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Quaternions and Elliptic Partial Differential Equations

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Abstract

We present an approach for the solution of boundary value problems for linear elliptic PDEs in four dimensions. We first derive integral representations of the solutions of the four dimensional Poisson and inhomogeneous Biharmonic equations using some novel quaternionic generalisations of an important formula in Complex Analysis, the so-called Dbar formula. As an application of our approach, we solve the Dirichlet and Neumann problems for the Poisson equation in the half space.

Keywords: Quaternions, Linear Elliptic PDEs

1. Introduction

A general method for the solution of boundary value problems for a large class of linear and integrable nonlinear equations was introduced in [1]. Among the equations that can be treated using this method are linear elliptic Partial Differential Equations in two dimensions. In [2] this method was extended from the case of *homogeneous* to the case of *inhomogeneous* linear elliptic PDEs. Furthermore, it was shown there that this method can be formulated in either the *physical* space (the complex $z-$ plane) or the complex *Fourier* space (the complex $k-$ plane). Here, we extend the physical space formulation from the case of two to the case of four dimensions (see also [3]). We first derive the quaternionic generalization of a fundamental problem in Complex Analysis, namely of the so-called Dbar problem [4]. In particular, in Section 3 we prove the the following Proposition:

Proposition 1.1 *Let the differentiable quaternion-valued function $\phi(x)$ satisfy the quatenionic Dbar equation*:

$$\overline{\partial_l}\phi(x) = f(x), \tag{1}$$

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in a simply connected and bounded domain D in R^4 with a smooth boundary ∂D, where $f(x)$ is a given quaternion-valued function with sufficient smoothness and decay.

Then, $\phi(x)$ admits the following integral representation:

$$\phi(x) = -\frac{1}{2\pi^2} \int_{\partial D} K(\xi;x) J^{\xi} \phi(\xi) + \frac{1}{2\pi^2} \int_D K(\xi;x) f(\xi) d\xi, \quad x \in D. \tag{2}$$

Furthermore, the function $\phi(x)$ satisfies the following global relation:

$$-\int_{\partial D} K(\xi;x) J^{\xi} \phi(\xi) + \int_D K(\xi;x) f(\xi) d\xi = 0, \quad x \notin D, \tag{3}$$

where $K(\xi;x)$ denotes the quaternionic generalisation of the Cauchy kernel

$$K(\xi;x) = \frac{(\zeta - z)^{-1}}{|\zeta - z|^2}, \tag{4}$$

the differential operator ∂_l is defined by

$$\partial_l \phi(\xi) = \frac{\partial \phi(\xi)}{\partial \xi_0} + e_i \frac{\partial \phi(\xi)}{\partial \xi_i}, \tag{5}$$

the scalar 4-form $d\xi$ is defined by

$$d\xi = d\xi_0 \wedge d\xi_1 \wedge d\xi_2 \wedge d\xi_3, \tag{6}$$

and J^{ξ} denotes the following quaternionic 3-form:

$$J^{\xi} = d\xi_1 \wedge d\xi_2 \wedge d\xi_3 - e_1 d\xi_2 \wedge d\xi_3 \wedge d\xi_0 + e_2 d\xi_3 \wedge d\xi_0 \wedge d\xi_1 - e_3 d\xi_0 \wedge d\xi_1 \wedge d\xi_2. \tag{7}$$

Using the identities

$$\partial_r\left(\frac{1}{|z|^2}\right) = -2K(x;0), \quad \partial_r \frac{1}{z} = \frac{2}{|z|^2}, \tag{8}$$

it is possible to derive certain variations of the basic equation (2), which are given in the following Propositions. The relevant proofs can be found in [5], (see also [3]).

