.

2.2. *The corresponding Cauchy problem of Sobolev type*

Using a six vector notation, we write the set of equations (2) in the more compact form of a Cauchy problem of Sobolev type in $\mathbf{H}$:

$$\left.\begin{array}{rl} \frac{\partial}{\partial t}(L\mathscr{E}) & = M\mathscr{E} + F \\ \mathscr{E}(x,0) & = \mathscr{E}_0 \end{array}\right\} \tag{3}$$

where:

$$\mathscr{E} = \begin{pmatrix} E \\ H \end{pmatrix}, \qquad \mathscr{E}_0 = \begin{pmatrix} E_0 \\ H_0 \end{pmatrix}, \qquad F = \begin{pmatrix} -\varepsilon^{-1}J_e \\ -\mu^{-1}J_m \end{pmatrix},$$

$$\text{and} \quad L = \begin{pmatrix} \ell & \mathbf{0} \\ \mathbf{0} & \ell \end{pmatrix}, \qquad M = \begin{pmatrix} \mathbf{0} & \varepsilon^{-1} curl \\ -\mu^{-1} curl & \mathbf{0} \end{pmatrix} \qquad \text{with}$$

$\ell = I + \beta curl$, $D(\ell) = \mathbf{W}_S$ and $D(L) = D(M) = \mathbf{W}_S \times \mathbf{W}_S$.

A Sobolev-type Cauchy problem is often treated in the literature as follows: If the operator M is closed and the operator L has a bounded inverse, then using the transformation $\mathscr{X}_t = L\mathscr{E}_t \Leftrightarrow \mathscr{E}_t = L^{-1}\mathscr{X}_t$, we can study the problem as a standard Cauchy one, governed by the bounded - due to the closed graph theorem - operator ML^{-1}. So, if we want to go on with the solvability of this Sobolev-type Cauchy problem, we need to know if the operator M is closed, and for which values of the chirality measure β the operator L has a bounded inverse $L^{-1} : \mathbf{H} \to D(L)$. To this direction we need a proposition that has been proved in [18]:

Proposition:

In the space $\boldsymbol{W} = H(div0;\Omega) \cap H_0(div;\Omega)$, *we consider the dense subspace* $\boldsymbol{W}_S = \{U \in \boldsymbol{W} : curlU \in \boldsymbol{W}\}$ *and we define the operator* $S : \boldsymbol{W}_S \to \boldsymbol{W}$, $S(U) = curlU$.

The following hold:

The operator S is closed and self-adjoint. Furthermore, S is invertible and its inverse $S^{-1} : \boldsymbol{W} \to \boldsymbol{W}_S$ is compact. The spectrum $\sigma(S)$ consists only of point spectrum $\sigma_\rho(S) \subset \mathbb{R}$, and the set of the corresponding eigenfunctions gives an orthogonal complete basis of the space $\boldsymbol{W}$.

Since S^{-1} is a compact operator, its spectrum consists only of point spectrum which does not accumulate besides 0 (in the case that this set is infinite). Furthermore, since the spectrum $\sigma(S)$ consists of only point spectrum $\sigma_\rho(S) \subset \mathbb{R}$ and the set of the corresponding eigenfunctions gives an orthogonal complete basis of the separable (as a subset of $L^2(\Omega)$) infinite dimensional space $\mathbf{W}$, we have that both sets $\sigma_\rho(S), \sigma_\rho(S^{-1})$ are countably infinite. Hence, $\sigma_\rho(S^{-1})$ accumulates to 0 and $\sigma_\rho(S)$ accumulates to $+\infty$ or $-\infty$ or both of them.

Let now $\lambda \in \sigma_\rho(S)$. Then $\lambda = \int_\Omega U \cdot curlU \;\; dx \Big/ \int_\Omega ||U||^2_{\mathbb{R}^3} \;\; dx$, where the quantity $\int_\Omega U \cdot curlU \;\; dx$ is the "helicity" of a function $U \in \mathbf{W}_S$. As we can always find a function $U \in \mathbf{W}_S$ so that its helicity can be as big positive or big negative as we want, the set $\sigma_\rho(S)$ accumulates to both $-\infty$ and $+\infty$.

