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A spectral bound for a class of Schrödinger operators

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Abstract

We present a different proof and a generalization of a result of Fusco and Pignotti [FP] concerning the lower bound of the spectrum of a class of Schrödinger operators of the form $L^{\varepsilon} := -\varepsilon^2 \frac{d^2}{ds^2} + V$, where $V : \mathbb{R} \to \mathbb{R}$ is a nonnegative function singular (vanishes) at finite points $s_1 < \ldots < s_N$. We also examine the case of multidimensional Schrödinger operators possessing radial symmetry and the case of V being singular at infinity.

Keywords: Schrödinger operators, singular perturbations, Schwarz Rearrangement

Consider, in $L^2(\mathbb{R})$, the Schrödinger operator

$$L^{\varepsilon} w := -\varepsilon^2 w'' + q^2 w$$

under the hypotheses on $q \in C(\mathbb{R})$:

(H1) $q(s) > 0$, $s \neq s_k$, $q(s_k) = 0$, $k = 1, \ldots, N$ (N independent of ε),

(H2) $q(s) \geq c|s - s_k|^p$ if $|s - s_k| < \delta$, $k = 1, \ldots, N$, for some $c, p, \delta > 0$ indep. of ε,

(H3) $q(s) \geq \tilde{c}$ if $s \in \mathbb{R} - \cup_{k=1}^{N} B(s_k, \delta)$ for some $\tilde{c} > 0$ independent of ε, where $B(s_k, \delta) = (s_k - \delta, s_k + \delta)$.

The case $N = 1$ was studied in [FP], where (among other things) a lower bound for the principal eigenvalue of L^{ε} was obtained. This followed from sharp upper bounds for the fundamental solution of L^{ε}; with $q \in C(\mathbb{R})$ satisfying (H1) with $N = 1$ ($s_1 = 0$), $q(s) = c|s|^p$, $|s| < \delta$, and (H3).

The purpose of this note is to give a simple proof of this spectral bound which applies directly to the case $N \geq 1$. If we assume that $q(s) = c|s|^p$ for all $s \in \mathbb{R}$, then, by setting $s = \varepsilon^{\frac{1}{p+1}} r$ and $E_n = \varepsilon^{\frac{2p}{p+1}} c_n$, the eigenvalue equation $L^{\varepsilon} w = Ew$ can be rescaled to $-\frac{d^2w}{dr^2} + c^2|r|^{2p}w = c_n w$ which is independent of ε. The idea is to apply this re-scaling to the general case (see also [SF]). Our result is

Theorem 1 If $\varepsilon > 0$ is sufficiently small, the spectrum $\sigma(L^\varepsilon)$ of L^ε satisfies

$$\sigma(L^\varepsilon) \subseteq \left[c_0 \varepsilon^{\frac{2p}{p+1}}, \infty\right)$$

for some $c_0 > 0$ independent of ε.

Proof. By the Friedrichs extension, L^ε defines a self-adjoint operator in $L^2(\mathbb{R})$ with domain $D(L^\varepsilon) \subseteq H^1(\mathbb{R})$ (cf. [Ze] Ch. 5). Also (cf. [HS] pg. 55):

$$\sigma(L^\varepsilon) \subseteq \left[\inf_{0 \neq w \in D(L^\varepsilon)} \frac{(L^\varepsilon w, w)}{(w,w)}, \infty\right).$$

Moreover, given $w \in D(L^\varepsilon)$, there exist $\varphi_n \in C_0^\infty(\mathbb{R})$ such that $\varphi_n \overset{L^2(\mathbb{R})}{\rightarrow} w$ and $(L^\varepsilon \varphi_n, \varphi_n) \to (L^\varepsilon w, w)$ as $n \to \infty$.

