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Stability of triple junctions on the plane

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Abstract

Abstract. We begin by introducing the geometric evolution problem of networks of triple junctions in planar domains. Then we identify a few basic steady states in bounded and smooth domains and discuss their stability in terms of the geometry of the boundary.

Keywords: stability,triple junctions,network

1. Introduction

About 20 years ago,motivated by dynamical models in materials science describing phase separation and the motion of interfaces separating phases, Bronsard and Reitich [1] introduced the problem of networks of curves in a planar domain with normal velocity proportional to the curvature and fixed angle conditions at the junction.
They derived from the underlying model the equations of motion, as well as the boundary conditions: the angles formed by the curves at a triple junction are constant throughout the evolution and intersecting the boundary of the domain orthogonally at all times.
So,our interest is in studying a network of curves that form triple junctions and also is in motion with the normal velocity equal to the curvature. We begin with the simplest case: A single triple junction. We begin with the parametrization of the curves. The situation can be formulated mathematically as follows:
Let Ω be a bounded and smooth domain on the plane, t$\geq$0 time, s arclength parameter and $G_i(s,t)$, (i=1,2,3) embeddings contained in Ω that meet at one point and intersect with $\partial\Omega$ at the other ends. Then the evolution of $G_i(s,t)$ is described as follows:
(We denote: $G_{is} = \frac{\partial G_i}{\partial s}, G_{iss} = \frac{\partial^2 G_i}{\partial s^2}$ and $G_{it} = \frac{\partial G_i}{\partial t}$. $\partial\Omega$ the boundary of Ω, $L_i(t)$ the length of curve G_i (i=1,2,3), $b(\cdot,\cdot)$ a C^1 real function of two varables that describes locally the boundary $\partial\Omega$ and $<\cdot,\cdot>$ Euclidean inner product).
For i=1,2,3 in $0 \leq s \leq L_i(t)$

$$G_{it} = G_{iss} \qquad \textbf{(I)}$$

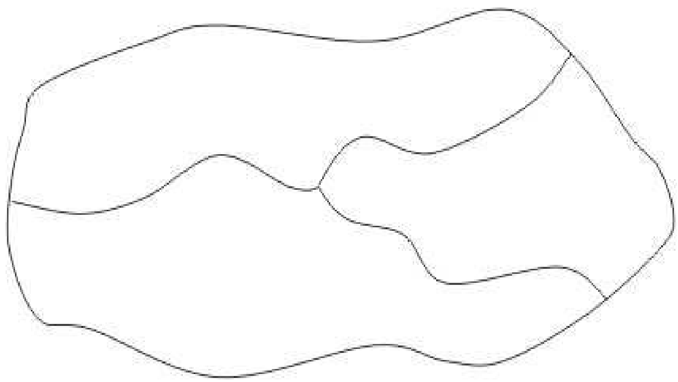

Figure 1: A single triple junction

And conditions :

1.1. Incidence at the junction:

$$G_1(0,t) = G_2(0,t) = G_3(0,t)$$

1.2. Angle conditions at the junction:

$$G_{is}(0,t) \cdot G_{(i+1)s}(0,t) = cos120^o \quad , \quad i = 1,2$$

1.3. Incidence at $\partial\Omega$:

$$b(G_i(L_i(t),t) = 0 \quad , \quad i = 1,2,3$$

1.4. Angle conditions at $\partial\Omega$:

$$< G_{is}(L_i(t),t), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla b(G_i) >= 0 \ \ , \ \ i = 1,2,3$$

Comments:

1.5. $G_i(s,t)$ are embeddings in the plane and the network is in motion with the normal velocity equal to the curvature law,

$$V_i^N = k_i \ \ i = 1,2,3$$

Here $V_i^N = G_{it} \cdot N_i$ is the normal velocity of the curve G_i and N_i is the unit normal vector to G_i , T_i=G_{is} is the unit tangent vector . N_i,T_i with the orientation of the coordinate system.

V_i^T is the tangential velocity of curve G_i and k_i is the curvature of curve G_i.Moreover,the velocity V_i of the curve G_i is given from:

$$V_i = (V_i^N, V_i^T) \Leftrightarrow \; V_i = (G_{it} \cdot T_i, G_{it} \cdot N_i).$$

But note that $G_{it} \cdot T_i = 0$ because G_{iss} is perpendicular to T_i .So,

$$V_i = (0, G_{it} \cdot N_i)$$

and therefore:

$$G_{it} \cdot N_i = G_{iss} \cdot N_i \Leftrightarrow V_i^N = k_i$$

1.6. The curves G_i (i=1,2,3) meet at one point (the triple junction) and (1.1) is describing that property.(1.2) describe the Plateau angle conditions: the three angles formed at the junction are 120^o each. We note that these angles can be replaced from arbitrary prearranged values $\vartheta_1, \vartheta_2, \vartheta_3$ as long as $\vartheta_1 + \vartheta_2 + \vartheta_3 = 360^o$.(1.3) describe the contact of each curve to the boundary $\partial\Omega$ of the domain Ω.Finally the orthogonal intersection of each curve G_i to the boundary $\partial\Omega$ is described at (1.4).

1.7. The network reduces its perimeter (the total length) along the evolution:

$$\frac{d}{dt}L(t) = -\int_G kV^N = -\sum_{i=1}^{3}\int_{G_i} k^2 \leq 0$$

1.8. Note that the system (I) consists of 6 parabolic equations of second order and 12 boundary conditions (4 from the incidence at the junction,2 from the angle conditions at the junction,3 from the incidence at the boundary and 3 from the angle conditions at the boundary).

It would be more convenient to formulate the problem,so that the parameter that represents the arclength takes its values in a domain independent from time t. For this purpose:

Let $\Gamma_1 = (g_1(x,t), g_2(x,t)), x \in [0,1]$ and $G_1 = (g_1, g_2)$ then:

$$G_{1s} = G_{1x}\frac{dx}{ds} \Leftrightarrow G_{iss} = \frac{\partial G_{1x}}{\partial s}\frac{dx}{ds} + G_{1x}\frac{d^2s}{dx^2} \Leftrightarrow$$

$$\Leftrightarrow G_{iss} = G_{1xx}(\frac{dx}{ds})^2 + G_{1x}\frac{d^2s}{dx^2} \Leftrightarrow \Gamma_{iss} = \frac{\Gamma_{1xx}}{|\Gamma_{1x}|^2} + \Gamma_{1x}\frac{d^2x}{ds^2}$$

.

