BULLETIN OF THE
GREEK MATHEMATICAL SOCIETY
Volume 54, 2007 (257–267)

The Fluid-Core Model in Electromagnetic Brain Activity

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Received 21 December 2006   Accepted 24 September 2007

Abstract

The direct problem of Magnetoencephalography for an ellipsoidal inhomogeneous one- shell-model of the brain with a dipole source in the shell is studied in the present work. The inhomogeneity is due to the attendance of a confocal ellipsoidal shell exhibiting different conductivity than the one of the brain tissue. The magnetic field in the exterior of the conductor is derived. It is shown that it depends strongly on the anisotropy imposed by the use of the ellipsoidal geometry, on the inhomogeneity dictated by the shell and also on the position and on the moment of the dipole source.

1. Introduction

The Electoencephalography (EEG) and Magnetoengephalography (MEG) are widely used non invasive methods for studying the human brain activity. It is well-known that an electrochemical source in the interior of the brain issue produces an electric field and a magnetic field, both in the interior and in the exterior of the brain. The produced electric field is measured on the head surface via EEG and the exterior magnetic field is measured via MEG.

The calculation of the electric field and of the magnetic field that a given source produces, consists the so called forward EEG and the forward MEG problem respectively. In dealing analytically with any of these problems, certain assumptions have to be made, concerning the physical and geometrical characteristics of the brain-model that is used. The most popular model for the source is that of a point dipole current and this is also used in the present work. As far as the geometrical model for the brain is concerned, a lot of work has been made using the spherical homogeneous model [5, 10, 14], the spheroidal homogeneous model [16] and the ellipsoidal homogeneous model [3, 12]. Taking under consideration the layers of different conductivities that cover the human brain, such as the scalp, the scull and the fluid layer, efforts have been made on dealing with the head as an inhomogeneous conductor [1, 4, 6, 7, 8, 11].

In all these works, the source is considered to be located in the interior of the homogeneous core that models the cerebrum issue. In the present work we study the one shell inhomogeneous model, where the source lies in the shell. In this way one can model the possible existence of an area with different conductivity inside the cerebrum issue, as is for example the area of a tumour. The geometry chosen is the ellipsoidal geometry, as this is the one that models the human brain most realistically [15].

So, in the present work we consider an inhomogeneous ellipsoidal conductor which consists of two confocal ellipsoids. An electric dipole source is located in the interior of the ellipsoidal shell. We calculate analytically, the electric potential field and the magnetic field that this source produces in the interior and in the exterior of the conductor.

In Section 2, the mathematical formulation of the forward EEG problem for our model is presented. The exterior and the interior electric potential are given in Section 3, while in Section 4 we evaluate of magnetic field in the exterior non conductive space.

2. Statement of the problem

Let's denote by V_c the region occupied by the cerebrum issue which is characterized by the constant conductivity σ_c and by V_f the fluid-core which is characterized by the conductivity σ_f. The exterior boundary surface S_c of the cerebrum is modelled with the triaxial ellipsoid

$$\frac{x_1^2}{c_1^2} + \frac{x_2^2}{c_2^2} + \frac{x_3^2}{c_3^2} = 1 \tag{1}$$

while the interior boundary surface $S_f = \partial V_f$ of the cerebrum is modelled with the confocal ellipsoid, defined by

$$\frac{x_1^2}{f_1^2} + \frac{x_2^2}{f_2^2} + \frac{x_3^2}{f_3^2} = 1. \tag{2}$$

The semiaxes are ordered as follows

$$\begin{cases} c_3 < c_2 < c_1 \\ f_3 < f_2 < f_1 < c_1. \end{cases} \tag{3}$$

The ellipsoids (1) and (2) belong into an ellipsoidal system with coordinates (ρ, μ, ν) and semifocal distances h_1, h_2, h_3. The ellipsoidal coordinates are connected with the Cartesian ones by

$$\begin{cases} h_2 h_3 x_1 = \rho\mu\nu \\ h_1 h_3 x_2 = \sqrt{\rho^2 - h_3^2}\sqrt{\mu^2 - h_3^2}\sqrt{h_3^2 - \nu^2} \\ h_1 h_2 x_3 = \sqrt{\rho^2 - h_2^2}\sqrt{h_2^2 - \mu^2}\sqrt{h_2^2 - \nu^2} \end{cases} \tag{4}$$

and the semifocal distances h_1, h_2, h_3 are given by

$$h_1^2 = c_2^2 - c_3^2 = f_2^2 - f_3^2 \tag{5}$$

$$h_2^2 = c_1^2 - c_3^2 = f_1^2 - f_3^2 \tag{6}$$

$$h_3^2 = c_1^2 - c_2^2 = f_1^2 - f_2^2. \tag{7}$$

The surface S_c corresponds to $\rho = c_1$, the surface S_f to $\rho = f_1$, the core-domain V_f to $\rho \epsilon [h_2, f_1)$ and the shell domain V_c to $\rho \epsilon (f_1, c_1)$. The exterior domain V is described by $\rho > c_1$ and is characterized by zero conductivity.

At the point $\mathbf{r}_0 = (\rho_0, \mu_0, \nu_0)$, in the interior of the ellipsoidal shell V_c, an equivalent dipole current source, with moment $\mathbf{Q}$, produce an electromagnetic field, which is assumed to be quasistatic. Consequently, the primary current

$$\mathbf{J}^{\mathbf{P}}(\mathbf{r}) = \mathbf{Q}\delta(\mathbf{r} - \mathbf{r}_0) \tag{8}$$

generates an electric potential u and a magnetic field $\mathbf{B}$ in the interior and in the exterior of the conductor.

