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The Greatest Common Divisor of a set of Polynomials, Control Theory and Approximate Algebraic Computations

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Abstract

The theory of algebraic and geometric invariants in Linear Systems is instrumental in describing system properties and it is linked to solvability of fundamental Control Theory problems. The computation of the Greatest Common Divisor (GCD) is central to the development of algebraic synthesis method and their robust computation is a key issue. Engineering models are not exact and they are always characterised by parameter uncertainty; thus, there is need to develop appropriate algebraic computations, on models characterised by parameter uncertainty and this requires the transformation of algebraic concepts to an equivalent analytic set up. Invariance properties of the GCD and Control Theory based concepts and tools are considered which enable the development of novel methodologies for GCD computation and for defining approximate solutions. The paper provides an overview of such methodologies leading to new GCD algorithms and focuses on the definition of "approximate notions" for the GCD. A distinguishing feature of the current approach for the computation of "approximate" solutions is their reduction to the computation of distance between points of a projective space and certain algebraic varieties. The study of such problems provides the means for computing "optimal approximate" solutions, as well as evaluating the strength of ad-hoc approximations derived from different algorithms.

Keywords: Greatest Common Divisor of Polynomials, Control Theory, Computations, Approximate Solutions

1. Introduction

The theory of algebraic and geometric invariants in Linear Systems is instrumental in describing system properties and it is linked to solvability of fundamental Control Theory problems [23], [9]. These invariants are defined on rational, polynomial matrices and matrix pencils under different transformation groups (coordinate, compensation, feedback type) and their computation relies on algebraic algorithms, whereas symbolic tools are used for their implementation. Engineering models are not exact and they are always characterised by parameter uncertainty. This introduces some considerable problems with any framework based on exact symbolic tools, given that the underlined models are always characterised by parameter uncertainty. The central challenge is the transformation of algebraic notions to an appropriate analytic setup within which meaningful "approximate" solutions to exact algebraic problems may be sought. This motivates the need for transforming the algebraic problems into equivalent linear algebra problems and then develop approximate algebraic computations, which are appropriate for the case of computations on models characterised by parameter uncertainty. A number of important invariants for Linear Systems rely on the notion of Greatest Common Divisor (GCD) of many polynomials. The link between Control Theory and the GCD problem is very strong; in fact, the GCD is instrumental in defining system notions such as zeros, decoupling zeros, zeros at infinity, notions of Minimality of system representations etc. On the other hand, Systems and Control Methods provide concepts and tools which enable the development of new computational procedures for GCD. This paper focuses on the links between GCD and Control Theory and the definition of appropriate "approximate" solutions for GCD [19], [21], [16], [15] which are meaningful in the setup of uncertain models. The numerical analysis and computational issues are not treated here, but they are dealt with in the cited references.

The existence of certain types and/or values of invariants and system properties may be classified as generic or nongeneric [7], [23], [17] on a family of linear models. Computing, or evaluating nongeneric types, or values of invariants and thus associated system properties on models with numerical inaccuracies is crucial for applications. For such cases, symbolic tools fail, since "almost always" lead to a generic solution, which does not represent the "approximate presence" of the value property on the set of models under considerations. The formulation of a methodology for robust computation of nongeneric algebraic invariants, or nongeneric values of generic ones, has as prerequisites: **(a)** The development of a numerical linear algebra characterisation of the invariants, which may allow the measurement of degree of presence of the property on every point of the parameter set. **(b)** The development of special numerical tools, which avoid the introduction of additional errors. **(c)** The formulation of appropriate criteria which allow the termination of algorithms at certain steps and the definition of meaningful approximate solutions to the algebraic computation problem. It is clear that the formulation of the algebraic problem as an equivalent numerical linear algebra problem, is essential in transforming concepts of algebraic nature to equivalent concepts of analytic character and thus setup up the right framework for approximations.

The GCD related work described in this paper goes back to the attempt to introduce the notion of almost zero of a set of polynomials [10] and study the properties of such zeros from the feedback viewpoint. This work was subsequently developed to a methodology for computing the approximate gcd of polynomials using numerical linear algebra methods, such as the ERES [19] and matrix pencil methods [12]. In this paper the definition of the "Approximate GCD" is considered as a distance problem in a projective space. This formulation is central in the development of a meaningful algebraic systems framework for models characterised by parameter uncertainty and their solution is linked to a large range of related problems such as: **(i)** Almost non-coprimeness and solutions of polynomial Diofantine Equations**. (ii)** Characterisation of Almost uncontrollability and Almost unobservability**. (iii)** Approximate factorisation of rational transfer function models. The new distance framework given for the "approximate GCD" provides the means for computing optimal solutions, as well as evaluating the strength of ad-hoc approximations derived from different algorithms.

Throughout the paper, $\mathbb{R}^{m\times n}$ denotes the set of all $m\times n$ real matrices and $\mathbb{R}[s]$ denotes the ring of real polynomials. If $A\in\mathbb{R}^{m\times n}$, then $\rho(A)$ denotes its rank, $n_r(A)$ denotes it right nullity and $N_r(A), N_l(A)$ denote its right, left nullspace. Capital letters denote matrices and small underlined letters denote vectors. $C_r(A)$ denotes the r-th order compound matrix of A [18]. $\partial[a(s)]$ denotes the degree of a given polynomial $a(s)$. Throughout the paper if a a property is said to be true for $i\in \underset{\sim}{k}, k\in\mathbb{Z}^+$, this means it is true for all $1\leq i\leq k$. The proofs of the results are given in the references.