Note that (H1), (H2), (H3) imply that, for $\varepsilon > 0$ sufficiently small,

$$q(s) \geq q_1(s) = \begin{cases} c|s - s_k|^p, & |s - s_k| \leq \varepsilon^{\frac{1}{p+1}},\ k = 1, \ldots, N, \\ \\ c\varepsilon^{\frac{p}{p+1}}, & \text{otherwise.} \end{cases}$$

Thus, from the above, it suffices to show that

$$\int_{-\infty}^{\infty} \varepsilon^2 \varphi'^2 + q_1^2(s)\varphi^2 ds \geq c_0 \varepsilon^{\frac{2p}{p+1}} \int_{-\infty}^{\infty} \varphi^2 ds, \quad \forall \varphi \in C_0^\infty(\mathbb{R}). \tag{1}$$

Let $\varphi \in C_0^\infty(\mathbb{R})$, then

$$\int_{-\infty}^{\infty} \varepsilon^2 \varphi'^2 + q_1^2(s)\varphi^2 ds =$$

$$= \sum_{k=1}^{N} \int_{B(s_k, \varepsilon^{\frac{1}{p+1}})} \varepsilon^2 \varphi'^2 + c^2 |s - s_k|^{2p} \varphi^2 ds + \int_{\mathbb{R} - \cup_{k=1}^N B(s_k, \varepsilon^{\frac{1}{p+1}})} \varepsilon^2 \varphi'^2 + c^2 \varepsilon^{\frac{2p}{p+1}} \varphi^2 ds \geq$$

$$\left(\text{Set} \quad r = \varepsilon^{-\frac{1}{p+1}}(s - s_k) \quad \text{and write} \quad \dot{} = \frac{d}{dr}\right)$$

$$\geq \sum_{k=1}^{N} \varepsilon^{\frac{1}{p+1}} \int_{B(0,1)} \varepsilon^{2 - \frac{2}{p+1}} \dot{\varphi}^2 + c^2 \varepsilon^{\frac{2p}{p+1}} |r|^{2p} \varphi^2 dr + c^2 \varepsilon^{\frac{2p}{p+1}} \int_{\mathbb{R} - \cup_{k=1}^N B(s_k, \varepsilon^{\frac{1}{p+1}})} \varphi^2 ds =$$

$$= \sum_{k=1}^{N} \varepsilon^{\frac{2p}{p+1}} \varepsilon^{\frac{1}{p+1}} \int_{B(0,1)} \dot{\varphi}^2 + c^2 |r|^{2p} \varphi^2 dr + c^2 \varepsilon^{\frac{2p}{p+1}} \int_{\mathbb{R} - \cup_{k=1}^N B(s_k, \varepsilon^{\frac{1}{p+1}})} \varphi^2 ds \geq$$

(Let $\mu_1 > 0$ be the principal eigenvalue of $-\ddot{\varphi} + c^2|r|^{2p}\varphi = \mu\varphi,\ r \in (-1,1)$ with Neumann B.C's)

$$\geq \sum_{k=1}^{N} \varepsilon^{\frac{2p}{p+1}} \varepsilon^{\frac{1}{p+1}} \mu_1 \int_{B(0,1)} \varphi^2 dr + c^2 \varepsilon^{\frac{2p}{p+1}} \int_{\mathbb{R}-\cup_{k=1}^N B(s_k, \varepsilon^{\frac{1}{p+1}})} \varphi^2 ds =$$

$$= \sum_{k=1}^{N} \varepsilon^{\frac{2p}{p+1}} \mu_1 \int_{B(s_k, \varepsilon^{\frac{1}{p+1}})} \varphi^2 ds + c^2 \varepsilon^{\frac{2p}{p+1}} \int_{\mathbb{R}-\cup_{k=1}^N B(s_k, \varepsilon^{\frac{1}{p+1}})} \varphi^2 ds \geq$$

$$\geq c_0 \varepsilon^{\frac{2p}{p+1}} \int_{-\infty}^{\infty} \varphi^2 ds.$$

Hence, (1) is true.

Remark 1 In the case that L^ε has essential spectrum, (H3) yields $\sigma_{ess}(L^\varepsilon) \subseteq [\tilde{c}^2, \infty)$ (cf. [Da]).