Let's denote by u, u_c and u_f the electric potential in V, V_c and V_f respectively. The electric potentials u and u_f satisfy the Laplace equation in V and V_f, respectively, due to the absence of sources in their domains. The presence of the source in V_c dictates the replacement of the Laplace equation by the Poisson equation, which u_c has to solve in V_c. Therefore, we have

$$\Delta u(\mathbf{r}) = 0, \qquad \mathbf{r} \in V \tag{9}$$

$$\Delta u_c(\mathbf{r}) = \frac{1}{\sigma_c} \nabla \cdot \mathbf{J}^P(\mathbf{r}), \qquad \mathbf{r} \in V_c, \tag{10}$$

$$\Delta u_f(\mathbf{r}) = 0, \qquad \mathbf{r} \in V_f. \tag{11}$$

Continuity conditions on the boundary surfaces impose that the fields u, u_c and u_f are related as follows

$$u_c(\mathbf{r}) = u(\mathbf{r}), \qquad \mathbf{r} \in S_c, \tag{12}$$

$$\partial_n u_c(\mathbf{r}) = 0, \qquad \mathbf{r} \in S_c, \tag{13}$$

$$u_f(\mathbf{r}) = u_c(\mathbf{r}), \qquad \mathbf{r} \in S_f, \tag{14}$$

$$\sigma_f \partial_n u_f(\mathbf{r}) = \sigma_c \partial_n u_c(\mathbf{r}), \qquad \mathbf{r} \in S_f, \tag{15}$$

where ∂_n stands for the outward normal differentiation on the corresponding surface.

Moreover, in the unbounded region V we assume the asymptotic behavior for u

$$u(\mathbf{r}) = O(\frac{1}{r}), \qquad r \to \infty \tag{16}$$

so as the related exterior problem to be well-posed.

The magnetic field $\mathbf{B}$ in the exterior space V , as a consequence of Ampere's law [13] and of Geselowitz formulae [2], assumes the representation

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{Q} \times \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} - \sigma_c \int_{S_c} u_c(\mathbf{r}')\hat{\boldsymbol{\rho}}' \times \nabla_{\mathbf{r}'} \frac{1}{|\mathbf{r}-\mathbf{r}'|} ds(\mathbf{r}')$$

$$+(\sigma_c - \sigma_f) \int_{S_f} u_f(\mathbf{r}')\hat{\boldsymbol{\rho}}' \times \nabla_{\mathbf{r}'} \frac{1}{|\mathbf{r}-\mathbf{r}'|} ds(\mathbf{r}'). \tag{17}$$

3. The exterior and the interior electric potential

The exterior electric potential u solves the boundary value problem (9) , (12), (16) and in terms of ellipsoidal harmonics assumes the form

$$u(\mathbf{r}) = g_0^1 \frac{I_0^1(\rho)}{I_0^1(c_1)} + \frac{1}{\sigma_c C_n^m} \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \left\{ \left[I_n^m(c_1, f_1) - \frac{1}{C_n^m} - \frac{\sigma_c}{(\sigma_f - \sigma_c)} \frac{1}{F_n^m} \right]^{-1} \right.$$

$$\left. \left[\frac{\mathbf{Q} \cdot \nabla \mathbb{E}_n^m(\mathbf{r}_0)}{\gamma_n^m} \left(-I_n^m(f_1) - \frac{\sigma_c}{(\sigma_f - \sigma_c)} \frac{1}{F_n^m} \right) + \frac{\mathbf{Q} \cdot \nabla \mathbb{F}_n^m(\mathbf{r}_0)}{(2n+1)\gamma_n^m} \right] \right\} \frac{I_n^m(\rho)}{I_n^m(c_1)} \mathbb{E}_n^m(\rho, \mu, \nu) \tag{18}$$

for $\rho > c_1$. Similarly, the electric potential u_c solves the problem (10) , (12), (13) and assumes the form

$$u_c^+(\mathbf{r}) = g_0^1 + \frac{1}{\sigma_c} \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \left\{ \left[I_n^m(c_1, f_1) - \frac{1}{C_n^m} - \frac{\sigma_c}{(\sigma_f - \sigma_c)} \frac{1}{F_n^m} \right]^{-1} \right.$$

$$\left[\frac{\mathbf{Q} \cdot \nabla \mathbb{E}_n^m(\mathbf{r}_0)}{\gamma_n^m} \left(I_n^m(f_1) + \frac{\sigma_c}{(\sigma_f - \sigma_c)} \frac{1}{F_n^m} \right) - \right.$$

$$\left. \left. - \frac{\mathbf{Q} \cdot \nabla \mathbb{F}_n^m(\mathbf{r}_0)}{(2n+1)\gamma_n^m} \right] \right\} \left[I(c_1, \rho) - \frac{1}{C_n^m} \right] \mathbb{E}_n^m(\rho, \mu, \nu) \tag{19}$$

for $\rho_0 < \rho < c_1$ and

$$u_c^-(\mathbf{r}) = g_0^1 + \frac{1}{\sigma_c} \sum_{n=1}^{\infty} \sum_{m=1}^{2n+1} \left\{ \left[I_n^m(c_1, f_1) - \frac{1}{C_n^m} - \frac{\sigma_c}{\sigma_f - \sigma_c} \frac{1}{F_n^m} \right]^{-1} \right.$$

$$\left[\frac{\mathbf{Q} \cdot \nabla \mathbb{E}_n^m(\mathbf{r}_0)}{\gamma_n^m} \left(I_n^m(c_1) - \frac{1}{C_n^m} \right) - \right.$$