Now, it is easy to see that the operator

$$L = \begin{pmatrix} \ell & \mathbf{0} \\ \mathbf{0} & \ell \end{pmatrix}, \quad D(L) = \mathbf{W}_s \times \mathbf{W}_s$$

has a bounded inverse $L^{-1} : \mathbf{H} \to D(L)$ for these values of the chirality measure β that the operator $\ell = I + \beta curl : \mathbf{W}_S \to \mathbf{W}$ has a bounded inverse.

We have that $\ell = I + \beta curl = \beta\ [\frac{1}{\beta}\ I - (-S)]$, which has a densely defined and continuous inverse if $\frac{1}{\beta} \in \rho(-S)$.
Furthermore, since $-S$ is closed its resolvent operator $[\lambda I - (-S)]^{-1}$, for $\lambda \in \rho(-S)$, is not only densely defined and continuous but also bounded (see [6] p. 3).

Therefore, if $-\frac{1}{\beta} \in \sigma_\rho(S)$ then ℓ is not invertible and when the chirality measure $\beta \neq -\frac{1}{\lambda}$, where $\lambda \in \sigma_\rho(S)$, then ℓ has a bounded inverse. In the latter case the operator L has also a bounded inverse:

$$L^{-1} : \mathbf{H} \to D(L) \quad \text{and} \quad L^{-1} = \begin{pmatrix} (I+\beta curl)^{-1} & \mathbf{0} \\ \mathbf{0} & (I+\beta curl)^{-1} \end{pmatrix}.$$

Let $\beta \neq -\frac{1}{\lambda}$, where $\lambda \in \sigma_\rho(S)$. Using the transformation $\mathscr{X} = L\mathscr{E} \Leftrightarrow \mathscr{E} = L^{-1}\mathscr{X}$, problem (3) takes the form

$$\left.\begin{aligned} \frac{\partial}{\partial t}\mathscr{X} &= ML^{-1}\mathscr{X} + F \\ \mathscr{X}(x,0) &= L\mathscr{E}_0 \end{aligned}\right\} \tag{4}$$

which is a standard Cauchy problem governed by the bounded - due to the closed graph theorem - operator ML^{-1}.

Another way to handle problem (3) taking advantage of the fact that $D(L) = D(M)$ is the following:
Since $L^{-1} : \mathbf{H} \to D(L)$ is bounded, we rewrite problem (3) in the equivalent form:

$$\left.\begin{aligned} \frac{\partial}{\partial t}\mathscr{E} &= L^{-1}M\mathscr{E} + L^{-1}F \\ \mathscr{E}(x,0) &= \mathscr{E}_0 \end{aligned}\right\} \tag{5}$$

where we observe that (5) is posed in the space $D(L)$. Since $D(L)$ is dense in $\mathbf{H}$, it cannot be a closed subspace of $\mathbf{H}$ under the restriction of the inner product in $\mathbf{H}$. But, as we can see the operator M is closed, so the space $D(M)$ equipped with the inner product introduced by its graph norm

$$\left(\begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix} \cdot \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix} \right)_{\mathbf{D}} =$$

$$\int_\Omega (\varepsilon\phi_1 \cdot \phi_2 + \mu\psi_1 \cdot \psi_2 + \mu^{-1} curl\phi_1 \cdot curl\phi_2 + \varepsilon^{-1} curl\psi_1 \cdot curl\psi_2)\ dx$$

is a Hilbert space which will be noted hereafter as $\mathbf{D}$; since $D(L) = D(M)$ and by our assumptions $\mathscr{E}_0 \in D(L)$, we can consider that problem (5) is a standard Cauchy problem in $\mathbf{D}$.

Remarks. As we have already observed, the operators L, M are closed since S is closed. This means that both standard Cauchy problems (4), (5) are governed by a bounded operator (i.e an infinitesimal generator of a uniformly continuous semigroup of operators in $\mathbf{H}$ and $\mathbf{D}$ respectively). As we will see in the next section, problem (4) is uniquely solvable but not well posed. This is the reason why we rewrite problem (3) into (5), which is set in the Hilbert space $\mathbf{D}$, where well posedness can be confirmed.