Proposition 1.2 *(ϕ in terms of $\Delta\phi$ and ϕ in terms of $\partial_l \Delta\phi$)*

Let D be a bounded simply connected domain in R^4 with a smooth boundary ∂D. Let $\partial_l, d\xi, \partial_l, \Delta$ be defined by equations (5), (6), (20) and (21) below, respectively. Let $\phi(x)$ be a twice differentiable quaternion-valued function. Then the function $\phi(x)$ admits the following integral representation for $x \in D$:

$$\phi(x) = -\frac{1}{4\pi^2} \int_D \frac{\Delta\phi(\xi)}{|\zeta - z|^2} d\xi + \frac{1}{2\pi^2} \int_{\partial D} \left\{ K(\xi;x) J^{\xi} \phi(\xi) + \frac{\frac{1}{2} J^{\xi} \partial_l \phi(\xi)}{|\zeta - z|^2} \right\}. \tag{9}$$

Furthermore, if $\phi(x)$ is three times differentiable, then it also admits the integral representation

$$\phi(x) = \frac{1}{8\pi^2}\int_D (\zeta - z)^{-1}(\partial_l \Delta)\phi(\xi)d\xi + \frac{1}{2\pi^2}\int_{\partial D}\left\{K(\xi;x)J^\xi\phi(\xi) + \frac{\frac{1}{2}J^\xi\partial_l\phi(\xi)}{|\zeta - z|^2}\right\}$$

$$-\frac{1}{8\pi^2}\int_{\partial D}(\zeta - z)^{-1}J^\xi\Delta\phi(\xi), \qquad x \in D. \tag{10}$$

Proposition 1.3 *(an alternative representation for ϕ in terms of $\Delta\phi$ and $\partial_l\Delta\phi$)*

Let D be a bounded simply connected domain in R^4 with a smooth boundary ∂D. Let $\partial_l, d\xi, \partial_l, \Delta$ be defined as in Proposition 1.2. Let $\phi(x)$ be a three times differentiable real function. Then the function $\phi(x)$ admits the following integral representations for $x \in D$:

$$\phi(x) = -\frac{1}{4\pi^2}\int_D \frac{\Delta\phi(\xi)}{|\zeta - z|^2}d\xi$$
$$+\frac{1}{2\pi^2}\int_{\partial D}\left\{\frac{\left[(\xi_0 - x_0)J_0^\xi + (\xi_j - x_j)J_j^\xi\right]\phi}{|\zeta - z|^4} + \frac{\frac{1}{2}(\phi_{\xi_0}J_0^\xi + \phi_{\xi_j}J_j^\xi)}{|\zeta - z|^2}\right\} \tag{11}$$

and

$$\phi(x) = \frac{1}{8\pi^2}\int_D (\zeta - z)^{-1}(\partial_l\Delta)\phi(\xi)d\xi$$
$$+\frac{1}{2\pi^2}\int_{\partial D}\left\{\frac{[(\xi_0 - x_0)J_0^\xi + (\xi_j - x_j)J_j^\xi]\phi}{|\zeta - z|^4} + \frac{\frac{1}{2}(\phi_{\xi_0}J_0^\xi + \phi_{\xi_j}J_j^\xi)}{|\zeta - z|^2}\right\}$$
$$-\frac{1}{8\pi^2}\int_{\partial D}(\zeta - z)^{-1}J^\xi\Delta\phi(\xi), \qquad x \in D, \tag{12}$$

where $J^\xi = J_0^\xi + e_jJ_j^\xi$.

In the following Section we introduce the basic Definitions and Notation.

2. Definitions and Notation.

Let $x_{0,}\{x_i\}_1^3$ scalars. Let $\{e_i\}_1^3$ be the elements satisfying

$$e_1^2 = e_2^2 = e_3^2 = -1,\ e_1e_2 = e_3,\ e_2e_1 = -e_3, \tag{13}$$

as well as the expressions obtained from the above by the cyclic permutation $\{0 \to 0, 1 \to 2, 2 \to 3, 3 \to 1\}$.

The *quaternionic* variable z is defined by

$$z = x_0 + x_je_j, \quad j = 1, 2, 3, \tag{14}$$

Also, the variable ζ is defined by

$$\zeta = \xi_0 + \xi_j e_j, \tag{15}$$

where $\xi_0, \{\xi_j\}_1^3$ are scalars.