$$\left. \left. - \frac{\mathbf{Q} \cdot \nabla \mathbb{F}_n^m(\mathbf{r}_0)}{2n+1} \right] \right\} \left[I(f_1, \rho) + \frac{\sigma_c}{\sigma_f - \sigma_c} \frac{1}{F_n^m} \right] \mathbb{E}_n^m(\rho, \mu, \nu) \tag{20}$$

for $f_1 < \rho < \rho_0$. Finally u_f solves the problem (11) , (14), (15) and assumes the form

$$u_f(\mathbf{r}) = g_0^1 + \frac{1}{\sigma_c}\sum_{n=1}^{\infty}\sum_{m=1}^{2n+1}\left\{\left[I_n^m(c_1,f_1) - \frac{1}{C_n^m} - \frac{\sigma_c}{\sigma_f-\sigma_c}\frac{1}{F_n^m}\right]^{-1}\right.$$
$$\left[\frac{\mathbf{Q}\cdot\nabla\mathbb{E}_n^m(\mathbf{r}_0)}{\gamma_n^m}\left(I_n^m(c_1) - \frac{1}{C_n^m}\right) - \right.$$
$$\left.\left.-\frac{\mathbf{Q}\cdot\nabla\mathbb{F}_n^m(\mathbf{r}_0)}{2n+1}\right]\right\}\frac{\sigma_c}{\sigma_f-\sigma_c}\frac{1}{F_n^m}\mathbb{E}_n^m(\rho,\mu,\nu) \tag{21}$$

for $h_2 < \rho < f_1$ where

$$C_n^m = E_n^m(c_1)E_n^{m'}(c_1)c_2c_3 \tag{22}$$

$$F_n^m = E_n^m(f_1)E_n^{m'}(f_1)f_2f_3. \tag{23}$$

The interior Lamé function of degree n and m is denoted by E_n^m, while $\mathbb{E}_n^m$ and $\mathbb{F}_n^m$ are the corresponding interior and exterior ellipsoidal harmonics [9] and I_n^m is the elliptic integral defined by

$$I_n^m(\rho) = \int_\rho^\infty \frac{dt}{\left[E_n^m(t)\right]^2\sqrt{t^2-h_2^2}\sqrt{t^2-h_3^2}} \tag{24}$$

and

$$I_n^m(x,y) = I_n^m(x) - I_n^m(y) = \int_x^y \frac{dt}{\left[E_n^m(t)\right]^2\sqrt{t^2-h_2^2}\sqrt{t^2-h_3^2}}. \tag{25}$$

The constants γ_n^m are the L_2 norms of the surface ellipsoidal harmonics defined by

$$\gamma_n^m = \iint_{\rho=\rho_0}[E_n^m(\mu)E_n^m(\nu)]^2 l_{\rho_0}(\mu,\nu)ds = 0 \tag{26}$$

for each $n = 1, 2, \ldots$ and $m = 1, 2, \ldots, 2n+1$, and l_{ρ_0} is the ellipsoidal weighting function

$$l_{\rho_0}(\mu,\nu) = [(\rho_0^2-\mu^2)(\rho_0^2-\nu^2)]^{-1/2} \tag{27}$$

4. The exterior magnetic field

We now proceed with the evaluation of the magnetic field in the exterior space V. According to (17), the magnetic field $\mathbf{B}$ can be rewritten as

$$\mathbf{B}(\mathbf{r}') = \frac{\mu_0}{4\pi}\mathbf{Q}\times\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} - \frac{\mu_0}{4\pi}\sigma_c\mathbf{I}_c(\mathbf{r}) + \frac{\mu_0}{4\pi}(\sigma_c-\sigma_f)\,\mathbf{I}_f(\mathbf{r}) \tag{28}$$

where $\mathbf{I}_i$, $i = c, f$ are the following integrals

$$\mathbf{I}_c\left(\mathbf{r}\right) = \int_{S_c} u^{+}(\mathbf{r}')\hat{\boldsymbol{\rho}}' \times \nabla_{\mathbf{r}'} \frac{1}{|\mathbf{r}-\mathbf{r}'|} ds(\mathbf{r}') \tag{29}$$

$$\mathbf{I}_f\left(\mathbf{r}\right) = \int_{S_f} u^{-}(\mathbf{r}')\hat{\boldsymbol{\rho}}' \times \nabla_{\mathbf{r}'} \frac{1}{|\mathbf{r}-\mathbf{r}'|} ds(\mathbf{r}') \tag{30}$$

and $\hat{\boldsymbol{\rho}}'$ stands for the outward unit normal vector at the point $\mathbf{r}'$ on the corresponding ellipsoidal surface. We start with the evaluation of the integral $\mathbf{I}_c\left(\mathbf{r}\right)$. On the surface S_c the expansion of the quantity $\hat{\boldsymbol{\rho}}' \times \nabla_{\mathbf{r}'} \mid \mathbf{r}-\mathbf{r}' \mid^{-1}$ reads as

$$\hat{\boldsymbol{\rho}}' \times \nabla_{\mathbf{r}'} \frac{1}{\mid \mathbf{r}-\mathbf{r}' \mid}\bigg|_{\rho'=c_1}$$

$$= \ell_{c_1}\left(\mu',\nu'\right)\left[\sum_{m=1}^{3} \boldsymbol{\beta}_m^c\left(\mathbf{r}\right) E_1^m\left(\mu'\right) E_1^m\left(\nu'\right) + \sum_{m=1}^{5} \boldsymbol{\delta}_m^c\left(\mathbf{r}\right) E_2^m\left(\mu'\right) E_2^m\left(\nu'\right)\right] + O\left(el_3\right) \tag{31}$$

where l_{c_1} is obtained from (27) by substituting $\rho_0 = c_1$. The vector coefficients are given by beginequation