2. Models with numerical inaccuracies and classification of algebraic computation problems

The development of robust computational procedures for engineering type models always has to take into account that the models have certain accuracy and that it is meaningless to continue computations beyond the accuracy of the original data set. In fact, engineering computations are defined not on a single model of a system S, but on a ball of system models $\Sigma(S_0, r(\varepsilon))$, where S_0 is a nominal system and $r(\varepsilon)$ is some radius defined by the data error order ε. The result of computations has thus to be representative for the family $\Sigma(S_0, r(\varepsilon))$ and not just the particular element of this family. From this viewpoint, symbolic computations carried out on an element of the $\Sigma(S_0, r(\varepsilon))$ family may lead to results, which do reveal the desired properties of the family. Numerical computations have to stop, when we reach the original data accuracy and an approximate solution to the computational task has to be given. To motivate the significance of the approximate notions we consider first a simple example.

Example 2.1 [10] Consider the polynomials $t_1(s) = s + 1.1$ and $t_2(s) = s\ (s + 1)$. Clearly, the polynomials are coprime and any symbolic procedure for the computation of GCD will lead to this result. It is worth pointing out that the two polynomials have roots $-1,\ -1.1$, which are very close to each other and thus the notion of an "approximate common zero" emerges. For such a notion to make sense, there must be a continuity with some important properties of the exact zero. A key property of the common zero z of two polynomials $t_1(s), t_2(s)$ is that any combination $k_1t_1(s)+k_2t_2(s)$

for all $k_1, k_2 \in \mathbb{R}$ will have also z as a common zero. Thus exact zeros correspond to fixed zeros of the polynomial combinants of $(t_1(s), t_2(s))$. For our example the zeros of $k_1(s+1.1) + k_2 s(s+1)$ are defined by the root locus diagram of the single input single output system with transfer function g(s) and root locus diagram defined below:

$$g(s) = \frac{k(s+1.1)}{s(s+1)}, \;\; k = k_1/k_2$$

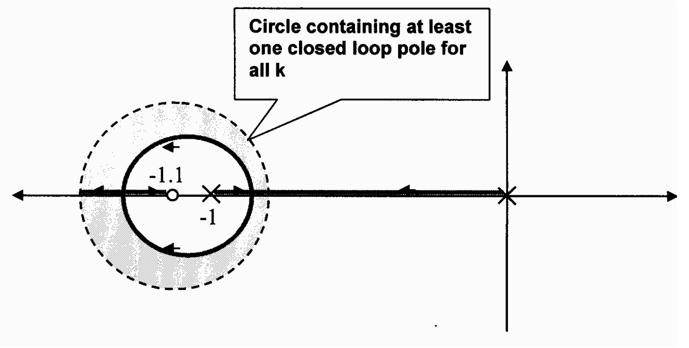

Figure 1: Root Locus Diagram

and this reveals the presence of circle that contains at least one of the zeros of the combinant for all values of k. The fixed point interpretation of the exact zero, now becomes a circle with the same property, as far as distribution of zeros of the corresponding combinants; the size of disc depends on the proximity of the zeros of the two polynomials. The continuity of properties between the exact and the approximate case (root distribution of combinants) allows us to say that an "almost zero" exists in the neighborhood of -1.

This notion of "almost zero", is not an algebraic property any more, but an analytic and it depends on distance between root sets of polynomials. The above example demonstrates that the fundamental notion of common divisor may be extended to an "approximate" sense. Although the existence of GCD of polynomials is nongeneric property [7], [17], [23], "almost common divisors" may be introduced under certain conditions.

To make the idea of genericity precise [7], [23], we borrow some terminology from algebraic geometry. Consider polynomials $\phi(\lambda_1, \ldots, \lambda_n)$ with coefficients in $\mathbb{R}$. A variety $V \subset \mathbb{R}^N$ is defined to be the locus of common zeros of a finite number of polynomials $\phi_1, \ldots, \phi_k$:

$$V = \{P \in \mathbb{R}^N : \phi_i(P_1, \ldots, P_N) = 0, \; i \in \underset{\sim}{k}\}$$

For example one can prove that the set of all (A, B, C, D) of fixed dimensions modulo coordinate state transformations is a variety. A property Π on V is merely a function $\Pi : V \to \{0, 1\}$, where $\Pi(P) = 1$(or 0) means Π holds (or fails) at P. Let V be a proper variety, we shall say that Π is *generic relative* to V provided $\Pi(P) = 0$ only for points $P \in V' \subset V$ where V' is a proper subvariety of V; and that Π is *generic* provided such a V' exists. If Π is generic, we sometimes write $\Pi = 1(g)$. As V' is a locus of zeros of polynomials in V, the subset of V such that the property is not true is a negligible set (measure zero).

Definition 2.1 Numerical computations dealing with the derivation of an approximate value of a property, function, which is nongeneric on a given model set, will be referred as *nongeneric computations* (NGC). If the value of a function always exists on every element of the model set and depends continuously on the model parameters, then the computations leading to the determination of such values will be called *normal numerical* (NNC). Computational procedures aiming at defining the generic value of a property, function on a given model set will be called *generic* (GC).