We may also note, that the set $\sigma_\rho(S)= \{\lambda_i\}_{i\in\mathbb{N}}$, accumulates to both $-\infty$ and $+\infty$, so if $\dfrac{1}{\beta} \to +\infty$ or $\dfrac{1}{\beta} \to -\infty$ (therefore if $\beta \to 0$), then the operators ℓ^{-1}, L^{-1} are not defined.

3. The stochastic model

A stochastic model for differential equations of Sobolev type allows us to describe phenomena subject to various forms of uncertainty in space and time, that the deterministic model can not describe. In this paper we will be interested in studying phenomena that are related to stochastic densities of electric and magnetic currents J_e ad J_m respectively. These will be modelled as random fields, i.e. as random variables indexed by spatial and time coordinates. We will consider Gaussian random fields, which may be modelled as an infinite dimensional Wiener process. Therefore, the evolution of the electromagnetic fields in the medium will be given by a stochastic Sobolev type evolution equation. While the literature on deterministic Sobolev type equations is extended see e.g. [16] and references therein, there is still relatively little work done on stochastic Sobolev type equations (e.g [11], [12]). For the necessary notions and results from Probability Theory and Stochastic Analysis we refer to [5], [8], [13], [17].

3.1. *The setting of the stochastic Cauchy problems*

Let U be a real separable and infinite dimensional Hilbert space and consider the real and separable Hilbert space $\mathbf{H} = \mathbf{W} \times \mathbf{W}$, the probability space $(\Omega, \mathscr{F}, P)$ with a normal filtration $\mathscr{F}_t$, $t \geq 0$, and the predictable σ-field $\mathscr{P}_T$ in the space $\Omega_T = [0,T] \times \Omega$. Let us also consider the measurable spaces $(U, \mathscr{B}(U))$, $(\mathbf{H}, \mathscr{B}(\mathbf{H}))$, $(\Omega_T \times \mathbf{H}, \mathscr{P}_T \times \mathscr{B}(\mathbf{H}))$, $(L_2^0, \mathscr{B}(L_2^0))$, where by L_2^0 we denote the space of all Hilbert-Schmidt operators in $L_2(U_0, \mathbf{H})$ with $U_0 = Q^{1/2}(U)$, and $Q \in \mathcal{L}(U)$ a nonnegative, nuclear operator ($Tr[Q] < \infty$). The Cauchy problem for the linear equation of Sobolev type (4) with additive noise has the form:

$$\left.\begin{aligned} d(L\mathscr{E}_t) &= [M\mathscr{E}_t + F(t)] \ \ dt + B \ \ dW_t, \quad t \geq 0 \\ Y_0 &= \xi \end{aligned}\right\} \tag{6}$$

The noise $W_t, t \geq 0$ is a $U-$valued Q-Wiener process, with $Tr[Q] < \infty$. As this problem is a stochastic model for problem (3), for L, M, F, B, ξ we consider the following assumptions:

(A1) $M : D(M) \subset \mathbf{H} \to \mathbf{H}$ is closed and linear operator.

(A2) $L : D(L) \subseteq D(M) \to \mathbf{H}$ is a linear, closed, invertible operator and $L^{-1} : \mathbf{H} \to D(L)$ is bounded.

(A3) $B \in \mathcal{L}(U, \mathbf{H})$

(A4) F is a $\mathbf{H}-$valued predictable process with Bochner integrable, on arbitrary finite interval [0,T], trajectories

(A5) ξ is a $D(L)-$valued, $\mathcal{F}_0-$measurable random variable and $\mathscr{E}_t,\ t \geq 0$ is the unknown $D(L)-$valued process.

Definition 3.1 An $\mathbf{H}-$ valued predictable process $\mathscr{E}_t,\ t \in [0,T]$, is called a mild solution of (6) if:

1 $\displaystyle\int_0^T \|\mathscr{E}_s\|_{\mathbf{H}} \ \ ds < \infty,$ P-a.s.

2 $$\mathscr{E}_t = L^{-1}T(t)L\xi + \int_0^t L^{-1}T(t-s)F(s) \ \ ds + \int_0^t L^{-1}T(t-s)B \ \ dW_s, \text{ P-a.s.}, \ \ t \in [0,T],$$

where $T(t),\ t \geq 0$ is a uniformly continuous semigroup in $\mathbf{H}$ generated by the bounded operator $ML^{-1} : \mathbf{H} \to \mathbf{H}$.