$$\boldsymbol{\beta}_m^c = 3\frac{c_1c_2c_3}{h_1h_2h_3}\frac{h_m}{c_m}\hat{\mathbf{x}}_m \otimes \mathbf{r} \underset{\cdot}{\times} \tilde{\mathbf{H}}_1(\rho), \qquad m = 1,2,3 \tag{32}$$

$$\boldsymbol{\delta}_{1}^c = -\frac{c_1c_2c_3}{3(\Lambda_c - \Lambda_c')}\tilde{\boldsymbol{\Lambda}}_c \underset{\cdot}{\times} \tilde{\mathbf{F}}_c(\mathbf{r}) \tag{33}$$

$$\boldsymbol{\delta}_2^c = \frac{c_1c_2c_3}{3(\Lambda_c - \Lambda_c')}\tilde{\boldsymbol{\Lambda}}_c' \underset{\cdot}{\times} \tilde{\mathbf{F}}_c(\mathbf{r}) \tag{34}$$

$$\boldsymbol{\delta}_3^c = \frac{c_1c_2c_3}{h_1h_2h_3^2}\left[\frac{c_2}{c_1}\hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_2 + \frac{c_1}{c_2}\hat{\mathbf{x}}_2 \otimes \hat{\mathbf{x}}_1\right] \underset{\cdot}{\times} \tilde{\mathbf{F}}_c(\mathbf{r}) \tag{35}$$

$$\boldsymbol{\delta}_4^c = \frac{c_1c_2c_3}{h_1h_2^2h_3}\left[\frac{c_3}{c_1}\hat{\mathbf{x}}_1 \otimes \hat{\mathbf{x}}_3 + \frac{c_1}{c_3}\hat{\mathbf{x}}_3 \otimes \hat{\mathbf{x}}_1\right] \underset{\cdot}{\times} \tilde{\mathbf{F}}_c(\mathbf{r}) \tag{36}$$

$$\boldsymbol{\delta}_5^c = \frac{c_1c_2c_3}{h_1^2h_2h_3}\left[\frac{c_3}{c_2}\hat{\mathbf{x}}_2 \otimes \hat{\mathbf{x}}_3 + \frac{c_2}{c_3}\hat{\mathbf{x}}_3 \otimes \hat{\mathbf{x}}_2\right] \underset{\cdot}{\times} \tilde{\mathbf{F}}_c(\mathbf{r}) \tag{37}$$

where the cross-dot dyadic product is defined by

$$(\mathbf{a} \otimes \mathbf{b}) \underset{\cdot}{\times} (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \times \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) \tag{38}$$

and the related polyadics are defined as follows

$$\tilde{\boldsymbol{\Lambda}}_c = \sum_{m=1}^{3} \frac{\hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m}{\Lambda_c - c_m^2} \tag{39}$$

$$\tilde{\mathbf{\Lambda}}'_c = \sum_{m=1}^{3} \frac{\hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m}{\Lambda'_c - c_m^2} \tag{40}$$

$$\tilde{\mathbf{H}}_1(\rho) = \sum_{m=1}^{3} I_1^m(\rho)\hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m \tag{41}$$

$$\tilde{\mathbf{H}}_2(\rho) = \sum_{\substack{i,j=1 \\ i\neq j}}^{3} I_2^{i+j}(\rho)\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j \tag{42}$$

$$\tilde{\mathbf{F}}_c(\mathbf{r}) = -\frac{I\!F_2^1(\mathbf{r})}{\Lambda_c - \Lambda'_c}\tilde{\mathbf{\Lambda}}_c + \frac{I\!F_2^2(\mathbf{r})}{\Lambda_c - \Lambda'_c}\tilde{\mathbf{\Lambda}}'_c + 15\mathbf{r}\otimes\mathbf{r} : \tilde{\mathbf{H}}_2(\rho). \tag{43}$$

Now we rewrite the expression (18) for the potential u_c at $\rho = c_1$ in the form

$$u_c(c_1, \mu', \nu') = u^+(\mathbf{r}) = g_0^1 + \sum_{m=1}^{3} \zeta_m^c E_1^m(\mu')E_1^m(\nu') + \sum_{m=1}^{5} \theta_m^c E_2^m(\mu')E_2^m(\nu') + O\left(el'_3\right) \tag{44}$$

Using connection formulae between the Lamé functions E_n^m and the ellipsoidal harmonics $I\!E_n^m$ in Cartesian coordinates, for $n \leq 2$ [3], we obtain, after some long algebraic manipulations, the following expressions for the coefficients ζ_m^c, θ_m^c

$$\zeta_m^c = \frac{3h_m}{4\pi h_1 h_2 h_3}\frac{G_1^m}{A_1^m}\mathbf{Q}\cdot\hat{\mathbf{x}}_m$$

$$-\frac{3h_m}{4\pi h_1 h_2 h_3}\frac{\rho_0 K_1^m}{\left[E_1^m(\rho_0)\right]^2\sqrt{\rho_0{}^2-\mu_0{}^2}\sqrt{\rho_0{}^2-\nu_0{}^2}}x_{0m}\sum_{k=1}^{3}\frac{\mathbf{x}_k\mathbf{Q}\cdot\hat{\mathbf{x}}_k}{\rho^2-c_1^2+c_i^2}, \tag{45}$$

$$\theta_1^c = -\frac{5}{4\pi(\Lambda_c-\Lambda'_c)}\frac{G_2^1}{A_2^1}\frac{x_{0m}\mathbf{Q}\cdot\hat{\mathbf{x}}_m}{\Lambda_c - c_m^2}$$