On a set of polynomials with coefficients taking values from a certain parameter set, the existence of GCD in nongeneric; numerical procedures that aim to produce an approximate nontrivial value by exploring the numerical properties of the parameter set are typical examples of NG computations and two approximate GCD procedures will be considered subsequently. NG computations refer to both continuous and discrete type system invariants. On the other hand, the eigenvalues of a square matrix, or the zeros of a square polynomial matrix are always defined on any model set and their numerical values continuously depend on the numerical values of the parameter set. For unstructured model sets the computation of the generic value of discrete invariants follows from the dimensionality of the matrices involved and the genericity arguments [17]. For model sets characterised by some underlying graph structure, the computation of the generic values can benefit by exploiting the genericity argument. The computation of discrete invariants such as McMillan degree, order of infinite zeros are problems within the class of generic computations; such problems have been recently considered in [25] and are referred to as problems of structural identification. We consider here the fundamental problem of approximate algebraic computations, and in particular the problem of defining the approximate GCD [19], [21], [16], [15] which is central in the development of algebraic control synthesis methodologies.

3. Matrix Based GCD Methods

The problem of finding the greatest common divisor (GCD) of a polynomial set has been a subject of interest for a very long time and has widespread applications. Since the existence of a nontrivial common divisor of polynomials is a property that holds for specific sets, extra care is needed in the development of efficient numerical algorithms calculating correctly the required GCD. Several numerical methods for the computation of the GCD of a set $P_{m,d}$, of m polynomials of $\mathbb{R}[s]$ of maximal degree d, have been proposed, see ([1], [2], [12], [19] etc). These methods can be classified as: **(i)** Numerical methods based on Euclid's algorithm and its generalizations. **(ii)**

Numerical methods based on procedures involving matrices (matrix - based). The matrix - based methods usually perform specific transformations to a matrix formed directly from the coefficients of the given polynomials. Two of the matrix based methods leading to approximate solutions are considered in the following.

3.1. *The ERES method*

The ERES method is an iterative matrix - based method developed in [19], [20] and is based on the properties of the GCD as an invariant of $P_{m,d}$ under extended - row - equivalence and shifting (ERES) operations [11]. The principles behind the development of this method are considered next.

Let $P_{m,d} = \{p_i(s) : p_i(s) \in \mathbb{R}[s]\,,\ i = 1, \ldots, m,\ d_i = deg\{p_i(s)\},\ d = max\{d_i, i = 1, \ldots, m\}\}$ be the set of m polynomials of $\mathbb{R}[s]$ of maximal degree d. For any $P_{m,d}$ set we define a vector representative $p_m(s)$ and a basis matrix P_m for $e_d(s) = [1, s, \ldots, s^d]^t$ vector by $p_m(s) = [p_1(s), \ldots, p_m(s)]^t = [p_0, \ldots, p_d]\, e_d(s) = = P_m\, e_d(s)$where $P_m \in \mathbb{R}^{m\times(d+1)}$ and we denote by GCD $\{P_{m,d}\} = \phi(s)$ the GCD of the set. If c is the integer for which $p_0 = \ldots = p_{c-1} = 0$, $p_c \neq 0$, then $c = w(P_{m,d})$ is called the order of $P_{m,d}$ and s^c is an elementary divisor of the GCD. $P_{m,d}$ will be called proper if $c = 0$, and nonproper if $c \geq 1$.

In the ERES (Extended - Row - Equivalence and Shifting) method, for a given $P_{m,d}$ with a basis matrix P_m the following operations are defined:

(i) Elementary row operations with scalars from $\mathbb{R}$ on P_m.

(ii) Addition or elimination of zero rows on P_m.

(iii) If $a^t = [0, \ldots, 0, a_\varepsilon, \ldots, a_{d+1}] \in \mathbb{R}^{1\times(d+1)}$, $a_\varepsilon \neq 0$ is a row of P_m then we define the shifting operation

$$shf : shf(a^t) = (a^*)^t = [a_\varepsilon, \ldots, a_{d+1}, 0, \ldots, 0].$$

By $shf(P_{m,d}) = P^*_{m,d}$, we shall denote the set obtained from $P_{m,d}$ by applying shifting on every row of P_m. Type **(i), (ii), (iii)** operations are referred to as Extended - Row - Equivalence and Shifting (ERES) operations. The following theorem describes the properties characterising the GCD of any given $P_{m,d}$ [11]. If the set $P_{m,d}$ is nonproper with $w(P_{m,d}) = c$, then $GCD\{P_{m,d}\} = s^c \cdot GCD\{P_{m,d'}\}$.

Theorem 3.1 *For any set $P_{m,d}$, with a basis matrix P_m, $\rho(P_m) = r$ and $GCD\{P_{m,d}\} = \phi(s)$ we have the following properties:*

(i) If P is the row space of P_m, then $\phi(s)$ is an invariant of P (e.g. $\phi(s)$ remains invariant after the execution of elementary row operations on P_m). Furthermore if r=dim(P) $= d + 1$, then $\phi(s) = 1$.

(ii) If $w(P_{m,d}) = c \geq 1$, $shf(P_{m,d}) = P^*_{m,d}$, then $\phi(s) = GCD\{P_{m,d}\} = s^c \cdot GCD\{P^*_{m,d}\}$

(iii) If $P_{m,d}$ is proper, then $\phi(s)$ is invariant under the combined ERES set of operations.

