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On the scattering of 2D–elastic waves generated by dyadic point–sources

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Abstract

The problem of scattering of a time–harmonic elastic wave, due to a dyadic point–source field by a rigid body in two–dimensional linear elasticity is considered. Some scattering theorems for elastic waves emanating from point–sources are presented. The solution for the scattering of elastic waves by a rigid circular scatterer in the form of infinite series is established. Finally, for this scatterer and in view of low–frequency results, a method for a far–field inverse problem is described, where an approximation of the known exact solution is established.

Keywords: Linear elasticity, dyadic scattering, point–sources, scattering relations

1. Introduction

In this paper scattering of elastic waves related to point–generated wave fields is considered. The interest in problems of scattering for point–source incidence, is connected due to the variety of applications coming from the theoretical analysis of biological studies at the cell level, from geophysics, from modelling in medicine and health sciences, e.t.c.

For acoustic scattering, similar results concerning point–source excitation can be found in [1, 2] and the references therein, for electromagnetism in [3] and for the case of elastic scattering, in [4].

In this paper, the scattering of a time–harmonic elastic point–source field by a rigid body in two dimensions is considered. We organize our paper as follows. In Section 2 the problem of scattering of elastic waves by a rigid body in dyadic form is formulated. In Section 3, dyadic far–field pattern generators are defined, scattering relations are presented and expressions for the differential and the scattering cross–section, due to point–source dyadic incidence are given. Finally, in Section 4 the

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case of a rigid circular scatterer is considered. In particular, an incident elastic wave due to a point–source, as well as the corresponding scattered field in terms of Navier eigenvectors are given, and a method for low-frequency results by approximating the exact solution is presented.

2. Setting up the problem

Let B denote an open, bounded and simply connected subset of R^2 with boundary ∂B, which is assumed to be a bounded Lyapunov surface. The set B will be referred to as the scatterer. The exterior domain $B_e = \mathrm{R}^2 \setminus B$, where $B = B \cup \partial B$, is characterized by the Lamé constants λ and μ and mass density ρ. The Lamé constants λ and μ are assumed to satisfy the strong ellipticity conditions $\mu > 0,\ \lambda + 2\mu > 0$, in order for the medium to sustain longitudinal as well as transverse waves. In what follows we consider the scattering problem in dyadic formulation. As Twersky [12] pointed out for electromagnetic waves, the dyadic scattering problem – because of its higher symmetry – is easier than the corresponding vector scattering problem.

Assuming harmonic time depedence $e^{-i\omega t}$, where ω denotes the angular frequency, the governing equation of linearized elasticity which the displacement field satisfies in the region B_e, is the well known spectral Navier equation [11]

$$(\Delta^* + \rho\,\omega^2)\,\widetilde{\mathbf{u}}(\mathbf{r}) = \widetilde{\mathbf{0}}, \quad \mathbf{r} \in B_e, \tag{1}$$

where Δ^* is the differential operator given by $\Delta^* = \mu\,\Delta + (\lambda + \mu)\,\nabla\nabla\cdot$.

We irradiate our object by an incident elastic wave due to a source located at a point with position vector $\mathbf{a}$, [4], given by,

$$\begin{aligned}\widetilde{\mathbf{u}}_a^{inc}(\mathbf{r}) \;&=\; -\frac{i}{4\omega^2}\,\nabla_{\mathbf{r}}\nabla_{\mathbf{r}} H_0^{(1)}(k_p\,|\mathbf{r}-\mathbf{a}|) \\ &\quad + \frac{i}{4\omega^2}\,(\nabla_{\mathbf{r}}\nabla_{\mathbf{r}} + k_s^2\,\widetilde{\mathbf{I}})\,H_0^{(1)}(k_s\,|\mathbf{r}-\mathbf{a}|), \quad \mathbf{r} \neq \mathbf{a},\end{aligned} \tag{2}$$

where $\widetilde{\mathbf{I}}$ is the identity dyadic, and $H_0^{(1)}(z)$, is the Hankel function of first kind and zero order. This is actually like the fundamental solution with a singularity at the point $\mathbf{a}$, [4], and k_p, k_s are the wave numbers of the longitudinal and transverse wave, given by $k_p = \omega\,\sqrt{\varrho/(\lambda + 2\mu)}$ and $k_s = \omega\,\sqrt{\varrho/\mu}$, respectively.