$$+\frac{5}{8\pi(\Lambda_c-\Lambda'_c)}\frac{\rho_0 K_2^1\left(\frac{x_{0m}}{\Lambda_c-c_m^2}+1\right)}{\left[E_1^2(\rho_0)\right]^2\sqrt{\rho_0{}^2-\mu_0{}^2}\sqrt{\rho_0{}^2-\nu_0{}^2}}\sum_{k=1}^{3}\frac{\mathbf{x}_k\mathbf{Q}\cdot\hat{\mathbf{x}}_k}{\rho^2-c_1^2+c_i^2} \tag{46}$$

$$\theta_2^c = \frac{5}{4\pi(\Lambda_c-\Lambda'_c)}\frac{G_2^2}{A_2^2}\frac{x_{0m}\mathbf{Q}\cdot\hat{\mathbf{x}}_m}{\Lambda'_c - c_m^2} -$$

$$-\frac{5}{8\pi(\Lambda_c-\Lambda'_c)}\frac{\rho_0 K_2^2\left(\frac{x_{0m}}{\Lambda'_c-c_m^2}+1\right)}{\left[E_2^2(\rho_0)\right]^2\sqrt{\rho_0{}^2-\mu_0{}^2}\sqrt{\rho_0{}^2-\nu_0{}^2}}\sum_{k=1}^{3}\frac{\mathbf{x}_k\mathbf{Q}\cdot\hat{\mathbf{x}}_k}{\rho^2-c_1^2+c_i^2} \tag{47}$$

$$\theta_{i+j}^c = \frac{15h_ih_j}{4\pi(h_1h_2h_3)^2}\frac{G_2^{i+j}}{A_2^{i+j}}\mathbf{Q}\cdot(x_{0i}\hat{\mathbf{x}}_j + x_{0j}\hat{\mathbf{x}}_i) -$$

$$-\frac{15h_ih_j}{4\pi(h_1h_2h_3)^2}\frac{\rho_0 K_1^{i+j}}{\left[E_1^{i+j}(\rho_0)\right]^2\sqrt{\rho_0{}^2-\mu_0{}^2}\sqrt{\rho_0{}^2-\nu_0{}^2}}\mathbf{Q}\cdot x_{0i}x_{0j}\sum_{k=1}^{3}\frac{\mathbf{x}_k\hat{\mathbf{x}}_k}{\rho^2-c_1^2+c_i^2}, \tag{48}$$

for $i, j \in 1, 2, 3$ and $i \neq j$. The constants A_n^m, G_n^m and K_n^m are given by

$$A_n^m = \left[I_n^m(c_1, f_1) - \frac{1}{C_n^m} - \frac{\sigma_c}{(\sigma_f - \sigma_c)} \frac{1}{F_n^m} \right] F_n^m \tag{49}$$

$$G_n^m = \left[-I_n^m(f_1) - \frac{\sigma_c}{(\sigma_f - \sigma_c)} \frac{1}{F_n^m} + I_n^m(\rho_0) \right] \frac{E_n^m(c_1)}{\sigma_c} \tag{50}$$

$$K_n^m = \frac{E_n^m(c_1)}{\sigma_c A_n^m}. \tag{51}$$

By using orthogonality of the surface ellipsoidal harmonics and inserting the expressions (31) and (44) into (29) we obtain

$$\mathbf{I}_c(\mathbf{r}) = \sum_{m=1}^{3} \zeta_m^c \gamma_1^m \boldsymbol{\beta}_m^c(\mathbf{r}) + \sum_{m=1}^{5} \theta_m^c \gamma_2^m \boldsymbol{\delta}_m^c(\mathbf{r}) + O(el_3). \tag{52}$$

where the notation $O(el_3)$ stands for ellipsoidal terms in $\mathbf{r}$ of order higher or equal to three. Replacing (45)-(48) into (52) we rewrite the integral $\mathbf{I}_c(\mathbf{r})$ as

$$\mathbf{I}_c(\mathbf{r}) = \sum_{m=1}^{3} \frac{3c_1c_2c_3}{c_m} \frac{G_1^m}{A_1^m} \mathbf{Q} \cdot \hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m \otimes \mathbf{r} \times \tilde{\mathbf{H}}_1(\rho)$$

$$- \sum_{m=1}^{3} \sum_{k=1}^{3} \frac{3c_1c_2c_3}{c_m} \frac{\rho_0 K_1^m}{[E_1^m(\rho_0)]^2 \sqrt{\rho_0{}^2 - \mu_0{}^2}\sqrt{\rho_0{}^2 - \nu_0{}^2}} \frac{x_{0m} x_k \mathbf{Q} \cdot \hat{\mathbf{x}}_k}{\rho^2 - c_1^2 + c_k^2} \otimes \hat{\mathbf{x}}_m \otimes \mathbf{r} \times \tilde{\mathbf{H}}_1(\rho),$$

$$- \frac{(\Lambda_c - c_1^2)(\Lambda_c - c_2^2)(\Lambda_c - c_3^2)}{3(\Lambda_c - \Lambda_c')} 2c_1c_2c_3 \frac{G_2^1}{A_2^1} \mathbf{Q} \otimes \mathbf{r}_0 : \tilde{\boldsymbol{\Lambda}}_{\mathbf{c}} \otimes \tilde{\boldsymbol{\Lambda}}_{\mathbf{c}} \times \tilde{\mathbf{F}}_{\mathbf{c}}(\mathbf{r})$$