From Theorem (3.1) it is evident that ERES operations preserve the GCD of any $P_{m,d}$ and thus can be easily applied in order to obtain a modified basis matrix with much simpler structure. After successive applications of ERES operations on an initial basis matrix, the maximal degree of the resulting set of polynomials is reduced and after a finite number of steps the resulting basis matrix has rank one. In that stage, any row of the matrix specifies the coefficients of the required GCD of the set. A useful criterion deciding when the iterative process applying ERES operations on a specific basis matrix, will be terminated is given in [19]:

Lemma 3.1 *Let $A = [r_1, r_2, \ldots, r_m]^t \in \mathbb{R}^{m\times n}$, $r_1 \neq 0$, $i = 1, 2, \ldots, m$. Then $\rho(A) = 1$, if and only if, the singular values $\sigma_m \leq \sigma_{m-1} \leq \ldots \leq \sigma_1$ of the normalization $A_N = [v_1, v_2, \ldots, v_m]^t \in \mathbb{R}^{m\times n}$, $v_i = r_i \,/\, \|\, r_i \,\|_2$, of A satisfy the conditions: $\sigma_1 = \sqrt{m}$, $\sigma_i = 0$, $i = 2, 3, \ldots, m$.*

The development of an effective numerical algorithm for the ERES method involves the following numerical problems: **(i)** Use of the most reliable and stable numerical method performing ERES operations is the method of Gaussian elimination with partial pivoting. **(ii)** The successive implementation of Gaussian elimination and shifting will be terminated when the resulting matrix has numerical rank one. **(iii)** Selection of the representative row in rank one matrix leading to the definition of the coefficients of the GCD. The numerical aspects for the above procedures, algorithms and relevant results may be found in the references [19], [20].

3.2. *GCD Method Based on Matrix Pencils*

The computation of GCD may be alternatively achieved by using concepts from systems theory, which lead to an alternative formulation using matrix pencils. The process involves the transformation of GCD computation to a linear systems problem as shown below [12]: For any set $P_{m,d}$ we may define an associated linear system such that the GCD of the set is the output decoupling zero polynomial [12] of the system.

Theorem 3.2 *Let $P_{m,d} \in \{P_d\}, P_m$ be a basis matrix, $\rho(P_m) = r < d+1, M \in \mathbb{R}^{(d+1)\times\mu}$ $, \mu = d - r + 1$ be a basis matrix for $N_r(P_m) = M$ and $M_1 \in \mathbb{R}^{d\times\mu}$ be the submatrix of M obtained by deleting the last row of M. If $p(s) \in P_{m,d}$ is any monic polynomials of degree $d, \widehat{A} \in \mathbb{R}^{d\times d}$ is the associated companion matrix and $\widehat{C} \in \mathbb{R}^{(r-1)\times d}$, $\rho(\widehat{C}) = r-1$ is such that $\widehat{C}M_1 = 0$, then the unobservable modes of the system: $S(\widehat{A}, \widehat{C})$: $\dot{x} = \widehat{A}x$, $y = \widehat{C}x$ with multiplicities included define the roots of the GCD of $P_{m,d}$.*

$S(\widehat{A}, \widehat{C})$ will be called the *associated system* of $P_{m,d}$ and the observability matrix [9], [23]:

$$Q(\widehat{A}, \widehat{C}) = [\widehat{C}^t, \widehat{A}^t\, \widehat{C}, ..., (\widehat{A}^t)^{d-1}\, \widehat{C}^t]^t \in \mathbb{R}^{d(r-1)\times d}$$

will be referred to as a *reduced resultant* of $P_{m,d}$. From the above we have:

Remark 3.1 If $N_r\{P_m\} = \{0\}$, then the set $P_{m,d}$ is coprime. If $\rho(P_m) = 1$, then $\mu = d$ and any polynomial in $P_{m,d}$ defines the GCD.

Remark 3.2 With any set $P_{m,d}$ there is a family of associated systems $S(\widehat{A}, \widehat{C})$. Furthermore, if any $S(\widehat{A}, \widehat{C})$ is observable, then $P_{m,d}$ is coprime.

Corollary 3.1 *[12] Let $P_{m,d} \in \{P_d\}, \rho(P_m) = r < d + 1, S(\widehat{A}, \widehat{C})$ be the associated system of $P_{m,d}$ and let $\widehat{P}_{r-1,d'}$ be the set of $r-1$ polynomials of degree $d' \leq d-1$ defined by $\widehat{P}_{r-1}(s) = \widehat{C}e_{d-1}(s)$. Then the sets $P_{m,d}$ and $\widehat{P}_{r-1,d'}$ have the same GCD.*

The set $\widehat{P}_{r-1,d'}$ defined above is equivalent to $P_{m,d}$ as far as the GCD and there are clear advantages in deploying $\widehat{P}_{r-1,d'}$ instead of $P_{m,d}$ for computing the GCD. Successive application of the above result leads to equivalent sets of smaller or equal number of elements and degree. Using basic properties from systems and the above results we derive a new characterisation of GCD. Let $P_{m,d} \in \{P_d\},\ \rho(P_m) = r < d+1$ and let $Q(\widehat{A}, \widehat{C})$ be the corresponding reduced resultant. Let $\rho\{Q(\widehat{A}, \widehat{C}) < d$ (when is equal to d it can be proved that the set is coprime) and $\mathcal{W} \equiv \mathcal{N}_r\{Q(\widehat{A}, \widehat{C})\} \neq \{0\},\ k = dim\{\mathcal{W}\}$. The pencil $T(s) = sW - \widehat{A}W$ characterises the set $P_{m,d}$ and it is called the *associated pencil* of the set. The following result forms a basis for the numerical computation of the GCD.