Where s(x)=$\int_0^x |\Gamma_x(p,t)|$dp and $\frac{ds}{dx} = |\Gamma_x(x,t)|$.So the equations (I) take the following form:

$$\Gamma_{it} = \frac{\Gamma_{1xx}}{|\Gamma_{1x}|^2} + \Gamma_{1x}\frac{d^2x}{ds^2} \quad , \quad i = 1,2,3$$

in D_t=$\{(x,t)|0 \leq x \leq 1\}$.Note that by multiplying with N_1 left and right we get

$$\Gamma_{it} \cdot N_1 = \frac{\Gamma_{1xx}}{|\Gamma_{1x}|^2} \cdot N_1 + \Gamma_{1x}\frac{d^2x}{ds^2} \cdot N_1 \Rightarrow \Gamma_{it} \cdot N_1 = \frac{\Gamma_{1xx}}{|\Gamma_{1x}|^2} \cdot N_1$$

since

$$\Gamma_{1x}\frac{d^2x}{ds^2}\cdot N_1 = 0$$

Remark. We note that the tangential term in the equation can be assigned at will without affecting the equation $V_i^N = k_i \;\; i = 1,2,3$. For example another equation that is compatible with motion by curvature is: $\Gamma_{it} = \frac{\Gamma_{1xx}}{|\Gamma_{1x}|^2}$, for i=1,2,3 , $x \in [0,1]$ which has to be supplemental with conditions 1.1-4 in (I).The condition $V_i^N = k_i \;\; i = 1,2,3$ by itself is not sufficient to determine the evolution. Different equations for the embedding are expected to lead to different evolutions for the curves. In the case withat junctions does not affect the evolution. If we use these equations the system will take the following form:

$$\Gamma_{it} = \frac{\Gamma_{ixx}}{|\Gamma_{ix}|^2}\;,\; i = 1,2,3 \quad x \in [0,1] \qquad \textbf{(II)}$$

And conditions:

1.9. Incidence at the junction at x=0

$$\Gamma_1(0,t) = \Gamma_2(0,t) = \Gamma_3(0,t)$$

1.10. Angle conditions at the junction for i=1,2 at x=0

$$\frac{\Gamma_{ix}}{|\Gamma_{ix}|}\cdot\frac{\Gamma_{(i+1)x}}{|\Gamma_{(i+1)x}|} = cos120^o$$

1.11. Incidence at $\partial\Omega$ for i=1,2,3 at x=1

$$b(\Gamma_i) = 0$$

1.12. Angle conditions at $\partial\Omega$ for i=1,2,3 at x=1

$$< \Gamma_{ix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla b(\Gamma_i) >= 0$$

2. Linearization

In this paragraph we linearize equations (II) in order to study the stability of the steady states. At first we define the family of petrubations $\tilde{\Gamma}_i^\epsilon = \tilde{\Gamma}_i$:

$$\tilde{\Gamma_i} = \Gamma_i + \epsilon(h_i^N N_i + h_i^T T_i) \quad ,0 < \epsilon << 1$$

Where $h_i^N, h_i^T : [0,l_i] \to \Re, (l_i$ length of curve $\tilde{\Gamma_i})$.Note that for i=1,2 $N_i N_{i+1} = T_i T_{i+1} = cos120^o$ and that x is arclength parameter for Γ_i but not for $\tilde{\Gamma}_i$.By computing $\tilde{\Gamma}_{ix} = \frac{\partial\tilde{\Gamma}_i}{\partial x}$, $\tilde{\Gamma}_{ixx} = \frac{\partial^2\tilde{\Gamma}_i}{\partial x^2}$ and using the Frenet formulas we show that:

$$\frac{d}{d\epsilon}\frac{\tilde{\Gamma}_{ixx}}{|\tilde{\Gamma}_{ix}|^2}|_{\epsilon=0} = (h_i''^N - h_i^T k_{ix} + h_i^N k_i^2)N_i + (-2h_i'^N k_i - h^N k_{ix} + h_i''^T - h_i^T k_i^2)T_i$$

Thus the relevant eigenvalue problem is **(III)**:

$$h_i''^N - h_i^T k_{ix} + h_i^N k_i^2 = -\lambda h_i^N$$

$$-2h_i'^N k_i - h^N k_{ix} + h_i''^T - h_i^T k_i^2 = -\lambda h_i^T$$

From (1.9) at x=0 (incidence at the junction) we have $\tilde{\Gamma_1} = \tilde{\Gamma_2} = \tilde{\Gamma_3} \Leftrightarrow \Gamma_1 + \epsilon(h_1^N N_1 + h_1^T T_1) = \Gamma_2 + \epsilon(h_2^N N_2 + h_2^T T_2) = \Gamma_3 + \epsilon(h_3^N N_3 + h_3^T T_3) \Leftrightarrow h_1^N N_1 + h_1^T T_1 = h_2^N N_2 + h_2^T T_2 = h_3^N N_3 + h_3^T T_3$ **(2.1')**. From here we see that **(2.1)**:

$$h_1^N + h_2^N + h_3^N = 0$$

$$h_1^T + h_2^T + h_3^T = 0$$

From (1.10) at x=0 for i=1,2 (angle conditions at the junction) we have $\frac{\tilde{\Gamma}_{ix}}{|\tilde{\Gamma}_{ix}|} \frac{\tilde{\Gamma}_{(i+1)x}}{|\tilde{\Gamma}_{(i+1)x}|} = cos120^o \Leftrightarrow$ from the Taylor expansion series around ϵ =0 we get:
$[\frac{\Gamma_{ix}}{|\Gamma_{ix}|} + (h_i'^N + h_i^T k_i)N_i\epsilon + O(\epsilon^2)][\frac{\Gamma_{(i+1)x}}{|\Gamma_{(i+1)x}|} + (h_{i+1}'^N + h_{i+1}^T k_{i+1})N_{i+1}\epsilon + O(\epsilon^2)] =$
$= cos120^o$.From here we can see that **(2.2)**:

$$h_1'^N + h_1^T k_1 = h_2'^N + h_2^T k_2 = h_3'^N + h_3^T k_3$$

From (1.11) at $\partial\Omega$ (incidence at the boundary) and for i=1,2,3 we have: $b(\tilde{\Gamma}_i) = 0 \Leftrightarrow \frac{d}{d\epsilon} b(\tilde{\Gamma}_i)|_{\epsilon=0} = 0$ and by computing we get **(2.3)**:

$$h_1^T = h_2^T = h_3^T = 0$$

Finally working again as previously from (1.12) at $\partial\Omega$ (angle conditions at the boundary) and for i=1,2,3 we have: $< \tilde{\Gamma}_{ix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla b(\tilde{\Gamma}_i) >= 0$ and we get **(2.4)**:

$$K_{\partial\Omega}^i h_i^N = h_i'^N$$

Where $K_{\partial\Omega}^i$ is the curvature of the boundary $\partial\Omega$ at the point that the curve Γ_i meets $\partial\Omega$.

Comments:
2.5. If $h_i^N(x) \equiv 0$ then $h_i^T(x) \equiv 0, x \in [0, l_i]$.

Proof. Case $k_i \neq c$, c=constant
From (III): $h_i''^N - h_i^T k_{ix} + h_i^N k_i^2 = -\lambda h_i^N \Rightarrow$
$h_i^T k_{ix} = 0 \Rightarrow h_i^T(x) \equiv 0$, since $k_{ix} \neq 0, (k_i \neq c)$.
Case $k_i = c$, c=constant
From (III):$-2h_i'^N k_i - h^N k_{ix} + h_i''^T - h_i^T k_i^2 = -\lambda h_i^T \Rightarrow h_i''^T = (c^2 - \lambda)h_i^T$
For $c^2 - \lambda < 0$ (It doesn't change anything for $c^2 - \lambda \geq 0$):

$$h_i^T = C_i^T cosh(x\sqrt{c^2 - \lambda}) + D_i^T sinh(x\sqrt{c^2 - \lambda})$$

C_i^T, D_i^T unknown real variables.From (2.1') at x=0:

$$h_1^N N_1 + h_1^T T_1 = h_2^N N_2 + h_2^T T_2 \Rightarrow$$

$$h_i^T(0) = 0 \Rightarrow C_i^T = 0 \qquad (1')$$

From (2.3): $h_i^T(l_i) = 0 \Rightarrow$

$$D_i^T = 0 \qquad (2')$$

From (1') + (2'):

$$h_i^T(x) \equiv 0$$

To sum up, the eigenvalue problem **(III)** is the following:

$$h_i''^N - h_i^T k_{ix} + h_i^N k_i^2 = -\lambda h_i^N$$

$$\text{i=1,2,3}$$

$$-2h_i'^N k_i - h^N k_{ix} + h_i''^T - h_i^T k_i^2 = -\lambda h_i^T$$

And conditions:
2.1. *Incidence at the junction* at x=0

$$h_1^N + h_2^N + h_3^N = 0$$

$$h_1^T + h_2^T + h_3^T = 0$$

2.2. *Angle conditions at the junction* at x=0

$$h_1'^N + h_1^T k_1 = h_2'^N + h_2^T k_2 = h_3'^N + h_3^T k_3$$

2.3. *Incidence at* $\partial\Omega$ for i=1,2,3

$$h_1^T = h_2^T = h_3^T = 0$$

2.4. *Angle conditions at* $\partial\Omega$ for i=1,2,3

$$K_{\partial\Omega}^i h_i^N = h_i'^N$$

3. The case of the single triple junction

As written in the previous section in order to study the stability of a triple junction we linearized the system equations (II) and (1.9-12). The eigenvalues of the linearized operator will give information about the stability of the network. As we will see stability depends on the geometry of the boundary Ω. We define the sign of a curvature as follows. At the points that a curve is convex we define the sign of the curvature

positive and at the points that the a curve is concave we define the sign of the curvature negative.
We begin by considering the simplest steady state:A single triple junction with flat branches on a disc.

Proposition 3.1. The linearized operator for a single flat triple junction on a disc has one negative eigenvalue. Thus this steady state is unstable.

Proof. Let Ω be a disk on the plane with range R.Then $K^i_{\partial\Omega} = \frac{1}{R}$ (i=1,2,3) and for $k_i = 0$ (i=1,2,3) the linearized operator will take the following form:

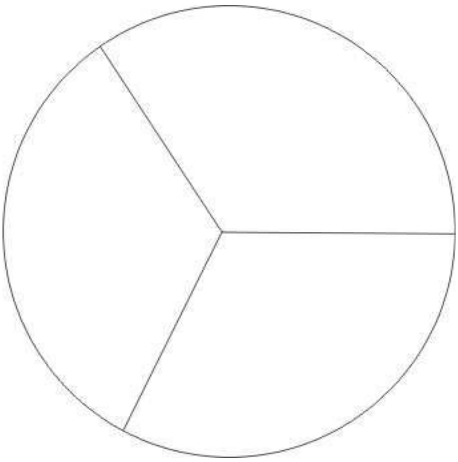

Figure 2: A single flat triple junction on a disc

$$h_i''^N = -\lambda h_i^N$$

i=1,2,3

$$h_i''^T = -\lambda h_i^T$$

Conditions:
3.1. *Incidence at the junction* at x=0

$$h_1^N + h_2^N + h_3^N = 0$$

$$h_1^T + h_2^T + h_3^T = 0$$

3.2. *Angle conditions at the junction* at x=0

$$h_1'^N = h_2'^N = h_3'^N$$

3.3. *Incidence at $\partial\Omega$* for i=1,2,3

$$h_1^T = h_2^T = h_3^T = 0$$

3.4. *Angle conditions at $\partial\Omega$* for i=1,2,3

$$\frac{1}{R} h_i^N = h_i'^N$$

We will establish the existence of a $\lambda < 0$ such that $h_i^N \neq 0$ (i=1,2,3).So if $\lambda < 0$ then $h_i^N = C_i cosh(x\sqrt{-\lambda}) + D_i sinh(x\sqrt{-\lambda})$, i=1,2,3.Where C_i,D_i unknown real variables.From (3.1),(3.2) and (3.4) we have:

$$C_1 + C_2 + C_3 = 0$$

$$D_1 = D_2 = D_3$$

$$C_i[\frac{1}{R} - \sqrt{-\lambda} tanh(R\sqrt{-\lambda})] - D_i(\sqrt{-\lambda} - \frac{1}{R} tanh(R\sqrt{-\lambda})) = 0 \ , \ \ i = 1,2,3$$