$$+ \sum_{k=1}^{3} \frac{(\Lambda_c - c_1^2)(\Lambda_c - c_2^2)(\Lambda_c - c_3^2) c_1c_2c_3 \rho_0 K_2^1 \left(\frac{x_{01}}{\Lambda_c - c_1^2} + 1 \right)}{3(\Lambda_c - \Lambda_c')[E_2^1(\rho_0)]^2 \sqrt{\rho_0{}^2 - \mu_0{}^2}\sqrt{\rho_0{}^2 - \nu_0{}^2}} \frac{x_k \mathbf{Q} \cdot \hat{\mathbf{x}}_k}{\rho^2 - c_1^2 + c_k^2} \tilde{\boldsymbol{\Lambda}}_{\mathbf{c}} \times \tilde{\mathbf{F}}_{\mathbf{c}}(\mathbf{r})$$

$$+ \frac{(\Lambda_c' - c_1^2)(\Lambda_c' - c_2^2)(\Lambda_c' - c_3^2)}{3(\Lambda_c - \Lambda_c')} 2c_1c_2c_3 \frac{G_2^2}{A_2^2} \mathbf{Q} \otimes \mathbf{r}_0 : \tilde{\boldsymbol{\Lambda}}_{\mathbf{c}}' \otimes \tilde{\boldsymbol{\Lambda}}_{\mathbf{c}}' \times \tilde{\mathbf{F}}_{\mathbf{c}}(\mathbf{r})$$

$$- \sum_{k=1}^{3} \frac{(\Lambda_c' - c_1^2)(\Lambda_c' - c_2^2)(\Lambda_c' - c_3^2) c_1c_2c_3 \rho_0 K_2^2 \left(\frac{x_{02}}{\Lambda_c - c_1^2} + 1 \right)}{3(\Lambda_c - \Lambda_c')[E_2^2(\rho_0)]^2 \sqrt{\rho_0{}^2 - \mu_0{}^2}\sqrt{\rho_0{}^2 - \nu_0{}^2}} \frac{x_k \mathbf{Q} \cdot \hat{\mathbf{x}}_k}{\rho^2 - c_1^2 + c_k^2} \tilde{\boldsymbol{\Lambda}}_{\mathbf{c}}' \times \tilde{\mathbf{F}}_{\mathbf{c}}(\mathbf{r})$$

$$+ \mathbf{Q} \otimes \mathbf{r}_0 : \left[\sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{(\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j + \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_i) \otimes (c_i^2 \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_i + c_j^2 \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j)}{c_i c_j} c_1c_2c_3 \frac{G_2^{i+j}}{A_2^{i+j}} \right] \times \tilde{\mathbf{F}}_{\mathbf{c}}(\mathbf{r})$$

$$- \mathbf{Q} \cdot \left[\sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{c_1c_2c_3 \rho_0 K_2^{i+j}}{\left[E_1^{i+j}(\rho_0)\right]^2 \sqrt{\rho_0{}^2 - \mu_0{}^2}\sqrt{\rho_0{}^2 - \nu_0{}^2}} \frac{(c_i^2 \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_i + c_j^2 \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j)}{c_i c_j} x_{0i} x_{0j} \right.$$

$$\otimes \sum_{k=1}^{3} \frac{x_k \hat{\mathbf{x}}_k}{\rho^2 - c_1^2 + c_k^2} \Bigg] \underset{\cdot}{\times} \tilde{\mathbf{F}}_{\mathbf{c}}(\mathbf{r}) + O\left(el_3\right) \tag{53}$$

We turn now to the calculation of the second integral $\mathbf{I}_f\left(\mathbf{r}\right)$. Following the same track of calculations as with the integral $\mathbf{I}_c\left(\mathbf{r}\right)$ we arrive at expression

$$\mathbf{I}_f\left(\mathbf{r}\right) = \sum_{m=1}^{3} \frac{3f_1f_2f_3}{f_m} \frac{R_1^m}{P_1^m} \frac{\mathbf{Q}}{\sigma_c} \cdot \hat{\mathbf{x}}_m \otimes \hat{\mathbf{x}}_m \otimes \mathbf{r} \underset{\cdot}{\times} \tilde{\mathbf{H}}_1(\rho)$$

$$+ \sum_{m=1}^{3} \sum_{k=1}^{3} \frac{3f_1f_2f_3}{\sigma_c f_m} \frac{\rho_0 W_1^m}{\left[E_1^m\left(\rho_0\right)\right]^2 \sqrt{\rho_0{}^2 - \mu_0{}^2}\sqrt{\rho_0{}^2 - \nu_0{}^2}} \frac{x_{0m} x_k \mathbf{Q} \cdot \hat{\mathbf{x}}_k}{\rho^2 - f_1^2 + f_k^2} \otimes \hat{\mathbf{x}}_m \otimes \mathbf{r} \underset{\cdot}{\times} \tilde{\mathbf{H}}_1(\rho),$$

$$- \frac{\left(\Lambda_f - f_1^2\right)\left(\Lambda_f - f_2^2\right)\left(\Lambda_f - f_3^2\right)}{3\left(\Lambda_f - \Lambda_f'\right)} \frac{2f_1f_2f_3}{\sigma_c} \frac{R_2^1}{P_2^1} \mathbf{Q} \otimes \mathbf{r}_0 : \tilde{\mathbf{\Lambda}}_{\mathbf{f}} \otimes \tilde{\mathbf{\Lambda}}_{\mathbf{f}} \underset{\cdot}{\times} \tilde{\mathbf{F}}_{\mathbf{f}}(\mathbf{r})$$

$$- \sum_{k=1}^{3} \frac{\left(\Lambda_f - f_1^2\right)\left(\Lambda_f - f_2^2\right)\left(\Lambda_f - f_3^2\right)}{3\sigma_c\left(\Lambda_f - \Lambda_f'\right)} \frac{f_1f_2f_3\rho_0 W_2^1 \left(\frac{x_{01}}{\Lambda_f - f_1^2} + 1\right)}{\left[E_2^1\left(\rho_0\right)\right]^2 \sqrt{\rho_0{}^2 - \mu_0{}^2}\sqrt{\rho_0{}^2 - \nu_0{}^2}}$$