Corollary 3.2 *[12] Let $T(s) = sW - \widehat{A}W \in \mathbb{R}^{d\times k}[s]$ be the associated pencil of $P_{m,d}$. If $\phi(s)$ is the GCD of $P_{m,d}$, then $C_k(T(s)) = \phi(s) \cdot C_k(W)$.*

Remark 3.3 Let $T(s) = sW - \widehat{A}W = sW - \widetilde{W}$ and let $sW_\alpha - \widetilde{W}_\alpha\ \ \alpha \in Q_{k,d}$, be any minor of maximal order such that $|W_\alpha| \neq 0$. Then, the GCD of $P_{m,d}$ is defined by $|sW_\alpha - \widetilde{W}_\alpha|$.

In order to derive an effective numerical algorithm for the MP method based on the above theoretical results we have to resolve the following numerical problems: Numerical interpretation of the notion of nullity, Numerical computation of right and left null spaces of matrices, Computation of compound matrices. The numerical algorithm of the method and its analysis can be found in [12].

4. Resultants and the Notion of Approximate GCD

4.1. *Definitions and Basic Properties*

The classical approaches for the study of coprimeness and determination of the GCD make use of the Sylvester Resultant which in the case of many polynomials is defined as shown below [1], [24].

Definition 4.1 For the set $P_{h+1,n} = \{a(s), b_i(s), i \in h, n = \deg\{a(s)\} \geq \deg\{b_i(s)\},$ $\forall i = 1,..,h,\ p = \max\{\deg\{b_i(s)\}, i = 1,\ldots,h\}\}$, where $a(s), b(s)$ are described as:

$$a(s) = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0, \quad b_i(s) = b_{i,p}s^p + ... + b_{i,1}s + b_{i,0}, \quad i = 1, 2, ...h \tag{1}$$

(i) We can define a $p \times (n+p)$ matrix associated with $a(s)$:

$$S_0 = \begin{bmatrix} 1 & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & 1 & a_{n-1} & \cdots & a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & & \ddots & & & & & & \vdots \\ 0 & 0 & \cdots & 1 & a_{n-1} & \cdots & \cdots & a_1 & a_0 \end{bmatrix} \tag{2}$$

(ii) and an $n \times (n+p)$ matrix associated with $b_i(s)$:

$$S_0 = \begin{bmatrix} b_{i,p} & b_{i,p-1} & b_{i,p-2} & \cdots & b_{i,1} & b_{i,0} & 0 & \cdots & 0 \\ 0 & b_{i,p} & b_{i,p-1} & \cdots & b_{i,2} & b_{i,1} & b_{i,0} & \cdots & 0 \\ \vdots & & \ddots & & & & & & \vdots \\ 0 & \cdots & 0 & b_{i,p} & b_{i,p-1} & \cdots & \cdots & b_{i,1} & b_{i,0} \end{bmatrix} \tag{3}$$

(iii) for each $i = 1, 2, \cdots, h$. An *extended Sylvester matrix* for the set P is then defined by:

$$S_P = \begin{bmatrix} S_0 \\ S_1 \\ \vdots \\ S_h \end{bmatrix} \in \mathbb{R}^{(p+hn)\times(n+p)} \tag{4}$$

(iv) The matrix S_P is the basis matrix of the set of polynomials

$$S\left[P\right] = \left\{a\left(s\right), sa\left(s\right), ..., s^{p-1}a\left(s\right); b_1\left(s\right), ..., b_h\left(s\right), sb_h\left(s\right), ..., s^{n-1}b_h\left(s\right)\right\} \tag{5}$$

which is also referred to as the *Sylvester Resultant set* of the given set P.

Proposition 4.1 *The GCD of P is the same as the GCD of $S\left[P\right]$, that is*

$$\gcd\left\{P\right\} = \gcd\left\{S\left[P\right]\right\} \tag{6}$$

The classical result is usually given for two polynomials [1] and this is also extended to the case of many polynomials [24].

Theorem 4.1 *(Generalised Resultant Theorem) [1], [3]: Given a set of polynomials*

$$P = \{a(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0,$$

$$b(s) = b_{i,n}s^n + \ldots + b_{i,1}s + b_{i,0}, i = 1, 2, \ldots, h\}$$

$\max\left\{\deg b_i\left(s\right)\right\} = p$ *with a generalised resultant S_P the following properties hold true:*

(i) Necessary and sufficient condition for a set of polynomials to be coprime is that:

$$\rho\left(S_P\right) = n + p \tag{7}$$

(ii) Let $\phi(s)$ be the g.c.d. of P. Then:

$$\rho\left(S_P\right) = n + p - \deg\varphi\left(s\right) \tag{8}$$

(iii) If we reduce S_P, by using elementary row operations, to its row echelon form, the last non vanishing row defines the coefficients of the g.c.d.

Certain properties of extraction of divisors from the set P, are equivalently expressed as factorisation of resultant matrices. This leads to establishing a link between factorisation of resultants and a matrix representation of the GCD [1]. The new representation of the GCD provides the means for establishing a new characterisation of the "Best Approximate GCD of many polynomials".

4.2. *Matrix Representation of GCD*

The following result provides a representation in matrix terms of the standard factorisation of the GCD of a set of polynomials.