We have a 6x6 system with C_i,D_i unknown real variables:

$$\sum_{i=1}^{3} C_i[\frac{1}{R} - \sqrt{-\lambda} tanh(R\sqrt{-\lambda})] = \sum_{i=1}^{3} D_i(\sqrt{-\lambda} - \frac{1}{R} tanh(R\sqrt{-\lambda})) \Rightarrow$$

since $D_1 = D_2 = D_3$,

$$[\frac{1}{R} - \sqrt{-\lambda} tanh(R\sqrt{-\lambda})] \sum_{i=1}^{3} C_i = 3D_1(\sqrt{-\lambda} - \frac{1}{R} tanh(R\sqrt{-\lambda})) \Rightarrow$$

But $\sum_{i=1}^{3} C_i = 0$,

$$3D_1(\sqrt{-\lambda} - \frac{1}{R} tanh(R\sqrt{-\lambda})) = 0$$

and since R$\sqrt{-\lambda}$ $\neq$tanh(R$\sqrt{-\lambda}$),

$$D_1 = D_2 = D_3 = 0$$

and so,

$$[\frac{1}{R} - \sqrt{-\lambda} tanh(R\sqrt{-\lambda})] C_i = 0 \quad i = 1,2,3$$

If $\lambda_1 < 0$ is the solution of the equation

$$\frac{1}{R} = \sqrt{-\lambda} tanh(R\sqrt{-\lambda})$$

Then for λ_1 the 6x6 system has non-zero solutions:$D_1 = D_2 = D_3 = 0$ and $C_1 = -C_2 - C_3$.So, for C_2=A,C_3=B,C_1=-A-B the eigenfunctions of λ_1 are:

$$\begin{pmatrix} h_1^N \\ h_2^N \\ h_3^N \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} A cosh(x\sqrt{-\lambda_1}) + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} B cosh(x\sqrt{-\lambda_1})$$

and the eigenspace of λ_1 is :

$$< \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} >$$

The steady states are unstable and the dimension of the eigenspace of the negative eigenvalue λ_1 is two.
Note: The linearized operator has also the zero eigenvalue.Moreover for λ=0 we have $\Rightarrow h_i^N$=A_ix+B_i , i=1,2,3. Where A_i,B_i unknown real variables.From (3.1),(3.2) and (3.4) we have:

$$B_1 + B_2 + B_3 = 0$$

$$A_1 = A_2 = A_3$$

$$B_i = 0 \ , \quad i = 1, 2, 3$$

We have a 6x6 system with A_i,B_i unknown real variables. By solving this system we find that $B_1 = B_2 = B_3 = 0$ and that $A_1 = A_2 = A_3 = C$.The eigenfunctions of the zero eigenvalue are:

$$\begin{pmatrix} h_1^N \\ h_2^N \\ h_3^N \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} Cx$$

and its eigenspace is:

$$< \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} >$$

The linearized operator has the zero eigenvalue with multiplicity 1.
*Remark.*The negative eigenvalues show how the network distabilizes after a pertubation on the curves while the zero eigenvalue (and the dimension of its eigenspace) show the way the network rotates (locally) after a pertubation on the curves.

Proposition 3.2. The single flat triple junction in a domain Ω which is non-degenerate concave at the points where the junction meets the boundary, is stable.

Proof. In order to show that the steady state is stable we will only have to prove that for $\lambda < 0 \Rightarrow h_i^N \equiv 0$ (i=1,2,3).(Note that from Comment 2.1 we have that if $h_i^N \equiv 0 \Rightarrow h_i^T \equiv 0$).The linearized operator takes the following form:

$$h_i''^N = -\lambda h_i^N \quad i = 1, 2, 3$$

$$h_1^N + h_2^N + h_3^N = 0 \quad at \quad x = 0$$

$$h_1'^N = h_2'^N = h_3'^N \quad at \quad x = 0$$

$$K_i^{\partial\Omega} h_i^N = h_i'^N \quad i = 1, 2, 3 \quad at \quad x = \partial\Omega, \quad K_i^{\partial\Omega} < 0$$

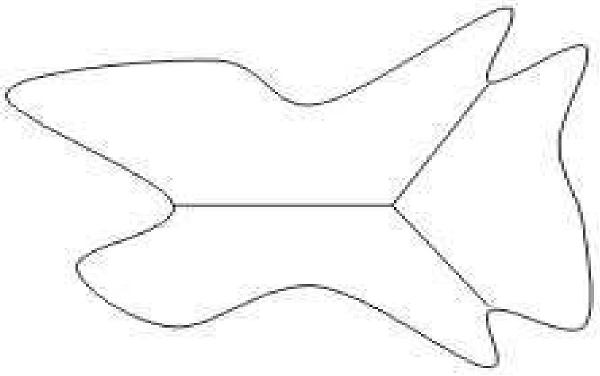

Figure 3: A single flat triple junction in a domain Ω which is non-degenerate concave at the points where the junction meets the boundary

So, if $\lambda < 0$ then $h_i^N = C_i \cosh(x\sqrt{-\lambda}) + D_i \sinh(x\sqrt{-\lambda})$, i=1,2,3.Where C_i,D_i unknown real variables.From (3.1),(3.2) and (3.4) we have:

$$C_1 + C_2 + C_3 = 0$$

$$D_1 = D_2 = D_3$$

$$C_i(K_i^{\partial\Omega} - \sqrt{-\lambda} tanh(l_i\sqrt{-\lambda}) - D_i(\sqrt{-\lambda} - K_i^{\partial\Omega} tanh(l_i\sqrt{-\lambda}) = 0 \quad , \quad i = 1,2,3$$