Definition 3.2 An $\mathbf{H}-$ valued predictable process $\mathscr{E}_t,\ t \in [0,T]$, is called a strong solution of (6) if:

1 $\mathscr{E}_t \in D(L)$, P-a.s., a.e. on $[0,T]$

2 $\displaystyle\int_0^T \|M\mathscr{E}_s\|_{\mathbf{H}} \ \ ds < \infty,$ P-a.s.

3 $\displaystyle\mathscr{E}_t = \xi + \int_0^t [L^{-1}M\mathscr{E}_s + L^{-1}F(s)]\ ds + L^{-1}BW_t,$ P-a.s., $t \in [0,T]$.

In [11], [12], using the transformation $\mathscr{X}_t = L\mathscr{E}_t \Leftrightarrow \mathscr{E}_t = L^{-1}\mathscr{X}_t$, which is well defined by (A2), it has been proved that Problem (6) under the assumptions (A1)-(A5) has a unique strong solution of the form

$$\mathscr{E}_t = L^{-1}T(t)L\xi + \int_0^t L^{-1}T(t-s)F(s) \ \ ds + \int_0^t L^{-1}T(t-s)B \ \ dW_s, \ t \in [0,T].$$

Note that (see [5] pg.119) a strong solution has a continuous modification. Also note, by the form of the first term of the solution, that since the operator L is closed but not bounded, well posedness for Problem (6) can not be confirmed. However, we observe that the above result stands even in the case that $D(L) \subset D(M)$. As we have already mentioned in section 2, taking in advantage that in our problem $D(L) = D(M)$ we can consider the following set up for a stochastic model which allows us to confirm well posedness.

In addition to the Hilbert spaces, the probability spaces, the predictable σ-fields and the measurable spaces mentioned in order to set up the stochastic problem (6), we consider the real and separable (as a subset of $L^2(V) \times L^2(V)$) Hilbert space $\mathbf{D}$ and the measurable spaces $(\mathbf{D}, \mathscr{B}(\mathbf{D}))$, $(\Omega_T \times \mathbf{D}, \mathscr{P}_T \times \mathscr{B}(\mathbf{D}))$, $(L_2^0, \mathscr{B}(L_2^0))$, but here we denote by L_2^0 the space of all Hilbert-Schmidt operators in $L_2(U_0, \mathbf{D})$, instead of $L_2(U_0, \mathbf{H})$.

Multiplying from the left by the bounded operator L^{-1}, problem (6) takes the form:

$$\left.\begin{aligned} d\mathscr{E}_t &= [L^{-1}M\mathscr{E}_t + L^{-1}F(t)] \;\; dt + L^{-1}B \;\; dW_t, \quad t \geq 0 \\ \mathscr{E}_0 &= \xi \end{aligned}\right\} \tag{7}$$

where $W_t,\ t \geq 0$, is again a $U-$valued Q-Wiener process, with $Tr[Q] < \infty$. We assume that (A1)-(A5) hold. Here, in particular we assume that (A2) holds for $D(L) = D(M)$. As far as the operator $L^{-1}M : \mathbf{D} \to \mathbf{D}$ is concerned, for every $y \in \mathbf{D}$, we have:

$$\|(L^{-1}M)y\|_{\mathbf{D}}^2 = \|L^{-1}My\|_{\mathbf{H}}^2 + \|ML^{-1}My\|_{\mathbf{H}}^2 \leq$$

$$\|L^{-1}\|_{\mathcal{L}(\mathbf{H})}^2 \|My\|_{\mathbf{H}}^2 + \|ML^{-1}\|_{\mathcal{L}(\mathbf{H})}^2 \|My\|_{\mathbf{H}}^2 \;\leq\; (\|L^{-1}\|_{\mathcal{L}(\mathbf{H})}^2 + \|ML^{-1}\|_{\mathcal{L}(\mathbf{H})}^2)\ \|y\|_{\mathbf{D}}^2,$$

so the operator $L^{-1}M : \mathbf{D} \to \mathbf{D}$ is bounded. For the operator $L^{-1}B : U \to \mathbf{D}$, for every $y \in U$, we have:

$$\|(L^{-1}B)y\|_{\mathbf{D}}^2 = \|L^{-1}By\|_{\mathbf{H}}^2 + \|ML^{-1}By\|_{\mathbf{H}}^2 \leq$$