Lemma 4.1 *[3] Let $\lambda\left(s\right) = \lambda_k s^k + ... + \lambda_1 s + \lambda_0$ be a polynomial and let $\hat{\Phi} \in \{T_n\}$, $k < n$, be a special Toeplitz matrix representation of $\lambda\left(s\right)$ defined by*

$$\hat{\Phi}_\lambda^n = \hat{\Phi} = \begin{bmatrix} \lambda_0 & 0 & \cdots & & & & & 0 \\ \lambda_1 & \lambda_0 & 0 & \cdots & & & & 0 \\ \lambda_2 & \lambda_1 & \lambda_0 & & & & & \\ \vdots & \vdots & & \ddots & & & & \\ \lambda_k & & & & \lambda_0 & 0 & & 0 \\ 0 & \ddots & & & & \ddots & \ddots & \vdots \\ \vdots & & & & & \lambda_1 & \lambda_0 & 0 \\ 0 & \cdots & 0 & \lambda_k & \cdots & \lambda_2 & \lambda_1 & \lambda_0 \end{bmatrix} \tag{9}$$

Then the inverse $\Phi = \hat{\Phi}^{-1}$ has the Toeplitz form:

$$\Phi = \begin{bmatrix} y_0 & 0 & \cdots & & & & & 0 \\ y_1 & y_0 & \ddots & & & & & \vdots \\ y_2 & y_1 & \ddots & & & & & \\ \vdots & \vdots & \ddots & y_0 & 0 & & & \\ & & & y_1 & y_0 & & & \\ & & & \vdots & \vdots & \ddots & & \\ y_{n-2} & y_{n-3} & \cdots & y_{n-j-2} & y_{n-j-3} & \cdots & y_0 & 0 \\ y_{n-1} & y_{n-2} & \cdots & y_{n-j-1} & y_{n-j-2} & \cdots & y_1 & y_0 \end{bmatrix} \tag{10}$$

where the y_i parameters satisfy the relationships

$$y_0 = \frac{1}{\lambda_0}\ ,\ y_1 = -\frac{\lambda_1}{\lambda_0} y_0\ ,\ \ldots\ ,\ y_j = -\frac{1}{\lambda_0} \sum_{i=1}^{\min\{j,k\}} \lambda_i y_{j-i}\ ,\ j = 2, \ldots, n-1 \tag{11}$$

Theorem 4.2 *[3] Let $P = \{a(s)\, b_1(s), \ldots, b_h(s)\}$ be a 0-order set, $\deg a(s) = n$, $\deg b_i(s) \le p \le n$, $i = 1, \ldots, h$ be a polynomial set, S_P the respective Sylvester matrix, $\varphi(s) = \lambda_k s^k + \cdots + \lambda_1 s + \lambda_0$ be the greatest common divisor of the set and let k be its degree. Then there exists transformation matrix $\Phi_\varphi \in \mathbb{R}^{(n+p)\times(n+p)}$ such that:*

$$\bar{S}_{P^*}^{(k)} = S_P \Phi_\varphi = \begin{bmatrix} 0_k & \bar{S}_{P^*} \end{bmatrix}, \tag{12}$$

where $\bar{S}_{P^*}^{(k)}$, Φ_φ are given by:

$$\Phi_\varphi = \begin{bmatrix} y_0 & 0 & \cdots & & & & \cdots & 0 \\ y_1 & y_0 & & & & & & \vdots \\ y_2 & y_1 & \ddots & \ddots & & & & \\ \vdots & \vdots & \ddots & y_0 & 0 & & & \\ & & & y_1 & y_0 & \ddots & & \\ & & & \vdots & \vdots & \ddots & & \\ y_{n+p-2} & y_{n+p-3} & \cdots & y_{n+p-j-2} & y_{n+p-j-3} & \cdots & y_0 & 0 \\ y_{n+p-1} & y_{n+p-2} & \cdots & y_{n+p-j-1} & y_{n+p-j-2} & \cdots & y_1 & y_0 \end{bmatrix} \tag{13}$$

where

$$y_0 = \frac{1}{\lambda_0}\ ,\ y_1 = -\frac{\lambda_1}{\lambda_0} y_0\ ,\ \ldots\ ,\ y_j = -\frac{1}{\lambda_0} \sum_{i=1}^{\min\{j,k\}} \lambda_i y_{j-i}\ , j = 2, \ldots, n+p-1 \tag{14}$$

where $\left[a_{p-k}^{(k)}, a_{p-k-1}^{(k)}, \ldots, a_0^{(k)}\right]$, $\left[b_{j,p-k}^{(k)}, b_{j,p-k-1}^{(k)}, \ldots, b_{j,0}^{(k)}\right], j = 1, \ldots, h$ are the coefficients of the coprime polynomials obtained from the original set after the division by the GCD, which define the set $P_{h+1,n-k}^*$ and $\bar{S}_{P^*}$ is the corresponding expanded resultant.

Corollary 4.1 *[3] Let $P = \{a(s), b_1(s), \ldots, b_h(s)\}$ be a 0-order set of polynomials, $\deg a(s) = n$, $\deg b_i(s) \le p \le n$, $i = 1, \ldots, h$ and let $\varphi(s)$ be the GCD of P, $\deg \varphi(s) = k$. If*

$$a(s) = a'(s)\, \varphi(s), \quad b_i(s) = b'_i(s)\, \varphi(s),\ i = 1, \ldots h \tag{15}$$

and $P^* = \{a'(s), b'_1(s), \ldots, b'_h(s)\}$, $\deg a'(s) = n-k$ $\deg b'_i(s) \le p-k$, $i = 1, \ldots h$ and S_P, $\bar{S}_{P^*}^{(k)}$ are the generalised resultants of P, P^*, where $\bar{S}_{P^*}^{(k)}$ is structured *by the indices of P (it is assumed that the structuring degrees are (n,p)). Then (15) is equivalent to*

$$S_P = \bar{S}^{(k)}_{P^*}\ \hat{\Phi}_\varphi = \begin{bmatrix} 0_k & \bar{S}_{P^*} \end{bmatrix}\ \hat{\Phi}_\varphi \tag{16}$$

where $\bar{S}_{P^*}$ is the (n,p)-expanded resultant of of P^* and $\hat{\Phi}_\varphi = \Phi_\varphi^{-1}$ has the form of (9) and it is defined by the GCD $\varphi(s)$.