Let now $a = |\mathbf{a}| \to \infty$, then, the incident point source field given by (2) reduces to a dyadic plane wave with direction of propagation $-\hat{\mathbf{a}}$. Indeed, using the asymptotic behaviour of the Hankel function of the first kind and zero order [9], we obtain

$$\widetilde{\mathbf{u}}^{inc}(\mathbf{r}; -\hat{\mathbf{a}}) = A_p\,(\hat{\mathbf{a}} \otimes \hat{\mathbf{a}})\,e^{-ik_p\,\mathbf{r}\cdot\hat{\mathbf{a}}} + A_s(\widetilde{\mathbf{I}} - \hat{\mathbf{a}} \otimes \hat{\mathbf{a}})\,e^{-ik_s\,\mathbf{r}\cdot\hat{\mathbf{a}}}, \tag{3}$$

where "$\otimes$" is the juxtaposition between two vectors (this gives a dyadic) and A_p, A_s are constant amplitudes given by

$$A_p := \frac{1}{\lambda + 2\mu}\,\frac{(1+i)\,e^{ik_p\,a}}{4\sqrt{\pi k_p\,a}} \quad \text{and} \quad A_s := \frac{1}{\mu}\,\frac{(1+i)\,e^{ik_s\,a}}{4\sqrt{\pi k_s\,a}}. \tag{4}$$

Due to the point source incident field at $\mathbf{a}$, the corresponding component of the scattered field is denoted by $\widetilde{\mathbf{u}}_a^{sct}$. Then the total field $\widetilde{\mathbf{u}}_a^{tot}$ in the exterior B of the scatterer is given by

$$\widetilde{\mathbf{u}}_a^{tot}(\mathbf{r}) = \widetilde{\mathbf{u}}_a^{inc}(\mathbf{r}) + \widetilde{\mathbf{u}}_a^{sct}(\mathbf{r}), \tag{5}$$

where the incident, the scattered and the total field satisfy Eq. (1). The Helmholtz decomposition [9] of $\widetilde{\mathbf{u}}_a^{sct}$ into the irrotational (P-wave) $\widetilde{\mathbf{u}}_a^{sct,\,p}$ and the solenoidal (S-wave) $\widetilde{\mathbf{u}}_a^{sct,\,s}$ is

$$\widetilde{\mathbf{u}}_a^{sct}(\mathbf{r}) = \widetilde{\mathbf{u}}_a^{sct,p}(\mathbf{r}) + \widetilde{\mathbf{u}}_a^{sct,s}(\mathbf{r}) \tag{6}$$

If now, the total displacement field vanishes on the surface of the scatterer, i.e.,

$$\widetilde{\mathbf{u}}_a^{tot}(\mathbf{r}) = \widetilde{\mathbf{0}}, \qquad \mathbf{r} \in \partial B, \tag{7}$$

then, we have the rigid body problem.

For the well-posedness of the problem, the well known radiation conditions for the irrotational and solenoidal components of the scattered field $\widetilde{\mathbf{u}}_a^{sct,p}$, $\widetilde{\mathbf{u}}_a^{sct,s}$, respectively, due to Kupradze [8], should also be satisfied given in [11].

We now present the relation that hold for the far-field patterns of the scattered field. Using dyadic formulation [10], exploiting Betti's formulae, and through asymptotic analysis, we obtain

$$\widetilde{\mathbf{u}}_a^{sct}(\mathbf{r}) = \widetilde{\mathbf{g}}_a^{r}(\hat{\mathbf{r}}) \frac{e^{ik_p r}}{\sqrt{r}} + \widetilde{\mathbf{g}}_a^{t}(\hat{\mathbf{r}}) \frac{e^{ik_s r}}{\sqrt{r}} + O(r^{-3/2}), \qquad r = |\mathbf{r}| \to \infty, \tag{8}$$

uniformly with respect to $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} \in \Omega$, where Ω is the unit circle in R^2. The coefficients of the terms $\frac{e^{ik_\beta r}}{\sqrt{r}}$, $\beta = p, s$, are the corresponding dyadic far–field patterns (defined on Ω) and are known as the longitudinal and the transverse far–field patterns, respectively, given in [11].