Let $\phi(x)$ be a quaternion valued function, then

$$\phi(x) = \phi_0(x) + \phi_j(x)e_j, \tag{16}$$

where $\phi_0(x), \{\phi_j(x)\}_1^3$ are real-valued functions and we use x to denote the ordered independent variables (x_0, x_1, x_2, x_3). We will refer to the functions $\phi_0(x)$ and $\{\phi_j(x)\}_1^3$ as the *components* of ϕ. The conjugate of $\phi(x)$ is defined by

$$\phi(x) = \phi_0(x) - \phi_j(x)e_j. \tag{17}$$

Then,

$$\phi\phi = \phi^2 = \phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2. \tag{18}$$

In general, the product is non-commutative , thus we define *both* the right and the left ∂ derivatives of a quaternion-valued function $\phi(x)$ given by (5) as well as

$$\partial_r\phi(x) = \frac{\partial\phi(x)}{\partial x_0} + \frac{\partial\phi(x)}{\partial x_i}e_i, \tag{19}$$

$$\partial_r\phi(x) = \frac{\partial\phi(x)}{\partial x_0} - \frac{\partial\phi(x)}{\partial x_i}e_i, \partial_l\phi(x) = \frac{\partial\phi(x)}{\partial x_0} - e_i\frac{\partial\phi(x)}{\partial x_i}. \tag{20}$$

Furthermore, the Laplacian operator in four dimensions can be factorised of the basic quaternionic derivatives (5), (19) and (20) as follows:

$$\partial_l\partial_l = \partial_l\partial_l = \partial_{x_0}^2 + \partial_{x_j}\partial_{x_j} \doteq \Delta, \quad \partial_r\partial_r = \partial_r\partial_r = \Delta. \tag{21}$$

3. The Poisson and inhomogeneous Biharmonic equations

We first prove Proposition 1.1, which we use later in this Section to construct integral representations.

Proof of Proposition 1.1. Define the quaternionic differential form

$$W(\xi;x) = K(\xi;x)J^\xi\phi(\xi). \tag{22}$$

Then,

$$\begin{aligned} dW(\xi;x) \quad &= \quad d\xi_0 \wedge \left[\frac{\partial K(\xi;x)}{\partial\xi_0}J^\xi\phi(\xi) + K(\xi;x)J^\xi\frac{\partial\phi(\xi)}{\partial\xi_0}\right] \\ &\quad + d\xi_j \wedge \left[\frac{\partial K(\xi;x)}{\partial\xi_j}J^\xi\phi(\xi) + K(\xi;x)J^\xi\frac{\partial\phi(\xi)}{\partial\xi_j}\right]. \end{aligned}$$

Combining terms, the above equation yields

$$dW(\xi;x) = \left[\partial_r K(\xi;x)\phi(\xi) + K(\xi;x)f(\xi)\right] d\xi. \tag{23}$$

Since ϕ satisfies the Poisson equation, using the identity

$$\partial_r \left(K(\xi;x)\right) = 2\pi^2\delta(\zeta - z), \quad z \in D \tag{24}$$

equation (23) yields

$$dW = K(\xi;x)f(\xi)d\xi + \begin{cases} 2\pi^2\delta(\zeta - z)\phi(\xi)d\xi, & z \in D \\ 0, & z \notin D. \end{cases} \tag{25}$$

Poincaré-Stokes lemma

$$\int_{\partial D} W = \int\int_D dW, \tag{26}$$

and equation (23a) yield equation (2). Similarly, equation (23b) implies equation (3). □

There exists a large class of equations that can be written in the form of equation (1). Here, we consider the Poisson and the inhomogeneous biharmonic equations. Using Proposition 1.1, we construct integral representations of the solutions of the four dimensional versions of these equations. These representations are given in the following Propositions.

Proposition 3.1 *Let the differentiable quaternion-valued function $\phi(x)$ satisfy the Poisson equation:*

$$\Delta\phi(x) = f(x), \tag{27}$$

in a simply connected and bounded domain D in $\mathbb{R}^4$ with a smooth boundary ∂D, where $f(x)$ is a given quaternion-valued function with sufficient smoothness and decay.

Then, $\phi(x)$ admits the following integral representation:

$$\partial_l\phi(x) = -\frac{1}{2\pi^2}\int\limits_{\partial D} K(\xi;x)J^\xi\partial\phi(\xi) + \frac{1}{2\pi^2}\int_D K(\xi;x)f(\xi), \quad k \in D. \tag{28}$$

Furthermore, the function $\phi(x)$ satisfies the global relation:

$$-\int\limits_{\partial D} K(\xi;x)J^\xi\phi(\xi) + \int_D K(\xi;x)f(\xi) = 0, \quad k \notin D. \tag{29}$$

Proof. The results of this Proposition follow immediately from Proposition 1.1 after replacing $\phi(x)$ with $\partial_l\phi(x)$. □