One can easily obtain (see also [FP])

Corollary 1 Let $g \in C(\mathbb{R})$ be such that

$$g(s) \geq -C_1 \varepsilon^{\frac{2p}{p+1}}, \quad s \in \mathbb{R}, \tag{2}$$

for some $C_1 > 0$ independent of ε. Then, if $C_1 > 0$ and $\varepsilon > 0$ are sufficiently small, we have

$$\sigma(L^\varepsilon + g) \subseteq \left[\tilde{c_0} \varepsilon^{\frac{2p}{p+1}}, \infty\right) \tag{3}$$

for some $\tilde{c_0}$ independent of ε.

Remark 2 As it can be seen from the proofs, the continuity assumption on q and g can be relaxed to $L^1_{loc}(\mathbb{R})$.

Remark 3 In [SF], for the analysis of the linearization of a singular perturbation problem, we studied a linear operator of the form $L^\varepsilon + g$; with g satisfying (2) $(p = \frac{1}{2})$ but with $C_1 > 0$ a fixed constant (*not* necessarily small). In that reference, more properties of q and g were used to show (3).

Remark 4 The proof of Theorem 1 seems to extend to the case of multidimensional Scrödinger operators of the form:

$$\mathbf{L}^\varepsilon u = -\varepsilon^2 \Delta u + q^2(|\mathbf{x}|)u, \quad \mathbf{x} \in \mathbb{R}^m,$$

with $q \in C[0,\infty)$ satisfying $q(0) = 0,\ q(s) \geq cs^p,\ 0 \leq s \leq \delta$ and $q(s) \geq \tilde{c},\ s \geq \delta$ $(c, p, \delta, \tilde{c}$ independent of $\varepsilon)$. If $\varepsilon > 0$ is sufficiently small, then $q(s) \geq q_1(s), s \geq 0,$

where $q_1(s) := cs^p,\ 0 \leq s \leq \varepsilon^{\frac{1}{p+1}}$, $q_1(s) := c\varepsilon^{\frac{p}{p+1}},\ s \geq \varepsilon^{\frac{1}{p+1}}$. The main observation is that to obtain the inequality

$$\int_{\mathbb{R}^m} \varepsilon^2 |\nabla u|^2 + q_1^2(|\mathbf{x}|)u^2 d\mathbf{x} \geq c_0 \varepsilon^{\frac{2p}{p+1}} \int_{\mathbb{R}^m} u^2 d\mathbf{x}, \quad \forall u \in C_0^\infty(\mathbb{R}^m),$$

it is enough to show it for the Schwarz Rearrangement u^* of u. Recall that u^* is radial, $u^* \in H^1(\mathbb{R}^m)$, $\int_{\mathbb{R}^m} |\nabla u^*|^2 d\mathbf{x} \leq \int_{\mathbb{R}^m} |\nabla u|^2 d\mathbf{x}$, $\int_{\mathbb{R}^m} q_1^2(|\mathbf{x}|) {u^*}^2 d\mathbf{x} \leq \int_{\mathbb{R}^m} q_1^2(|\mathbf{x}|)u^2 d\mathbf{x}$ (because q_1^2 is continuous and nondecreasing), and $\int_{\mathbb{R}^m} {u^*}^2 d\mathbf{x} = \int_{\mathbb{R}^m} u^2 d\mathbf{x}$ (cf. [LL] and Appendix D of [KP]).

By modifying the proof of Theorem 1, we can show

Theorem 2 Suppose that $q \in C(\mathbb{R})$ satifies

$$q(s) \geq c|s|^{-p} \text{ if } |s| \geq C \text{ and } q(s) \geq \tilde{c} \text{ if } |s| \leq C,$$

with $p > 1,\ c, C, \tilde{c} > 0$ constants independent of $\varepsilon > 0$. Then

$$\sigma(L^\varepsilon) \subseteq \left[c_0 \varepsilon^{\frac{2p}{p-1}}, \infty\right)$$

for some $c_0 > 0$ independent of ε, where $L^\varepsilon w = -\varepsilon^2 w'' + q^2(s)w$.

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