3. Scattering relations

In this section scattering relations for 2D–point generated dyadic fields will be presented. We follow the same ideas presented in [2, 3] for 3D–problems in acoustics and electromagnetics. So, for two point sources with position vectors $\mathbf{a}$ and $\mathbf{b}$, we define the following *dyadic longitudinal and transverse far–field pattern generators*, given by

$$\begin{aligned}\widetilde{\mathbf{G}}_b^{r}(\mathbf{a}) \;=\; & \frac{(1-i)\sqrt{k_p}}{4} \int_\Omega \left(\widetilde{\mathbf{g}}_b^{r}(\hat{\mathbf{r}})\right)^\top \cdot (\hat{\mathbf{r}} \otimes \hat{\mathbf{r}})\, e^{ik_p \mathbf{a}\cdot\hat{\mathbf{r}}}\, ds(\hat{\mathbf{r}}) \\ & + \frac{i\sqrt{\pi}}{2} \left(\widetilde{\mathbf{u}}_b^{sct}(\mathbf{a})\right)^\top \cdot (\hat{\mathbf{a}} \otimes \hat{\mathbf{a}}),\end{aligned} \tag{9}$$

and

$$\widetilde{\mathbf{G}}_b^t(\mathbf{a}) = \frac{(1-i)\sqrt{k_s}}{4}\int_\Omega \left(\widetilde{\mathbf{g}}_b^t(\hat{\mathbf{r}})\right)^\top \cdot (\widetilde{\mathbf{I}} - \hat{\mathbf{r}}\otimes\hat{\mathbf{r}})\, e^{ik_s\mathbf{a}\cdot\hat{\mathbf{r}}}\, ds(\hat{\mathbf{r}}) + \frac{i\sqrt{\pi}}{2}\frac{\lambda+\mu}{\mu}\left(\widetilde{\mathbf{u}}_b^{sct}(\mathbf{a})\right)^\top \cdot (\widetilde{\mathbf{I}} - \hat{\mathbf{a}}\otimes\hat{\mathbf{a}}), \tag{10}$$

respectively. We note here, that if we allow the point–source $\mathbf{a}$ recede to infinity, i.e. $\mathbf{a}\to\infty$, then the longitudinal far–field pattern generator (9), generates the far–filed pattern of the scattered filed $\widetilde{\mathbf{u}}_a^{sct}(\mathbf{r})$, in direction $-\hat{\mathbf{a}}$, and due to the source point $\mathbf{b}$. Similar result for the transverse one (10), holds, (see [4]).

The *general dyadic scattering theorem* for point–sources, now follows, (its proof can be found in [4]) :

Theorem 3.1 *Assume two point–source locations* $\mathbf{a}$ *and* $\mathbf{b}$ *in* B_e. *Then the following relation holds:*

$$\widetilde{\mathbf{G}}_b^r(\mathbf{a}) + \left(\widetilde{\mathbf{G}}_a^r(\mathbf{b})\right)^\top + \widetilde{\mathbf{G}}_b^t(\mathbf{a}) + \left(\widetilde{\mathbf{G}}_a^t(\mathbf{b})\right)^\top$$

$$= -k_p\,(\lambda+2\mu)\sqrt{\pi}\int_\Omega \left(\widetilde{\mathbf{g}}_b^r(\hat{\mathbf{r}})\right)^\top \cdot \widetilde{\mathbf{g}}_a^r(\hat{\mathbf{r}})\; ds(\hat{\mathbf{r}}) - k_s\,\mu\sqrt{\pi}\int_\Omega \left(\widetilde{\mathbf{g}}_b^t(\hat{\mathbf{r}})\right)^\top \cdot \widetilde{\mathbf{g}}_a^t(\hat{\mathbf{r}})\; ds(\hat{\mathbf{r}}). \tag{11}$$

In addition, a main reciprocity theorem follows; its proof can be found in [4], so we omit it here for brevity.