$$\cdot \frac{x_k \mathbf{Q} \cdot \hat{\mathbf{x}}_k}{\rho^2 - f_1^2 + f_k^2} \tilde{\mathbf{\Lambda}}_{\mathbf{f}} \underset{\cdot}{\times} \tilde{\mathbf{F}}_{\mathbf{f}}(\mathbf{r})$$

$$+ \frac{\left(\Lambda_f' - f_1^2\right)\left(\Lambda_f' - f_2^2\right)\left(\Lambda_f' - f_3^2\right)}{3\left(\Lambda_f - \Lambda_f'\right)} 2f_1f_2f_3 \frac{R_2^2}{P_2^2} \frac{\mathbf{Q}}{\sigma_c} \otimes \mathbf{r}_0 : \tilde{\mathbf{\Lambda}}'_{\mathbf{f}} \otimes \tilde{\mathbf{\Lambda}}'_{\mathbf{f}} \underset{\cdot}{\times} \tilde{\mathbf{F}}_{\mathbf{f}}(\mathbf{r})$$

$$+ \sum_{k=1}^{3} \frac{\left(\Lambda_f' - f_1^2\right)\left(\Lambda_f' - f_2^2\right)\left(\Lambda_f' - f_3^2\right)}{3\sigma_c\left(\Lambda_f - \Lambda_f'\right)} \frac{f_1f_2f_3\rho_0 W_2^2 \left(\frac{x_{02}}{\Lambda_f - f_1^2} + 1\right)}{\left[E_2^2\left(\rho_0\right)\right]^2 \sqrt{\rho_0{}^2 - \mu_0{}^2}\sqrt{\rho_0{}^2 - \nu_0{}^2}}$$

$$\cdot \frac{x_k \mathbf{Q} \cdot \hat{\mathbf{x}}_k}{\rho^2 - f_1^2 + f_k^2} \tilde{\mathbf{\Lambda}}'_{\mathbf{f}} \underset{\cdot}{\times} \tilde{\mathbf{F}}_{\mathbf{f}}(\mathbf{r})$$

$$+\mathbf{Q} \otimes \mathbf{r}_0 : \left[\sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{\left(\hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j + \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_i\right) \otimes \left(f_i^2 \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_i + f_j^2 \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j\right)}{f_i f_j} \frac{f_1f_2f_3}{\sigma_c} \frac{R_2^{i+j}}{P_2^{i+j}} \right] \underset{\cdot}{\times} \tilde{\mathbf{F}}_{\mathbf{f}}(\mathbf{r})$$

$$+ \frac{\mathbf{Q}}{\sigma_c} \cdot \left[\sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{f_1f_2f_3\rho_0 W_2^{i+j}}{\left[E_1^{i+j}\left(\rho_0\right)\right]^2 \sqrt{\rho_0{}^2 - \mu_0{}^2}\sqrt{\rho_0{}^2 - \nu_0{}^2}} \frac{\left(f_i^2 \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_i + f_j^2 \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j\right)}{f_i f_j} x_{0i} x_{0j} \right.$$

$$\left. \otimes \sum_{k=1}^{3} \frac{x_k \hat{\mathbf{x}}_k}{\rho^2 - f_1^2 + f_k^2} \right] \underset{\cdot}{\times} \tilde{\mathbf{F}}_{\mathbf{f}}(\mathbf{r}) + O\left(el_3\right) \tag{54}$$

The constants P_n^m, R_n^m and W_n^m are given by

$$P_n^m = \left[I_n^m(c_1, f_1) - \frac{1}{C_n^m} - \frac{\sigma_c}{\sigma_f - \sigma_c} \frac{1}{F_n^m} \right] \frac{\sigma_f - \sigma_c}{\sigma_c} F_n^m \tag{55}$$

$$R_n^m = \left[I_n^m(c_1, \rho_0) - \frac{1}{C_n^m} \right] E_n^m(c_1) \tag{56}$$

$$W_n^m = \frac{E_n^m(c_1)}{P_n^m}. \tag{57}$$

Finally, in order to calculate the magnetic field $\mathbf{B}$ we need the multipole expansion of the first term $\mathbf{Q} \times \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3}$ which has been obtained in [3] in the form

$$\mathbf{Q} \times \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} = 3\mathbf{Q} \times \tilde{\mathbf{H}}_1(\rho) \cdot \mathbf{r} + \mathbf{Q} \times \mathbf{r}_0 \cdot \tilde{\mathbf{F}}(\mathbf{r}) + O\left(el_3\right). \tag{58}$$

Inserting equations (53), (54) and (58) into (28), we calculate the magnetic field $\mathbf{B}(\mathbf{r})$. The calculation is not trivial and it ends up to the following expression