Since, $K_i^{\partial\Omega} - \sqrt{-\lambda} tanh(l_i\sqrt{-\lambda}) < 0$:

$$C_i = \frac{\sqrt{-\lambda} - K_i^{\partial\Omega} tanh(l_i\sqrt{-\lambda})}{K_i^{\partial\Omega} - \sqrt{-\lambda} tanh(l_i\sqrt{-\lambda})} D_i \Rightarrow$$

$$\sum_{i=1}^{3} C_i = \sum_{i=1}^{3} \frac{\sqrt{-\lambda} - K_i^{\partial\Omega} tanh(l_i\sqrt{-\lambda})}{K_i^{\partial\Omega} - \sqrt{-\lambda} tanh(l_i\sqrt{-\lambda})} D_i$$

and since $D_1 = D_2 = D_3$, $\sum_{i=1}^{3} C_i = 0$,

$$D_1 \sum_{i=1}^{3} \frac{\sqrt{-\lambda} - K_i^{\partial\Omega} tanh(l_i\sqrt{-\lambda})}{K_i^{\partial\Omega} - \sqrt{-\lambda} tanh(l_i\sqrt{-\lambda})} = 0$$

But $\sum_{i=1}^{3} \frac{\sqrt{-\lambda} - K_i^{\partial\Omega} tanh(l_i\sqrt{-\lambda})}{K_i^{\partial\Omega} - \sqrt{-\lambda} tanh(l_i\sqrt{-\lambda})} < 0$ and therefore,

$$D_1 = D_2 = D_3 = 0$$

and so,

$$[K_i^{\partial\Omega} - \sqrt{-\lambda} tanh(l_i\sqrt{-\lambda})] C_i = 0 \quad i = 1,2,3 \quad \Rightarrow$$

$$C_i = 0 \quad i = 1, 2, 3$$

because $K_i^{\partial\Omega} - \sqrt{-\lambda} tanh(l_i \sqrt{-\lambda}) < 0$.
$\forall \lambda < 0 \quad \Rightarrow C_i = D_i = 0$ for i=1,2,3.Which means that for $\lambda < 0$ we have $h_i^N = 0, \ \ i = 1, 2, 3$ and moreover that the linearized operator has no negative eigenvalues.
*Note:The linearized operator does not have the zero eigenvalue.*For λ=0 we have $\Rightarrow$ h_i^N=A_ix+B_i, i=1,2,3. Where A_i,B_i unknown real variables.From (3.2),(3.4) and (3.5) we have:

$$B_1 + B_2 + B_3 = 0$$

$$A_1 = A_2 = A_3$$

$$(K_i^{\partial\Omega} l_i - 1)A_i + K_i^{\partial\Omega} B_i = 0, \ \ i = 1, 2, 3$$

We have a 6x6 system with A_i,B_i unknown real variables. Since, $K_i^{\partial\Omega} < 0$,

$$B_i = \frac{1 - K_i^{\partial\Omega} l_i}{K_i^{\partial\Omega}} A_i$$

$$\sum_{i=1}^{3} B_i = \sum_{i=1}^{3} \frac{1 - K_i^{\partial\Omega} l_i}{K_i^{\partial\Omega}} A_i \Rightarrow$$

$$A_1 \sum_{i=1}^{3} \frac{1 - K_i^{\partial\Omega} l_i}{K_i^{\partial\Omega}} = 0$$

Because $\sum_{i=1}^{3} B_i = 0$, $A_1 = A_2 = A_3$.And while $\frac{1-K_i^{\partial\Omega} l_i}{K_i^{\partial\Omega}} < 0$,

$$A_1 = A_2 = A_3 = 0$$

and therefore,

$$B_i = 0, \ \ i = 1, 2, 3$$

That means for $\lambda = 0 \Rightarrow h_i^N = 0$ (i=1,2,3) and moreover that the linearized operator has no zero eigenvalues.

Proposition 3.3. The single flat triple junction in a domain Ω which is zero at the points where the junction meets the boundary,is neutrally stable with 0 eigenvalue with multiplicity 2.

Proof. The curvature of the boundary at the points that meets the network is flat.That means that $K_i^{\partial\Omega} = 0$, i=1,2,3.So the linearized operator will take the following form:

$$h_i''^N = -\lambda h_i^N \quad i = 1, 2, 3$$

$$h_1^N + h_2^N + h_3^N = 0 \quad at \quad x = 0$$

$$h_1'^N = h_2'^N = h_3'^N \quad at \quad x = 0$$

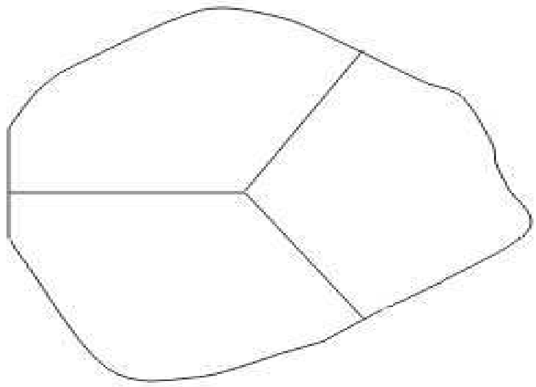

Figure 4: A single flat triple junction in a domain Ω which is zero at the points where the junction meets the boundary

$$h_i'^N = 0 \quad i = 1,2,3 \quad at \quad x = \partial\Omega$$

For $\lambda < 0$ we have h_i^N=$C_i$cosh(x$\sqrt{-\lambda}$)+$D_i$sinh(x$\sqrt{-\lambda}$) , i=1,2,3.Where C_i,D_i unknown real variables.From (3.2),(3.4) and (3.5) we get:

$$C_1 + C_2 + C_3 = 0$$

$$D_1 = D_2 = D_3$$

$$C_i tanh(l_i\sqrt{-\lambda}) + D_i = 0 \ , \ \ i = 1,2,3$$

We have a 6x6 system with C_i,D_i unknown real variables.
Since, $tanh(l_i\sqrt{-\lambda}) > 0$:

$$C_i = -\frac{1}{tanh(l_i\sqrt{-\lambda})}D_i$$

$$\sum_{i=1}^{3} C_i = \sum_{i=1}^{3} \frac{-1}{tanh(l_i\sqrt{-\lambda})}D_i \Rightarrow$$

and since $D_1 = D_2 = D_3$, $\sum_{i=1}^{3} C_i = 0$,

$$D_1 \sum_{i=1}^{3} \frac{-1}{tanh(l_i\sqrt{-\lambda})} = 0$$

But $\sum_{i=1}^{3} \frac{1}{tanh(l_i\sqrt{-\lambda})} < 0$,

$$D_1 = D_2 = D_3 = 0$$

and so,

$$tanh(l_i\surd{-\lambda})C_i = 0 \quad i = 1, 2, 3 \quad \Rightarrow$$

$$C_i = 0 \quad i = 1, 2, 3$$

So, $\forall\lambda < 0 \Rightarrow C_i = D_i = 0$ for i=1,2,3.Which means that for $\lambda < 0 \Rightarrow h_i^N = 0$ (i=1,2,3) and moreover that the linearized operator has no negative eigenvalues. *The linearized operator has the zero eigenvalue with multiplicity 2.*Moreover for λ=0 we have $\Rightarrow h_i^N = A_i x + B_i$, i=1,2,3. Where A_i,B_i unknown real variables.From (3.2),(3.4) and (3.5) we have:

$$B_1 + B_2 + B_3 = 0$$

$$A_1 = A_2 = A_3$$

$$A_i = 0 \ \ , \ \ i = 1, 2, 3$$

We have a 6x6 system with A_i,B_i unknown real variables. By solving this system we find that $A_i = 0$ i=1,2,3 and $B_1 = -B_2 - B_3$.That means that for $\lambda = 0$ we have:

$$\begin{pmatrix} h_1^N \\ h_2^N \\ h_3^N \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} B + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} C$$

and the eigenspace of the zero eigenvalue is:

$$< \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} >$$

Note that the eigenfunctions and the multiplicity of the zero eigenvalue show the two different ways that the network rotates.

4. Network of two single triple junctions

A network of one triple junction that is inside a domain on the plane contains three curves that meet at one end at the junction and at each other end with the boundary of the domain. A network of two triple junction that is inside a domain on the plane contains five curves and does not have all of its curves meeting at one end at the boundary of the domain. More specifically let Ω be a bounded and smooth domain on the plane that contains a network of two junctions.Also let Γ_i , i=1,2,3,4,5 for x$\in$[0,1] be curves contained on the network. Then Γ_3 has its one end meeting at the first junction and its other end meeting at the second junction.Also Γ_i , i=1,2 have their one end meeting at the first junction and their other meeting orthogonally at the boundary $\partial\Omega$ of the domain Ω. Finally Γ_i , i=4,5 have their one end meeting at the second junction and their other meeting orthogonally at the boundary $\partial\Omega$of the domain Ω. The angles formed by the curves at the triple junction are constant (120^o)

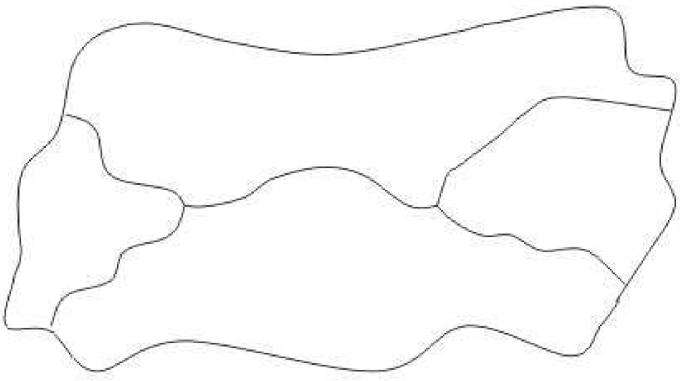

Figure 5: A network of two triple junction

throughout the evolution. The situation can be formulated mathematically as follows:

$$\Gamma_{it} = \frac{\Gamma_{ixx}}{|\Gamma_{ix}|^2}\ ,\ i = 1, 2, 3, 4, 5 \quad x \in [0, 1] \qquad \textbf{(IV)}$$

And conditions:

4.1. Incidence at the junction

$$\Gamma_1(0, t) = \Gamma_2(0, t) = \Gamma_3(0, t)$$

and

$$\Gamma_3(1, t) = \Gamma_4(0, t) = \Gamma_5(0, t)$$

4.2. Angle conditions at the junction

$$\frac{\Gamma_{1x}(0, t)}{|\Gamma_{1x}(0, t)|} \cdot \frac{\Gamma_{2x}(0, t)}{|\Gamma_{2x}(0, t)|} = cos120^o \quad , \quad \frac{\Gamma_{2x}}{|\Gamma_{2x}|} \cdot \frac{\Gamma_{3x}(0, t)}{|\Gamma_{3x}(0, t)|} = cos120^o$$

and

$$\frac{\Gamma_{3x}(1, t)}{|\Gamma_{3x}(1, t)|} \cdot \frac{\Gamma_{4x}(0, t)}{|\Gamma_{4x}(0, t)|} = cos120^o \quad , \quad \frac{\Gamma_{4x}(0, t)}{|\Gamma_{4x}(0, t)|} \cdot \frac{\Gamma_{5x}(0, t)}{|\Gamma_{5x}(0, t)|} = cos120^o$$

4.3. Incidence at $\partial\Omega$ for i=1,2,4,5 at x=1

$$b(\Gamma_i) = 0$$

4.4. Angle conditions at $\partial\Omega$ for i=1,2,4,5 at x=1

$$< \Gamma_{ix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla b(\Gamma_i) >= 0$$

And the linearized operator the (eigenvalue problem) will take the following form **(V)**:

$$h_i''^N - h_i^T k_{ix} + h_i^N k_i^2 = -\lambda h_i^N$$

i=1,2,3,4,5

$$-2h_i'^N k_i - h^N k_{ix} + h_i''^T - h_i^T k_i^2 = -\lambda h_i^T$$

Conditions:

4.5. *Incidence at the junction*

$$h_1^N(0) + h_2^N(0) + h_3^N(0) = 0$$

$$h_1^T(0) + h_2^T(0) + h_3^T(0) = 0$$

and

$$h_3^N(1) + h_4^N(0) + h_5^N(0) = 0$$

$$h_3^T(1) + h_4^T(0) + h_5^T(0) = 0$$

4.6. *Angle conditions at the junction*

$$h_1'^N(0) + h_1^T k_1(0) = h_2'^N(0) + h_2^T(0)k_2(0) = h_3'^N(0) + h_3^T(0)k_3(0)$$

and

$$h_3'^N(1) + h_3^T(1)k_3(1) = h_4'^N(0) + h_4^T(0)k_4(0) = h_5'^N(0) + h_5^T(0)k_5(0)$$

4.7. *Incidence at $\partial\Omega$* for i=1,2,4,5 at x=l

$$h_i^T = h_{i+1}^T = h_{i+2}^T = 0$$

4.8. *Angle conditions at $\partial\Omega$* for i=1,2,4,5 at x=l

$$K_{\partial\Omega}^i h_i^N = h_i'^N$$

Since we will study the stability of the steady states we will rewrite the linearized operator for $k_i = 0$.Also note that as already explained in previous section we will work again only with the functions h_i^N , i=1,2,3,4,5:

$$h_i''^N = -\lambda h_i^N$$

And conditions:

$$h_1^N(0) + h_2^N(0) + h_3^N(0) = 0$$

$$h_3^N(1) + h_4^N(0) + h_5^N(0) = 0$$

$$h_1'^N(0) = h_2'^N(0) = h_3'^N(0)$$

$$h_3'^N(1) = h_4'^N(0) = h_5'^N(0)$$

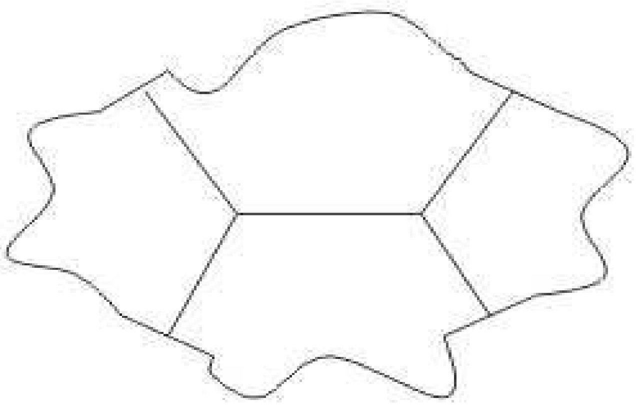

Figure 6: Two flat triple junctions in a domain Ω which is zero at the points where the junction meets the boundary

for i=1,2,4,5 at x=l

$$K^{i}_{\partial\Omega}h^{N}_{i} = h'^{N}_{i}$$

Proposition 4.1. Let Ω be a bounded and smooth domain on the plane that contains a steady state of two triple junctions. Then if the curvature of the boundary at the points that meets the steady state is zero then the steady state is neutrally stable.

Proof. (a) The curvature of the boundary at the points that meets the network is flat. That means that $K_i^{\partial\Omega} = 0$, i=1,2,3.

For $\lambda < 0$ we have h_i^N=$C_i$cosh(x$\sqrt{}-\lambda$)+$D_i$sinh(x$\sqrt{}-\lambda$)x, i=1,2,3. Where C_i,D_i unknown real variables. From (4.5),(4.6) and (4.8) we get:

$$C_1 + C_2 + C_3 = 0$$

$$C_3 cosh(l\sqrt{}-\lambda) + D_3 sinh(l\sqrt{}-\lambda) + C_4 + C_5 = 0$$

$$D_1 = D_2 = D_3$$

$$C_3 sinh(l\sqrt{}-\lambda) + D_3 cosh(l\sqrt{}-\lambda) = D_4 = D_5$$

$$C_i tanh(l\sqrt{}-\lambda) + D_i = 0 \ \ , \ \ i = 1, 2, 4, 5$$

We have a 10x10 system with C_i,D_i unknown real variables.By solving this system we find that $\forall\lambda < 0 \Rightarrow C_i = D_i = 0$ for i=1,2,3,4,5.Which means that for $\lambda < 0 \Rightarrow h_i^N = 0$ (i=1,2,3) and moreover that the linearized operator has no negative eigenvalues. *The linearized operator has the zero eigenvalue with multiplicity 2.*Moreover for λ=0 we have $\Rightarrow$ h_i^N=A_ix+B_i , i=1,2,3,4,5. Where A_i,B_i unknown real variables.From (4.5),(4.6) and (4.8) we have:

$$B_1 + B_2 + B_3 = 0$$

$$A_3 l + B_3 + B_4 + B_5 = 0$$

$$A_1 = A_2 = A_3$$

$$A_3 = A_4 = A_5$$

$$A_i = 0 \ \ , \ \ i = 1, 2, 4, 5$$

We have a 10x10 system with A_i,B_i unknown real variables. By solving this system we find that $A_i = 0$ i=1,2,3,$B_1 = -B_2 - B_3$ and $B_4 = -B_3 - B_5 B_3 = B_4 - B_5$.That means that for $\lambda = 0$ we have:

$$\begin{pmatrix} h_1^N \\ h_2^N \\ h_3^N \\ h_4^N \\ h_5^N \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} A + \begin{pmatrix} -1 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} B + \begin{pmatrix} -0 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} C$$

and the eigenspace of the zero eigenvalue is:

$$< \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} >$$

Note that the eigenfunctions and the multiplicity of the zero eigenvalue shows the two different ways that the network rotates.

Proposition 4.2. Let Ω be a strictly convex domain on the plane that contains a steady state of two triple junctions. Then the steady state is unstable.