Theorem 3.2 *Consider two point–source locations* $\mathbf{a}$ *and* $\mathbf{b}$ *in the exterior* B_e *of the scatterer, with* $\widetilde{\mathbf{u}}_\gamma^{inc}$, $\widetilde{\mathbf{u}}_\gamma^{sct}$, $\gamma = a, b$ *be the incident and scattered fields due to the source points* $\mathbf{a}$ *and* $\mathbf{b}$ *respectively. Then for any rigid body scatterer, the following relation holds*

$$\left(\widetilde{\mathbf{u}}_b^{sct}(\mathbf{a})\right)^\top \cdot (\hat{\mathbf{a}}\otimes\hat{\mathbf{a}}) + \frac{\lambda+\mu}{\mu}\left(\widetilde{\mathbf{u}}_b^{sct}(\mathbf{a})\right)^\top \cdot (\widetilde{\mathbf{I}} - \hat{\mathbf{a}}\otimes\hat{\mathbf{a}})$$

$$= \left(\left(\widetilde{\mathbf{u}}_a^{sct}(\mathbf{b})\right)^\top \cdot (\hat{\mathbf{b}}\otimes\hat{\mathbf{b}})\right)^T + \frac{\lambda+\mu}{\mu}\left(\left(\widetilde{\mathbf{u}}_a^{sct}(\mathbf{b})\right)^\top \cdot (\widetilde{\mathbf{I}} - \hat{\mathbf{b}}\otimes\hat{\mathbf{b}})\right)^\top. \tag{12}$$

In the sequel, an expression for the scattering cross–section σ_a^{sct}, due to an incident point–source field, will be presented. We deal with the energy triadic and the energy flux vector for the incident point–source field. In particular, we extend the ideas in [6, 7] in the two–dimensional elastic case. Since,

$$\hat{\mathbf{r}}\cdot\mathbf{P}_a^{inc}(\mathbf{r}) = \hat{\mathbf{r}}\cdot\omega\,\Im\left[\widetilde{\mathbf{u}}_a^{inc}(\mathbf{r})^\top \cdot \underset{\approx}{\widetilde{\mathbf{S}}}{}_a^{inc}(\mathbf{r})\right]^{213} : \mathbf{c}\otimes\mathbf{c} = \hat{\mathbf{r}}\cdot\underset{\approx}{\widetilde{\mathbf{E}}}{}_a^{inc}(\mathbf{r}) : \mathbf{c}\otimes\mathbf{c}, \tag{13}$$

where 213 denotes the order of the tensorial product in the corresponding triadic and $\mathbf{c}$ an arbitrary constant vector, then finding the asymptotic form of $\hat{\mathbf{r}} \cdot \widetilde{\widetilde{\mathbf{E}}}_a^{inc}(\mathbf{r})$ for $r \to \infty$, ($\widetilde{\widetilde{\mathbf{E}}}_a^{inc}(\mathbf{r})$ is the corresponding energy triadic), then, lengthy calculations lead to

$$\hat{\mathbf{r}} \cdot \mathbf{P}_a^{inc}(\mathbf{r}) = \frac{1}{r} \frac{\omega}{8\pi\rho} \left[\frac{\hat{\mathbf{r}} \otimes \hat{\mathbf{r}}}{c_p^2} + \frac{\widetilde{\mathbf{I}} - \hat{\mathbf{r}} \otimes \hat{\mathbf{r}}}{c_s^2} \right] : \mathbf{c} \otimes \mathbf{c} + O(r^{-3/2}), \qquad r \to \infty, \quad (14)$$

with the product $\hat{\mathbf{r}} \cdot \mathbf{P}_a^{inc}(\mathbf{r})$ being the normal energy flux of the incident point–source field in the radial direction.

Hence, we are ready now to define the *differential scattering cross–section* due to a point source at $\mathbf{a}$, as follows [4]:

$$\sigma(\hat{\mathbf{r}}) = \lim_{r \to \infty} \frac{2\pi r \, \hat{\mathbf{r}} \cdot \mathbf{P}_a^{inc}(\mathbf{r})}{\int_{\Omega_r} \hat{\mathbf{r}} \cdot \mathbf{P}_a^{inc}(\mathbf{r}) \, ds(\mathbf{r})}, \quad (15)$$

where $\hat{\mathbf{r}}$ is the direction of observation, Ω_r is a large circle of radius r, while the integral in the denominator is the total energy flux in all directions. After some calculations, and taking the integral for $\sigma(\hat{\mathbf{r}})$ over the unit circle Ω, we can define the *scattering cross–section* which is a measure of the disturbance caused by the scatterer to the propagation of the incident point–source field, i.e.,

$$\sigma_a^{sct} = \frac{1}{2\pi} \int_\Omega \sigma(\hat{\mathbf{r}}) \, ds(\hat{\mathbf{r}}). \quad (16)$$