$$\|L^{-1}\|_{\mathcal{L}(\mathbf{H})}^2 \|By\|_{\mathbf{H}}^2 + \|ML^{-1}\|_{\mathcal{L}(\mathbf{H})}^2 \|By\|_{\mathbf{H}}^2 \;\leq$$

$$\|B\|_{\mathcal{L}(U,\mathbf{H})}^2 (\|L^{-1}\|_{\mathcal{L}(\mathbf{H})}^2 + \|ML^{-1}\|_{\mathcal{L}(\mathbf{H})}^2)\ \|y\|_U^2,$$

so the operator $L^{-1}B : U \to \mathbf{D}$ is also bounded. So, for problem (7) the following statements hold:

(B_1) The operator $L^{-1}M : \mathbf{D} \to \mathbf{D}$ is the infinitesimal generator of a uniformly continuous semigroup of operators $S(t),\ t \geq 0$, in $\mathbf{D}$.

(B_2) $L^{-1}B \in \mathcal{L}(U, \mathbf{D})$

(B_3) Since the composition of measurable functions is measurable, $L^{-1}F$ is a $\mathbf{D}-$valued predictable process, with Bochner integrable, on arbitrary finite interval [0,T], trajectories since

$$\int_0^T \|L^{-1}F(s)\|_{\mathbf{D}} \;\; ds \leq (\|L^{-1}\|^2_{\mathcal{L}(\mathbf{H})} + \|ML^{-1}\|^2_{\mathcal{L}(\mathbf{H})})^{1/2} \int_0^T \|F(s)\|_{\mathbf{H}} \;\; ds < \infty,$$

P-as.

(B_4) ξ is a $\mathbf{D}-$valued, $\mathscr{F}_0-$measurable random variable and $\mathscr{E}_t$, $t \geq 0$, is the unknown $\mathbf{D}-$valued process.

Therefore, we have the following definitions for the solutions of problem (7).

Definition 3.3 A $\mathbf{D}-$valued predictable process $\mathscr{E}_t$, $t \in [0, T]$, is called a weak solution of problem (7) if:

1 $\int_0^T \|\mathscr{E}_s\|_{\mathbf{D}} \;\; ds < \infty$, P-a.s.

2 For every $\zeta \in \mathbf{D}$ holds
$$\langle \mathscr{E}_t, \zeta \rangle = \langle \xi, \zeta \rangle + \int_0^t [\langle \mathscr{E}_s, [L^{-1}M]^* \zeta \rangle + \langle L^{-1}F(s), \zeta \rangle] \;\; ds + \langle L^{-1}BW_t, \zeta \rangle,$$
P-as, for every $t \in [0, T]$.

Definition 3.4 A $\mathbf{D}-$valued predictable process $\mathscr{E}_t$, $t \in [0, T]$, is called a strong solution of problem (7) if:

1 $\int_0^T \|L^{-1}M\mathscr{E}_s\|_{\mathbf{D}} \;\; ds < \infty$, P-a.s.

2 $\mathscr{E}_t = \xi + \int_0^t [L^{-1}M\mathscr{E}_s + L^{-1}F(s)] \;\; ds + L^{-1}BW_t$, P-as, for every $t \in [0, T]$.

3.2. *Well posedness of the stochastic Cauchy problem*

We now have the following theorem:

Theorem 1 Problem (7) under the assumptions (B1)-(B4) is strongly well posed.

Proof. First, in order to prove that there is a unique weak solution for problem (7), under the assumptions (B1)-(B4), given by the form:

$$\mathscr{E}_t = S(t)\xi + \int_0^t S(t-s)L^{-1}F(s) \;\; ds + \int_0^t S(t-s)L^{-1}B \;\; dW_s, \quad t \in [0, T],$$

we only need to show (see [5]) that the stochastic convolution $\int_0^t S(t-s)B \;\; dW_s$ satisfies the property

$$\int_0^t \|S(s)L^{-1}B\|^2_{L_2^0} \;\; ds < \infty, \quad t \in [0,T].$$

Indeed, since $S(t)$, $t \geq 0$ is a uniformly continuous semigroup of operators, i.e $\|S(t)\|_{\mathcal{L}(\mathbf{D})} \leq e^{T\|L^{-1}M\|_{\mathcal{L}(\mathbf{D})}}$, for $t \in [0,T]$, we have:

$$\begin{aligned}
&\int_0^t \|S(s)L^{-1}B\|^2_{L_2^0} \;\; ds = \int_0^t \|S(s)L^{-1}B\|_{L_2(U_0,\mathbf{D})} \;\; ds = \\
&\int_0^t \|S(s)L^{-1}BQ^{1/2}\|^2_{L_2(U,\mathbf{D})} \;\; ds = \int_0^t \sum_{n=1}^{\infty} \|S(s)L^{-1}BQ^{1/2}e_n\|^2_{\mathbf{D}} \;\; ds \leq \\
&\int_0^t \|S(s)\|^2_{\mathcal{L}(\mathbf{D})} \|L^{-1}B\|^2_{\mathcal{L}(U,D)} \sum_{n=1}^{\infty} \|Q^{1/2}e_n\|^2_U \;\; ds \leq \\
&Te^{2T\|L^{-1}M\|_{\mathcal{L}(\mathbf{D})}} \|L^{-1}B\|^2_{\mathcal{L}(U,\mathbf{D})} \mathbf{Tr}[Q] < \infty, \quad t \in [0,T],
\end{aligned}$$

where by $\{e_n\}_{n=1}^{\infty}$ we denote an orthogonal basis in U. So, there is a unique weak solution for problem (7). Since $L^{-1}M$ is bounded, we can easily check that this weak solution satisfies the assertion 1, of definition 3.4, for a strong solution. Indeed, we have

$$\int_0^T \|L^{-1}M\mathscr{E}_s\|_{\mathbf{D}} \;\; ds \;\leq\; \|L^{-1}M\|_{\mathcal{L}(\mathbf{D})} \int_0^T \|\mathscr{E}_s\|_{\mathbf{D}} \;\; ds < \infty, \quad \text{P-as.}$$

Also, for every $\zeta \in \mathbf{D}$ holds:

$$\langle \mathscr{E}_t, \; \zeta\rangle = \langle \xi, \; \zeta\rangle + \int_0^t [\langle \mathscr{E}_s, \; [L^{-1}M]^*\zeta\rangle + \langle L^{-1}F(s), \; \zeta\rangle] \;\; ds + \langle L^{-1}BW_t, \; \zeta\rangle,$$

P-as, for every $t \in [0,T]$.

But $\mathscr{E}_s \in D(L^{-1}M) = \mathbf{D}$, for every $s \in [0,T]$, so

$$\langle \mathscr{E}_t, \; \zeta\rangle \;\; = \langle \xi, \; \zeta\rangle + \int_0^t \langle L^{-1}M\mathscr{E}_s + L^{-1}F(s), \; \zeta\rangle \;\; ds + \langle L^{-1}BW_t, \; \zeta\rangle,$$

P-as, for every $t \in [0,T]$.

Now, we have

$$\langle \mathscr{E}_t - \xi - \int_0^t \; [L^{-1}M\mathscr{E}_s + L^{-1}F(s)] \;\; ds - L^{-1}BW_t \;\; , \;\; \zeta\rangle = 0, \quad \text{P-as, for all } \; t \in [0,T]$$

and every $\zeta \in \mathbf{D}$. Consequently:

$$\mathscr{E}_t = \xi + \int_0^t \; [L^{-1}M\mathscr{E}_s + L^{-1}F(s)] \;\; ds + L^{-1}BW_t, \quad \text{P-as,} \quad \text{for every } \; t \in [0,T],$$

so this weak solution is also a strong solution. We can also check by the form of the solution, that problem (7) is strongly well posed. □

Remarks. We have treated problem (6) in two different ways. Using the transformation in section 3.1 we found that this problem has a unique strong solution. Multiplying from the left by the bounded operator L^{-1} and taking advantage of the fact that $D(L) = D(M)$, in section 3.2 we found again that this problem has a unique strong solution but well posedness also confirmed. Since these two solutions coincide in both spaces **D**, **H**, a property for the uniformly continuous semigroups $S(t), T(t)$ generated by the bounded operators $L^{-1}M, ML^{-1}$ in the Hilbert spaces **D** and **H** respectively, is obtained. In particular, this property is an operators equality in the case $D(L) = D(M)$:

$$S(t) = L^{-1}T(t)L \;\; in \;\; \mathcal{L}(\mathbf{D}) \;\; and \;\; S(t)L^{-1} = L^{-1}T(t) \;\; in \;\; \mathcal{L}(\mathbf{H}), \;\; for\ all \;\; t \in [0,T],$$

which is in accordance with the similarity of the operators $L^{-1}M,\ ML^{-1}$ presented in chapter 2 in [16].