$$\begin{aligned}
\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \Bigg\{ \Bigg[\mathbf{Q} \otimes \mathbf{r}_0 : \Bigg[\sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{\left(c_i^2 \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_i + c_j^2 \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j\right)}{I_2^{i+j}(c_1, f_1) - \frac{1}{C_2^{i+j}} - \frac{\sigma_c}{\sigma_f - \sigma_c} \frac{1}{F_2^{i+j}}} \\
\otimes \Bigg[\Bigg(I_2^{i+j}(f_1, \rho_0) + \frac{\sigma_c}{\sigma_f - \sigma_c} \frac{1}{F_2^{i+j}} \Bigg) \frac{\left(c_i^2 \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_i + c_j^2 \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j\right)}{c_i^2 + c_j^2} \\
- \Bigg(I_2^{i+j}(c_1, \rho_0) - \frac{1}{F_2^{i+j}} \Bigg) \frac{\left(f_i^2 \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_i + f_j^2 \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j\right)}{f_i^2 + f_j^2} \Bigg] \Bigg] \underset{\cdot}{\times} \tilde{\mathbf{F}}(\mathbf{r}) \Bigg] \\
+ \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{\rho_0 x_{0i} x_{0j}}{\left[E_1^{i+j}\left(\rho_0\right)\right]^2 \sqrt{\rho_0{}^2 - \mu_0{}^2} \sqrt{\rho_0{}^2 - \nu_0{}^2}} \frac{1}{I_2^{i+j}(c_1, f_1) - \frac{1}{C_2^{i+j}} - \frac{\sigma_c}{(\sigma_f - \sigma_c)} \frac{1}{F_2^{i+j}}} \\
\times \mathbf{Q} \cdot \left[\frac{\left(c_i^2 \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_i + c_j^2 \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j\right)}{c_i^2 + c_j^2} - \frac{\left(f_i^2 \hat{\mathbf{x}}_j \otimes \hat{\mathbf{x}}_i + f_j^2 \hat{\mathbf{x}}_i \otimes \hat{\mathbf{x}}_j\right)}{f_i^2 + f_j^2} \right] \otimes \\
\otimes \sum_{k=1}^{3} \frac{\mathbf{x}_k \hat{\mathbf{x}}_k}{\rho^2 - f_1^2 + f_k^2} \underset{\cdot}{\times} \tilde{\mathbf{F}}(\mathbf{r}) + \mathbf{Q} \otimes \mathbf{r}_0 \underset{\cdot}{\times} \tilde{\mathbf{F}} \Bigg\} + O\left(el_3\right)
\end{aligned} \tag{59}$$

The above result gives the magnetic field in the exterior of the head when this is modelled with an inhomogeneous conductor consist of an ellipsoidal fluid core, covered by a confocal ellipsoidal cerebrum shell. In this result similarities and differences can be observed, with the corresponding result for the magnetic field in the exterior of the ellipsoidal inhomogeneous model where the source lies in the cerebrum core and the shell is occupied by the cerebrospinal fluid. Our next step is to investigate further the differences between the two models, in order to estimate the effect of the location of the dipole source either inside or outside the homogeneous core.

Acknowledgement: We thank the European Social Fund (ESF), Operational Program for Educational and Vocational Training II (EPEAEK II), and particularly the Program PYTHAGORAS II, for funding the above work.

References

[1] B.N. Cuffin and D. Cohen, *Magnetic Fields of a Dipole in Special Volume Conductor Shapes*, IEEE Trans. Biomedical Eng., **BME-24**, pp. 372–381, 1997.
[2] G. Dassios and F. Kariotou, *On the Geselowitz Formula in Biomagnetics*, Quart. Appl. Math., **61**, 2003, pp. 387–400.
[3] G. Dassios and F. Kariotou, *Magnetoencephalography in Ellipsoidal Geometry*, J. Math. Phys., **44**, 2003, pp. 220–241.
[4] G. Dassios and F. Kariotou, *The Direct MEG Problem in the Presence of an Ellipsoidal Shell Inhomogeneity*, Quart. Appl. Math., LXIII, pp.601–618, 2005.
[5] A. S. Fokas, I. M. Gel'fand and Y. Kurylev, *Inversion Method for Magnetoencephalography*, Inverse Problems, **12**, 1996, L9–L11.
[6] J.C. de Munck, *The Potential Distribution in a Layered Anisotropic Spheroidal Volume Conductor*, J. Appl. Phys., **64**, pp. 464–470, 1988.
[7] D. B. Geselowitz, *On the Magnetic Field Generated Outside an Inhomogeneous Volume Conductor by Internal Current Sources*, IEEE Trans. Magn., **6**, 1970, pp. 346–347.
[8] S. N. Giapalaki and F. Kariotou, *The Complete Ellipsoidal Shell-Model in EEG Imaging*, Abstract and Applied Analysis, pp 1–18, Vol.2006, 2006.
[9] E. W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics*, Cambridge University Press, 1931.
[10] R. J. Ilmoniemi, M. S. Hamalainen and J. Knuutila *The Forward and Inverse Problems in the Spherical model*, pp. 278–282, Biomagnetism: Applications and Theory, edited by Harold Weinberg, Gerhard Stroink, and Toivo Katila, Pergamon Press, New York, 1985.
[11] G. Kamvyssas and F. Kariotou, *Confocal Ellipsoidal Boundaries in EEG Modeling*, Bulletin of the Greek Mathematical Society, **50**, 2005, pp. 119–133.
[12] F. Kariotou, *Electroencephalography in Ellipsoidal Geometry*, J. Math. Analysis and Applications, **290**, 2004, pp. 324–342.
[13] J. Malmivuo and R. Plonsey, *Bioelectromagnetism*, Oxford University Press, NY, 1995.
[14] J. Sarvas, *Basic Mathematical and Electromagnetic Concepts of the Biomagnetic Inverse Problem*, Phys. Med. Biol., **32**, 1987, pp. 11–22.
[15] W.S. Snyder, M.R. Ford, G.G. Warner and H.L. Fisher Jr., *Estimates of Absorbed Fractions for Monoenergetic Photon Sources Uniformly Distributed in Various Organs of a Heterogeneous Phantom*, Journal of Nuclear Medicine, Supplement Number 3, August 1969, Volume **10**, Pamphlet No. 5, Revised 1978
[16] W.X.Wang, *The Potential for a Homogeneous Spheroid in a Spheroidal Coordinate System: I. At an Exterior Point*, J. Phys. A. Math. Gen, 21, pp. 4245–4250, 1988.

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