Hence, with the aid of (14)–(16)

$$\sigma_a^{sct} = \frac{8\omega}{(c_p^{-2} + c_s^{-2}) \, \|\mathbf{c}\|^2} \int_\Omega \left(c_p \, \|\widetilde{\mathbf{g}}_a^r(\hat{\mathbf{r}}) \cdot \mathbf{c}\|^2 + c_s \, \left\|\widetilde{\mathbf{g}}_a^t(\hat{\mathbf{r}}) \cdot \mathbf{c}\right\|^2 \right) ds(\hat{\mathbf{r}}). \quad (17)$$

In addition, if we put in Theorem 3.1 $\mathbf{a} = \mathbf{b}$, contract (11) with a constant vector $\mathbf{c}$ from the left, and then from the right, we can arrive at:

$$\begin{aligned} & 2\,\Re\left[\mathbf{c} \cdot \widetilde{\mathbf{G}}_a^r(\mathbf{a}) \cdot \mathbf{c}\right] + 2\,\Re\left[\mathbf{c} \cdot \widetilde{\mathbf{G}}_a^t(\mathbf{a}) \cdot \mathbf{c}\right] \\ & = -k_p\,(\lambda + 2\mu)\sqrt{\pi} \int_\Omega \|\widetilde{\mathbf{g}}_a^r(\hat{\mathbf{r}}) \cdot \mathbf{c}\|^2 \, ds(\hat{\mathbf{r}}) - k_s\,\mu\sqrt{\pi} \int_\Omega \left\|\widetilde{\mathbf{g}}_a^t(\hat{\mathbf{r}}) \cdot \mathbf{c}\right\|^2 ds(\hat{\mathbf{r}}). \end{aligned} \quad (18)$$

So, taking into account (17) and (18) we can obtain for the rigid scatterer case, the following (*optical*) theorem:

Theorem 3.3 *For a point source at* $\mathbf{a}$*, and an arbitrary constant vector* $\mathbf{c}$*, we have*

$$\begin{aligned} \sigma_a^{sct} &= \frac{8}{(c_p^{-2} + c_s^{-2}) \, \|\mathbf{c}\|^2} \\ & \quad \times \left[-2\,\Re\left(\mathbf{c} \cdot \widetilde{\mathbf{G}}_a^r(\mathbf{a}) \cdot \mathbf{c}\right) - 2\,\Re\left(\mathbf{c} \cdot \widetilde{\mathbf{G}}_a^t(\mathbf{a}) \cdot \mathbf{c}\right) \right]. \end{aligned} \quad (19)$$

4. Green's function for an elastic rigid circular scatterer

In this section we study the solution for the scattering of elastic waves by a rigid circle, given in the form of infinite series. After the point—source excitation, we measure the corresponding P and S components of the scattered field at the receiver points. Hence, let us first consider a complete incident dyadic field, given in terms of Navier eigenvectors [5], as:

$$\widetilde{\mathbf{u}}_a^{inc}(\mathbf{r}) = -\frac{i}{4\mu k_s^2}\sum_{\sigma=1}^{2}\sum_{m=0}^{+\infty}\left[\Phi^i_{m,\sigma}(\mathbf{r})\otimes\Phi^e_{m,\sigma}(\mathbf{a}) + \Psi^i_{m,\sigma}(\mathbf{r})\otimes\Psi^e_{m,\sigma}(\mathbf{a})\right] \tag{20}$$

for $r < a$. The scattered field has a similar expression and takes the form:

$$\begin{aligned}\widetilde{\mathbf{u}}_a^{sct}(\mathbf{r}) = &\sum_{m=0}^{+\infty} a_m\, k_p \sqrt{\varepsilon_m}\, H_m^{(1)'}(k_p r)\; \hat{\mathbf{r}}\left[\cos(m\varphi)\otimes\Phi^e_{m,1}(\mathbf{a}) + \sin(m\varphi)\otimes\Phi^e_{m,2}(\mathbf{a})\right] \\ &+ \sum_{m=0}^{+\infty} \beta_m \frac{m}{r}\sqrt{\varepsilon_m}\, H_m^{(1)}(k_s r)\; \hat{\mathbf{r}}\left[\cos(m\varphi)\otimes\Psi^e_{m,2}(\mathbf{a}) - \sin(m\varphi)\otimes\Psi^e_{m,1}(\mathbf{a})\right] \\ &+ \sum_{m=0}^{+\infty} \gamma_m \frac{m}{r}\sqrt{\varepsilon_m}\; H_m^{(1)}(k_p r)\; \hat{\varphi}\left[\cos(m\varphi)\otimes\Phi^e_{m,2}(\mathbf{a}) - \sin(m\varphi)\otimes\Phi^e_{m,1}(\mathbf{a})\right] \\ &+ \sum_{m=0}^{+\infty} \delta_m\, k_s \sqrt{\varepsilon_m}\, H_m^{(1)'}(k_s r)\; \hat{\varphi}\left[-\cos(m\varphi)\otimes\Psi^e_{m,1}(\mathbf{a}) - \sin(m\varphi)\otimes\Psi^e_{m,2}(\mathbf{a})\right],\end{aligned} \tag{21}$$

where the coefficients a_m, β_m, γ_m and δ_m are to be determined and the Navier eigenvectors $\Phi^{e,i}_{m,\sigma}, \Psi^{e,i}_{m,\sigma}$ are given as:

$$\Phi^e_{m,1}(\mathbf{r}) \;=\; \nabla\left(H_m^{(1)}(k_p r)\, E_m^1(\varphi)\right) = \nabla\left(H_m^{(1)}(k_p r)\sqrt{\varepsilon_m}\,\cos(m\varphi)\right) \tag{22}$$

$$\Phi^i_{m,1}(\mathbf{r}) \;=\; \nabla\left(J_m(k_p r)\, E_m^1(\varphi)\right) = \nabla\left(J_m(k_p r)\sqrt{\varepsilon_m}\,\cos(m\varphi)\right) \tag{23}$$

$$\Phi^e_{m,2}(\mathbf{r}) \;=\; \nabla\left(H_m^{(1)}(k_p r)\, E_m^2(\varphi)\right) = \nabla\left(H_m^{(1)}(k_p r)\sqrt{\varepsilon_m}\,\sin(m\varphi)\right) \tag{24}$$

$$\Phi^i_{m,2}(\mathbf{r}) \;=\; \nabla\left(J_m(k_p r)\, E_m^2(\varphi)\right) = \nabla\left(J_m(k_p r)\sqrt{\varepsilon_m}\,\sin(m\varphi)\right) \tag{25}$$

and

$$\begin{aligned}
\Psi^e_{m,1}(\mathbf{r}) &= \nabla \times \left(H^{(1)}_m(k_s r)\, E^1_m(\varphi) \times \hat{\mathbf{k}} \right) = \nabla \left(H^{(1)}_m(k_s r)\, E^1_m(\varphi) \right) \times \hat{\mathbf{k}} \qquad (26)\\
&= \nabla \left(H^{(1)}_m(k_s r) \sqrt{\varepsilon_m}\, \cos(m\varphi) \right) \times \hat{\mathbf{k}},\\
\Psi^e_{m,2}(\mathbf{r}) &= \nabla \left(H^{(1)}_m(k_s r) \sqrt{\varepsilon_m}\, \sin(m\varphi) \right) \times \hat{\mathbf{k}}, \qquad (27)\\
\Psi^i_{m,1}(\mathbf{r}) &= \nabla \left(J_m(k_s r) \sqrt{\varepsilon_m}\, \cos(m\varphi) \right) \times \hat{\mathbf{k}}, \qquad (28)\\
\Psi^i_{m,2}(\mathbf{r}) &= \nabla \left(J_m(k_s r) \sqrt{\varepsilon_m}\, \sin(m\varphi) \right) \times \hat{\mathbf{k}}. \qquad (29)
\end{aligned}$$

where $\varepsilon_0 = 1$, $\varepsilon_m = 2$ for $m > 0$ and $\hat{\mathbf{k}}$ is a unit vector perpendicular to the (x_1, x_2)–plane.