Note that the density of $D(L), D(M)$ in **H**, is not necessary for our abstract models, so it is not assumed even if it holds. We will need the density of $D(M)$ in what follows:

Let us consider the case that $\beta = 0$. Then problem (6) takes the (achiral) form in **H**:

$$\left.\begin{aligned} d\mathscr{E}_t &= [M\mathscr{E}_t + F(t)] \;\; dt + B \;\; dW_t, \quad t \geq 0 \\ \mathscr{E}_0 &= \xi \end{aligned}\right\} \tag{8}$$

where $W_t,\ t \geq 0$, is an $\mathbf{H}-$valued Q-Wiener process, with $Tr[Q] < \infty$, and the assumptions (A1), (A3)-(A5) of problem (6) hold.

We will see that problem (8) is weakly well posed. We prove first that the operator M: $D(M) = \mathbf{W}_S \times \mathbf{W}_S \to \mathbf{H}$ is skew-adjoint ($M^* = -M$). For $u \in D(M),\ v \in \mathbf{H}$, we have:

$$< Mu,\ v >_{\mathbf{H}} = \left(\begin{pmatrix} \varepsilon^{-1}\ curl u_2 \\ -\mu^{-1}\ curl u_1 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)_{\mathbf{H}} =$$

$$\int_\Omega (curl u_2 \cdot v_1 - curl u_1 \cdot v_2) \;\; dx = < Su_2,\ v_1 >_W \; - < Su_1,\ v_2 >_W .$$

Now, for $u \in \mathbf{H},\ v \in\ D(M)$ we have:

$$< u,\ -Mv >_{\mathbf{H}} = \left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \begin{pmatrix} -\varepsilon^{-1}\ curl v_2 \\ \mu^{-1}\ curl v_1 \end{pmatrix} \right)_{\mathbf{H}} =$$

$$\int_\Omega (-u_1 \cdot curl v_2 + u_2 \cdot curl v_1) \;\; dx = < u_2,\ Sv_1 >_W - < u_1,\ Sv_2 >_W =$$

$< Su_2,\ v_1 >_W \; - < Su_1,\ v_2 >_W = < Mu,\ v >_{\mathbf{H}}$, since S is self adjoint.

Hence, $-M \subset M^* \Rightarrow iM \subset (iM)^*$ so the operator iM is Hermitian. Since S

is invertible with $R(S) = \mathbf{W}$, it is easy to show that there is a $\lambda \in \mathbb{C}$ such that $R(iM - \lambda I) = R(iM - \lambda I) = \mathbf{H}$. This means that the Hermitian operator iM is self adjoint, hence the densely defined operator M is skew-adjoint, therefore generates a unitary group $(U(t))_{t\in\mathbb{R}}$ on $\mathbf{H}$.

Consequently, by [5], problem (8) is weakly well posed and the unique weak solution is given by the form

$$\mathscr{E}_t = U(t)\xi + \int_0^t U(t-s)F(s) \quad ds + \int_0^t U(t-s)B \quad dW_s, \quad t \in [0,T].$$

An idea is to return to the solutions of problems (6), (7) and check if they converge to the solution of problem (8) as $\beta \to 0$. But, as we observe by the remark in section 2, L^{-1} is not defined as $\beta \to 0$, hence both types of solutions are not defined as $\beta \to 0$.

So, we conclude that the choice of the functional background of this problem came up by some very reasonable assumptions and led to a strongly well posed Cauchy problem in the Hilbert space $\mathbf{D}$. Nevertheless, this choice does not allow us to address the convergence of the solution of the chiral problem to the solution of the achiral problem, as the chirality measure β tends to zero. This important issue is under consideration and will be treated separately.

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