4.3. *The Notion of the Approximate GCD*

We are now in a position to address formally the notion of the "approximate GCD" and then consider the development of a computational procedure that allows the evaluation of how good is the given "approximate GCD". The essence of current methods for introduction of "approximate GCD" is the relaxation of conditions characterising the exact notion. We will define the strength or quality of a given "approximate GCD" by the size of the minimal perturbation required to make a chosen "approximate GCD" an exact GCD of a perturbed set of polynomials.

Let us denote by $\Pi(n,p;h+1)$ the set of all polynomial sets $P_{h+1,n}$ having $h+1$ elements with the two higher degrees (n,p), $n \geq p$; if $P_{h+1,n} = \{p_i(s),\ i = 0,1,...,h\} \in \Pi(n,p;h+1)$, then $\deg\{p_0(s)\} = n$, $\deg\{p_1(s)\} = p$,$\deg\{p_i(s)\} \leq p$, $p_0(s) = a_n s^n + ... + a_0 = a^t\, e_n(s)$, $p_i(s) = b_{i.p}s^p + ... + b_{i,0} = b_i^t\, e_p(s)$ $\quad i = 2,...,h$ where $e_k(s) = \left[s^k s^{k-1},...,s,1\right]$, then to the set $P_{h+1,n}$ we may correspond the vector

$$p_{h+1,n} = \begin{bmatrix} a^t & b_1^t & \cdots & b_n^t \end{bmatrix}^t \in \tag{17}$$

where $N = (n+1) + h(p+1)$, or alternatively a point $P_{h+1,n}$ in the projective space P^{N-1}. The set $\Pi(n,p;h+1)$ is clearly isomorphic with $\mathbb{R}^N$, or P^{N-1}. An important question relates to the characterisation of all points of P^{N-1}, which correspond to sets of polynomials with a given degree GCD. Such sets of polynomials correspond to certain varieties of P^{N-1}, which are defined below. We first note that an alternative representation of $P_{h+1,n}$ is provided by the generalised Sylvester resultant $S_P \in \mathbb{R}^{(p+hn)\times(n+p)}$ which is a matrix defined by the vector of coefficients $p_{h+1,n}$. If we denote by $C_k(\cdot)$ the k-th compound of S_P [18], then the varieties characterising the sets having, a given degree d, GCD are defined below [15]:

Proposition 4.2 *Let $\Pi(n,p;h+1)$ be the set of all polynomial sets $P_{h+1,n}$ with $h+1$ elements and with the two higher degrees (n,p), $n \geq p$ and let S_P be the Sylvester resultant of the general set $P_{h+1,n}$. The variety of P^{N-1} which characterise all sets $P_{h+1,n}$ having a GCD with degree d, $0 < d \leq p$ is defined by the set of equations*

$$C_{n+p-d+1}(S_P) = 0 \tag{18}$$

Conditions (18) define polynomial equations in the parameters of the vector $p_{h+1,n}$, or the point $P_{h+1,n}$ of P^{N-1}. The set of equations in (18) define a variety of P^{N-1}, which will be denoted by $\Delta_d(n,p;h+1)$ and will be referred to as the *d-GCD variety* of P^{N-1}. $\Delta_d(n,p;h+1)$ characterises all sets in $\Pi(n,p;h+1)$ which have a GCD with degree d.

Remark 4.1 The sets $\Delta_d(n,p;h+1)$ have measure zero [7] and thus the existence of a nontrivial GCD $d>0$ is a nongeneric property.

The important question that is posed, is how close the given set $P_{h+1,n}$is to given variety $\Delta_d(n,p;h+1)$. Defining the notion of the "approximate GCD" is linked to introducing an appropriate distance of $P_{h+1,n}$ from $\Delta_d(n,p;h+1)$. The diagram of Figure 1 illustrates the notion of "approximate GCD". In fact, if $Q^i_{h+1,n}$ is some perturbation set (to be properly defined) and assuming that $P^{',i}_{h+1,n}=P_{h+1,n}+Q^i_{h+1,n}$ such that $P^{',i}_{h+1,n}\in\Delta_d(n,p;h+1)$, then the GCD of $P^{',i}_{h+1,n}$, $\varphi(s)$, with degree d defines the notion of the approximate GCD" and its strength is defined by the "size" of the perturbation $Q^i_{h+1,n}$. Numerical procedures such as ERES, produce estimates of an "approximate GCD". Estimating the size of the corresponding perturbations provides the means to evaluate how good such approximations are. By letting the parameters of the GCD free and searching for the minimal size of the corresponding perturbations a distance problem is formulated that is linked to the definition of the "optimal approximate GCD". The key questions which have to be considered for such studies are:

Key Problems:

(i) Existence of perturbations of $P_{h+1,n}$ yielding $P'_{h+1,n}=P_{h+1,n}+Q_{h+1,n}\in\Delta_d(n,p;h+1)$

(ii) Parameterisations of all such perturbations.

(iii) Determine the minimal distance of $P_{h+1,n}$ from an element of $\Delta_d(n,p;h+1)$ with a given GCD $\varphi(s)$, and thus evaluation of strength of $\varphi(s)$.