Concerning the determination of the coefficients a_m, β_m, γ_m and δ_m, we use the rigid boundary condition (7), on $r = R$, (circle of radius R), and we have the following:

$$\begin{aligned}
&\sum_{m=0}^{+\infty} a_m\, k_p \sqrt{\varepsilon_m}\, H^{(1)'}_m(k_p R)\; \hat{\mathbf{r}} \left[\cos(m\varphi) \otimes \Phi^e_{m,1}(\mathbf{a}) + \sin(m\varphi) \otimes \Phi^e_{m,2}(\mathbf{a})\right]\\
&+ \sum_{m=0}^{+\infty} \beta_m \frac{m}{R} \sqrt{\varepsilon_m}\, H^{(1)}_m(k_s R)\; \hat{\mathbf{r}} \left[\cos(m\varphi) \otimes \Psi^e_{m,2}(\mathbf{a}) - \sin(m\varphi) \otimes \Psi^e_{m,1}(\mathbf{a})\right]\\
&+ \sum_{m=0}^{+\infty} \gamma_m \frac{m}{R} \sqrt{\varepsilon_m}\; H^{(1)}_m(k_p R)\; \hat{\varphi} \left[\cos(m\varphi) \otimes \Phi^e_{m,2}(\mathbf{a}) - \sin(m\varphi) \otimes \Phi^e_{m,1}(\mathbf{a})\right]\\
&+ \sum_{m=0}^{+\infty} \delta_m\, k_s \sqrt{\varepsilon_m}\, H^{(1)'}_m(k_s R)\; \hat{\varphi} \left[-\cos(m\varphi) \otimes \Psi^e_{m,1}(\mathbf{a}) - \sin(m\varphi) \otimes \Psi^e_{m,2}(\mathbf{a})\right]\\
= \quad &- \sum_{m=0}^{+\infty} k_p \sqrt{\varepsilon_m}\, J'_m(k_p R)\; \hat{\mathbf{r}} \left[\cos(m\varphi) \otimes \Phi^e_{m,1}(\mathbf{a}) + \sin(m\varphi) \otimes \Phi^e_{m,2}(\mathbf{a})\right]\\
&- \sum_{m=0}^{+\infty} \frac{m}{R} \sqrt{\varepsilon_m}\, J_m(k_s R)\; \hat{\mathbf{r}} \left[\cos(m\varphi) \otimes \Psi^e_{m,2}(\mathbf{a}) - \sin(m\varphi) \otimes \Psi^e_{m,1}(\mathbf{a})\right]\\
&- \sum_{m=0}^{+\infty} \frac{m}{R} \sqrt{\varepsilon_m}\; J_m(k_p R)\; \hat{\varphi} \left[\cos(m\varphi) \otimes \Phi^e_{m,2}(\mathbf{a}) - \sin(m\varphi) \otimes \Phi^e_{m,1}(\mathbf{a})\right]\\
&- \sum_{m=0}^{+\infty} k_s \sqrt{\varepsilon_m}\, J'_m(k_s R)\; \hat{\varphi} \left[-\cos(m\varphi) \otimes \Psi^e_{m,1}(\mathbf{a}) - \sin(m\varphi) \otimes \Psi^e_{m,2}(\mathbf{a})\right]. \qquad (30)
\end{aligned}$$

After some calculations in (30), using orthogonality of the vector functions $\cos(\cdot)\,\hat{\mathbf{r}}$, $\sin(\cdot)\,\hat{\varphi}$ and lengthy calculations, we find the coefficients in the expansion of $\widetilde{\mathbf{u}}^{sct}_a$ in (21), as

$$a_m = -\frac{J'_m(k_p R)}{H^{(1)'}_m(k_p R)}, \qquad \beta_m = -\frac{J_m(k_s R)}{H^{(1)}_m(k_s R)} \qquad (31)$$

$$\gamma_m = -\frac{J_m(k_p R)}{H^{(1)}_m(k_p R)}, \qquad \delta_m = -\frac{J'_m(k_s R)}{H^{(1)'}_m(k_s R)} \qquad (32)$$

A method now, for a far–field inverse problem is described, where an approximation of the known exact solution for a rigid circle is established. In particular, letting $r \to \infty$, the longitudinal and transverse far–field pattern of the scattered field (21) can be found, and via them, the scattering cross-section in (17) can be evaluated. In view now, of low–frequency results the coefficients of the scattered field can be computed. This leads to the solution of a far–field inverse problem, which is based on the knowledge of the leading order term in the low–frequency asymptotic expansion of the scattering cross–section. Hence, for far–field experiments, the scattering cross–section for various point–source locations is measured, and using this data the location and radius of the small rigid circular scatterer can be recovered.

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