Proposition 3.2 *Let D be a bounded simply connected domain in $\mathbb{R}^4$ with a smooth boundary ∂D and ∂_l, ∂_l and J^{ξ} be defined as in Proposition 1.2 . Let the quaternion-valued function $\phi(x)$ satisfy the inhomogeneous Biharmonic equation in four dimensions*

$$\Delta^2 \phi(x) = h(x), \quad x \in D, \tag{30}$$

where $h(x)$ is a given quaternion-valued function with sufficient smoothness and decay. Then, for $x \in D$, $\phi(x)$ admits the integral representation

$$\partial_l^2 \phi(x) = -\frac{1}{2\pi^2} \int_{\partial D} K(\xi; x) J^{\xi} \left(\partial_l^2 \phi + (x_0 - \xi_0) \partial_l \partial_l^2 \phi \right)(\xi) \\ - \frac{1}{2\pi^2} \int_D K(\xi; x)(\xi_0 - x_0) h(\xi) d\xi. \tag{31}$$

Furthermore, for $x \notin D$, the boundary values of $\phi(x)$ satisfy the global relations

$$\int_{\partial D} K(\xi; x) J^{\xi} \left(\partial_l^2 \partial_l \phi(\xi) \right) = \int_D K(\xi; x) h(\xi) d\xi, \tag{32}$$

$$\int_{\partial D} K(\xi; x) J^{\xi} \left(\partial_l^2 \phi - x_0 \partial_l \partial_l^2 \phi \right)(\xi) = -\int_D K(\xi; x) x_0 h(\xi) d\xi. \tag{33}$$

Proof. Replacing in equation (2) ϕ by $\partial_l^2 \partial_l \phi$, as well as ϕ by $\partial_l^2 \phi - x_0 \partial_l \partial_l^2 \phi$, we find the following integral representations:

$$\partial_l^2 \partial_l \phi(x) = -\frac{1}{2\pi^2} \int_{\partial D} K(\xi; x) J^{\xi} \left(\partial_l^2 \partial_l \phi(\xi) \right) + \frac{1}{2\pi^2} \int_D K(\xi; x) h(\xi) d\xi, \tag{34}$$

$$\left(\partial_l^2 \phi - x_0 \partial_l \partial_l^2 \phi \right)(x) = -\frac{1}{2\pi^2} \int_{\partial D} K(\xi; x) J^{\xi} \left(\partial_l^2 \phi - \xi_0 \partial_l \partial_l^2 \phi \right)(\xi) \\ - \frac{1}{2\pi^2} \int_D K(\xi; x) \xi_0 h(\xi) h\xi. \tag{35}$$

Multiplying equation (34) by x_0 and adding it to equation (35), we obtain equation (31).

The global relations (32) and (33) follow from the derivation of equations (34) and (35), using the fact that if $x \notin D$, then $\partial_r K = 0$. □

4. The Dirichlet and Neumann Problems in the Half Space

As an application of our approach, we solve certain boundary value problems for the Poisson equation in four dimensions. In particular, equation (11) immediately implies the following result.

Proposition 4.1 *Let D be a bounded simply connected domain in R^4 with a smooth boundary ∂D. Let the real-valued function $\phi(x)$ satisfy the Poisson equation in four dimensions*

$$\Delta\phi(x) = h(x), \quad x \in D, \tag{36}$$

where $h(x)$ is a given real-valued function with sufficient smoothness. Then, for $x \in D$, $\phi(x)$ admits the integral representation

$$\phi(x) = -\frac{1}{4\pi^2}\int_D \frac{h(\xi)}{|\zeta - z|^2} d\xi + \frac{1}{2\pi^2}\int_{\partial D}\left\{\frac{\left[(\xi_0 - x_0)J_0^\xi + (\xi_j - x_j)J_j^\xi\right]\phi}{|\zeta - z|^4} + \frac{(\phi_{\xi_0}J_0^\xi + \phi_{\xi_j}J_j^\xi)}{2\,|\zeta - z|^2}\right\}. \tag{37}$$