Proof. Ω is a strictly convex domain.That means $K_i^{\partial\Omega} > 0$, i=1,2,3,4,5.For $\lambda < 0$ we have h_i^N=$C_i(\cosh\sqrt{-\lambda})x$+$D_i(\sinh\sqrt{-\lambda})x$, i=1,2,3.Where C_i,D_i unknown real variables.From (4.5),(4.6) and (4.8) we get:

$$C_1 + C_2 + C_3 = 0$$

$$C_3 cosh(l\sqrt{-\lambda}) + D_3 sinh(l\sqrt{-\lambda}) + C_4 + C_5 = 0$$

$$D_1 = D_2 = D_3$$

$$C_3 sinh(l\sqrt{-\lambda}) + D_3 cosh(l\sqrt{-\lambda}) = D_4 = D_5$$

$$C_i(K_i^{\partial\Omega} - \sqrt{-\lambda} tanh\sqrt{-\lambda}) - D_i(\sqrt{-\lambda} - K_i^{\partial\Omega} tanh\sqrt{-\lambda}) = 0 \ \ , \ \ i = 1, 2, 4, 5$$

We have a 10x10 system with C_i,D_i unknown real variables.By solving this system we find that for $\lambda < 0$ we have non-zero solutions.Thus the linearized operator has negative eigenvalues and the steady state is.

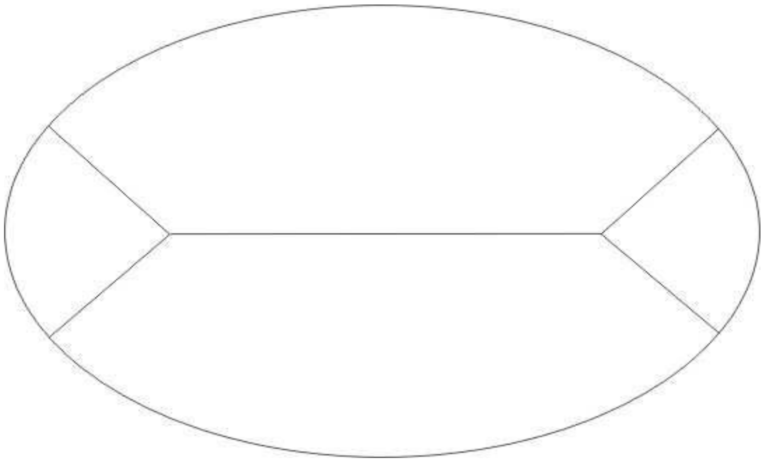

Figure 7: A strictly convex domain on the plane that contains a steady state of two triple junctions

Proposition 4.3. Let Ω be a bounded and smooth domain on the plane that contains a steady state of two triple junctions. Then if the curvature of the boundary at the points that meets the steady state is concave then the steady state is stable.

Proof. The curvature of the boundary at the points that meets the steady state is concave. That means $K_i^{\partial\Omega} < 0$, i=1,2,3,4,5.For $\lambda < 0$ we have $h_i^N = C_i cosh(x\sqrt{-\lambda}) + D_i sinh(x\sqrt{-\lambda})$, $i = 1, 2, 3$. Where C_i,D_i unknown real variables. From (4.5),(4.6) and (4.8) we get:

$$C_1 + C_2 + C_3 = 0$$

$$C_3 cosh(l\sqrt{-\lambda}) + D_3 sinh(l\sqrt{-\lambda}) + C_4 + C_5 = 0$$

$$D_1 = D_2 = D_3$$

$$C_3 sinh(l\sqrt{-\lambda}) + D_3 cosh(l\sqrt{-\lambda}) = D_4 = D_5$$

$$C_i(K_i^{\partial\Omega} - \sqrt{-\lambda} tanh\sqrt{-\lambda}) - D_i(\sqrt{-\lambda} - K_i^{\partial\Omega} tanh\sqrt{-\lambda}) = 0 \ , \ \ i = 1, 2, 4, 5$$

Again we have a 10x10 system with C_i,D_i unknown real variables.By solving this system we find that $\forall\lambda < 0 \Rightarrow C_i = D_i = 0$ for i=1,2,3,4,5.Which means that for $\lambda < 0 \Rightarrow h_i^N = 0$ (i=1,2,3,4,5) and moreover that the linearized operator has no negative eigenvalues.

The linearized operator has neither the zero eigenvalue,because for λ=0 we have $\Rightarrow$ h_i^N=A_ix+B_i , i=1,2,3,4,5 and from (3.2),(3.4) and (3.5) we have:

$$B_1 + B_2 + B_3 = 0$$

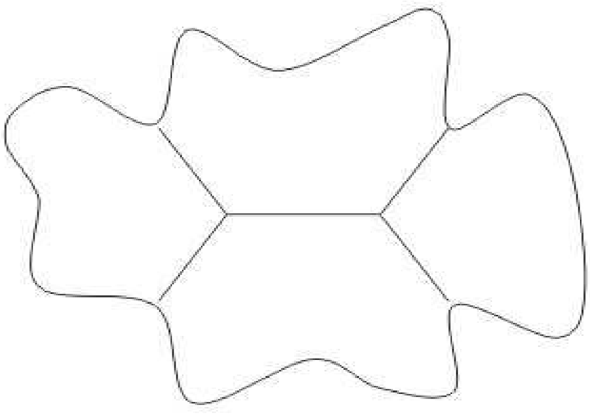

Figure 8: The curvature of the boundary at the points that meets the steady state is concave

$$A_3 l + B_3 + B_4 + B_5 = 0$$

$$A_1 = A_2 = A_3$$

$$A_3 = A_4 = A_5$$

$$A_i(1 - lK_i^{\partial\Omega}) - K_i^{\partial\Omega} B_i = 0 \ \ , \ \ i = 1, 2, 4, 5$$

We have a 10x10 system with A_i,B_i unknown real variables.By solving this system we find that $A_i = B_i = 0$ which means that for $\lambda = 0 \Rightarrow h_i^N = 0$ (i=1,2,3) and moreover that the linearized operator has no zero eigenvalues.

Remark. After the comletion of this work we found that E.Yanagida and R.Ikota have obtained the linearized equations.(A stability criterion for stationary curves to the curvature driven-motion with a triple junction. Differential and integral equations. 16-6.707-726/2003).

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