Furthermore, for $x \notin D$, the boundary values of ϕ satisfy the following global relation

$$0 = -\frac{1}{4\pi^2}\int_D \frac{h(\xi)}{|\zeta - z|^2} d\xi + \frac{1}{2\pi^2}\int_{\partial D}\left\{\frac{\left[(\xi_0 - x_0)J_0^\xi + (\xi_j - x_j)J_j^\xi\right]\phi}{|\zeta - z|^4} + \frac{(\phi_{\xi_0}J_0^\xi + \phi_{\xi_j}J_j^\xi)}{2\,|\zeta - z|^2}\right\}. \tag{38}$$

Proof. Equation (37) follows immediately from equation (11). Equation (38) follows from the derivation of equation (11) using the fact that if $x \notin D$, then $\partial_r K = 0$. □

Proposition 4.2 *Let the real-valued function $\phi(x)$ satisfy the Poisson equation*

$$\Delta\phi(x) = h(x), \quad x_0 \geq 0, \quad -\infty < x_j < \infty, \quad j = 1, 2, 3, \tag{39}$$

where $h(x)$ is a given real-valued function with sufficient smoothness and decay.
(a) Let $\phi(x)$ satisfy the Dirichlet boundary condition

$$\phi(0, x_1, x_2, x_3) = d(x_1, x_2, x_3), \tag{40}$$

where the function $d(x_1, x_2, x_3)$ has appropriate smoothness and decay. Then the function $\phi(x)$ is given by

$$\phi(x) = -\frac{1}{4\pi^2}\int_D \frac{h(\xi)}{(\xi_0 - x_0)^2 + (\xi_j - x_j)^2} d\xi + \frac{1}{4\pi^2}\int_D \frac{h(\xi)}{(\xi_0 + x_0)^2 + (\xi_j - x_j)^2} d\xi - \frac{x_0}{\pi^2}\int_{\{\xi_0 = 0\}} \frac{d(\xi_1, \xi_2, \xi_3)}{\left[{x_0}^2 + (\xi_j - x_j)^2\right]^2} d\xi_1 d\xi_2 d\xi_3. \tag{41}$$

(b) Let $\phi(x)$ satisfy the Neumann boundary condition

$$\phi_{x_0}(0, x_1, x_2, x_3) = n(x_1, x_2, x_3), \tag{42}$$

where the function $n(x_1, x_2, x_3)$ has appropriate smoothness and decay. Then the function $\phi(x)$ is given by

$$\phi(x) = -\frac{1}{4\pi^2}\int_D \frac{h(\xi)}{(\xi_0 - x_0)^2 + (\xi_j - x_j)^2} d\xi - \frac{1}{4\pi^2}\int_D \frac{h(\xi)}{(\xi_0 + x_0)^2 + (\xi_j - x_j)^2} d\xi$$
$$+ \frac{1}{2\pi^2}\int_{\{\xi_0=0\}} \frac{n(\xi_1, \xi_2, \xi_3)}{x_0{}^2 + (\xi_j - x_j)^2} d\xi_1 d\xi_2 d\xi_3. \tag{43}$$

Proof. The definition of J^ξ implies that for $\xi_0 = 0$, $J^\xi = d\xi_1 \wedge d\xi_2 \wedge d\xi_3 = J_0^\xi$. Hence equation (37) implies that for $x_0 > 0$, $\quad -\infty < x_j < \infty$, $\quad j = 1, 2, 3$, $\phi(x)$ is given by the equation

$$\phi(x) = -\frac{1}{4\pi^2}\int_D \frac{h(\xi)}{(\xi_0 - x_0)^2 + (\xi_j - x_j)^2} d\xi$$
$$+ \frac{1}{2\pi^2}\int_{\{\xi_0=0\}} \left\{ \frac{-x_0\phi(0, \xi_1, \xi_2, \xi_3)}{[x_0{}^2 + (\xi_j - x_j)^2]^2} + \frac{\frac{1}{2}\phi_{\xi_0}(0, \xi_1, \xi_2, \xi_3)}{x_0{}^2 + (\xi_j - x_j)^2} \right\} d\xi_1 d\xi_2 d\xi_3. \tag{44}$$

Equation (38) yields an equation similar to equation (44) where the left hand side is replaced by zero, and which is valid for $x_0 < 0$. Letting in this equation $x_0 \to -x_0$ we find that for $x_0 > 0$, $\quad -\infty < x_j < \infty$, $\quad j = 1, 2, 3$, the following equation is valid,

$$0 = -\frac{1}{4\pi^2}\int_D \frac{h(\xi)}{(\xi_0 + x_0)^2 + (\xi_j - x_j)^2} d\xi$$
$$+ \frac{1}{2\pi^2}\int_{\{\xi_0=0\}} \left\{ \frac{x_0\phi(0, \xi_1, \xi_2, \xi_3)}{[x_0^2 + (\xi_j - x_j)^2]^2} + \frac{\frac{1}{2}\phi_{\xi_0}(0, \xi_1, \xi_2, \xi_3)}{x_0{}^2 + (\xi_j - x_j)^2} \right\} d\xi_1 d\xi_2 d\xi_3. \tag{45}$$

If ϕ is given, then subtracting equations (44) and (45), we obtain (41). If ϕ_{ξ_0} is given, then adding equations (44) and (45), we obtain (43). □

Remark. The solution of the Dirichlet problem of the Laplace equation is derived in [6] using Green's functions and the method of images. The simple derivation presented here generalises this solution to the Poisson equation. Furthermore, at the same time it provides the solution of the Neumann problem.

5. Conclusions

Quaternions were discovered in 1843 by the famous Irish mathematician Sir William Rowan Hamilton (1805-1865), [7]. Hamilton was trying to obtain a generalisation of Complex Numbers. In particular, he wanted to find a number system which would model rotations in the three dimensional space in an analogous way that Complex Numbers model rotations on the plane. He finally succeeded in achieving this goal, at the expense of commutativity of multiplication. The discovery of Quaternions was published in its final form in Hamilton's famous book "Elements of Quaternions" in 1866 [8].

Based on the work of Hamilton, William Kingdom Clifford (1845-1879) introduced the so-called Clifford algebras in 1878 [9]. The Quaternion algebra is a special case of Clifford algebras, namely it is isomorphic to the four dimensional Clifford algebra. Wolfgang Pauli (1900-1958) rediscovered Quaternions as a matrix representation of the four dimensional Clifford algebra. In the 1930's the Swiss mathematician Rudolf Fueter (1880-1950), a student of David Hilbert (1862-1943), motivated by number theoretic problems, started research on hypercomplex function theory and introduced the analogue of the Cauchy-Riemann equations in four dimensions. Also, Paul Dirac (1902-1984) developed a quaternionic formulation for the equations of motion of a spinning electron in quantum mechanics.

Today, Quaternions and Clifford Algebras appear in many areas of physics, mathematics and engineering, from quantum theory to computer vision. However, the *analytic component* of the theory of Quaternions seems to remain underused in applications. In an attempt to enhance this aspect of the theory of Quaternions, we present here a novel application of this theory to the solution of boundary value problems for linear elliptic PDEs in four dimensions. In particular, we construct integral representations of the solutions of the Poisson and inhomogeneous Biharmonic equations (see equations (28) and (31)). Furthermore, using some novel generalisations of the quaternionic analogue of the Dbar (or Pompeiu or Cauchy-Green) formula, we solve the Dirichlet and Neumann problems for the Poisson equation in the half space (see equations (41) and (43)).

References

1. A.S. Fokas, On the integrability of linear and nonlinear partial differential equations, J. Math. Phys., **41**, 6, 4188 (2000)
2. A.S. Fokas and D.A. Pinotsis, The Dbar formalism for certain non homogeneous linear elliptic equations in two dimensions, Eur. J. Appl. Math., **17**, 3, 323-346 (2006)
3. A.S. Fokas and D.A. Pinotsis, Quaternions, Evaluation of Integrals and Boundary Value Problems, Comp. Meth. Fun. Theor. (to appear)
4. M.J. Ablowitz and A.S. Fokas, *Introduction and Applications of Complex Variables*, Cambridge University Press, 2nd Edition (2003)
5. D.A. Pinotsis, The Dbar formalism, Quaternions and Applications, *PhD Thesis*, University of Cambridge (2006)
6. P. R. Garabedian, *Partial differential equations*, Wiley (1964).
7. W.R. Hamilton, On a new Species of Imaginary Quantities connected with a theory of Quaternions, Proc. R. Irish. Acad., **2** 424 (1843)
8. W.R. Hamilton, *Elements of Quaternions,* Langmans Green, London (1866)
9. W.K. Clifford, Applications of Grassmann's extensive algebra, Amer. J. Math, **1**, 350 (1878)

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