(iv) Determine the minimal distance of $P_{h+1,n}$ from $\Delta_d(n,p;h+1)$ and thus compute the "optimal approximate GCD.

4.4. *Parametrization of GCD varieties and the Computation of Strength of the Approximate GCD*

The characterisation of the $\Delta_d(n,p;h+1)$ variety in a parametric form, as well as subvarieties of it, is a crucial issue for the further development of the topic. The subset of $\Delta_d(n,p;h+1)$, characterised by the property that all $P_{h+1,n}$ in it have a given GCD $v(s)\in\mathbb{R}[s]$, $\deg\{v(s)\}=d$, can be shown to be a subvariety of $\Delta_d(n,p;h+1)$ and shall be denoted by $\Delta^v_d(n,p;h+1)$. In fact $\Delta^v_d(n,p;h+1)$ is characterised by the equations of $\Delta_d(n,p;h+1)$ and a set of additional linear relations amongst the parameters of the vector $p_{h+1,n}$.

Proposition 4.3 *Consider the set* $\Pi(n,p;h+1)$*,* P^{N-1} *be the associated projective space,* $P_{h+1,n}\in\Pi(n,p;h+1)$ *and let* S_P *be the associated resultant. Then,* *(i) The variety $\Delta_d(n,p;h+1)$ of P^{N-1} is expressed parametrically by the resultant*

$$S_P = \begin{bmatrix} 0_d & \bar{S}_{P^*} \end{bmatrix} \hat{\Phi}_\upsilon \tag{19}$$

where $\hat{\Phi}_\upsilon$ is the $(n+p)\times(n+p)$ Toeplitz representation of an arbitrary $\upsilon(s) \in \mathbb{R}[s]$ with $\deg\{\upsilon(s)\}$ and $\bar{S}_{P^*} \in \mathbb{R}^{(p+hn)\times(n+p-d)}$ is the (n,d)-expanded resultant of an arbitrary set of polynomials $P^* \in \Pi(n-d,p-d;h+1)$.

(ii) The variety $\Delta_d^u(n,p;h+1)$ of P^{N-1} is defined by (19) with the additional constraint that $\upsilon(s) \in \mathbb{R}[s]$ is given.

Clearly, the free parameters in $\Delta_d(n,p;h+1)$ are the coefficients of the polynomials of $\Pi(n-d,p-d;h+1)$. Having defined the description of these varieties we consider next the perturbations that transfer a general set $P_{h+1,n}$ on a set $P'_{h+1,n}$ on them. If $P_{h+1,n} \in \Pi(n,p;h+1)$ we can define an (n,p)-ordered perturbed set by:

$$P'_{h+1,n} = P_{h+1,n} - Q_{h+1,n} \in \Pi(n,p;h+1) \quad :$$

$$P'_{h+1,n} = \{p'_i(s) = p_i(s) - q_i(s) : \deg\{q_i(s)\} \leq \deg\{p_i(s)\},\ i = 0,1,...,h\} \tag{20}$$

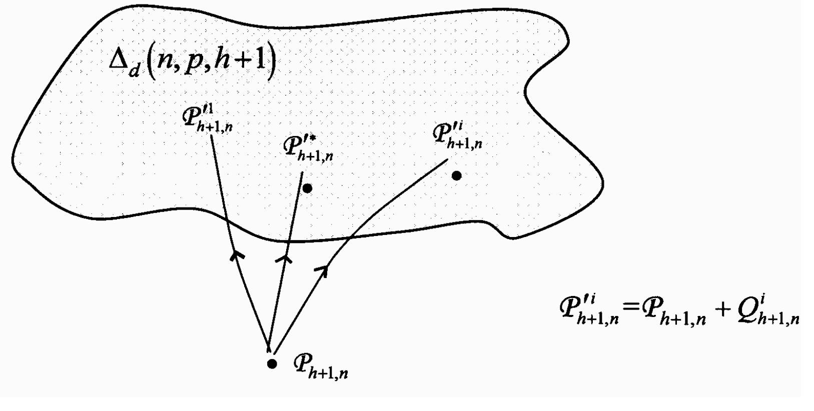

Figure 2: The notion of "approximate GCD"

Using the set of perturbations defined above we may now show that any polynomial from a certain class may become an exact GCD of a perturbed set under a family of perturbations.

Proposition 4.4 *[15] Given a set $P_{h+1,n}$ with maximal degrees (n,p), $n \geq p$ and a polynomial $\omega(s) \in \mathbb{R}[s]$ with $\deg\{\omega(s)\} \leq p$. There always exists a family of (n,p)-ordered perturbations $Q_{h+1,n}$ such that for every element of this family $P'_{h+1,n} = P_{h+1,n} - Q_{h+1,n}$ has a GCD which is divisible by $\omega(s)$.*

The above result establishes the existence of perturbations making *ω(s)* an exact GCD of the perturbed set and motivates the following definition, which defines *ω(s)* as an approximate GCD in an optimal sense.

Definition 4.2 Let $P_{h+1,n} \in \Pi(n,p;h+1)$ and $\omega(s) \in \mathbb{R}[s]$ be a given polynomial with $\deg\{\omega(s)\} = r \leq p$. Furthermore, let $\Sigma_\omega = \{Q_{h+1,n}\}$ be the set of all (n,p)-order perturbations such that

$$P'_{h+1,n} = P_{h+1,n} - Q_{h+1,n} \in \Pi(n,p;h+1) \tag{21}$$

with the property that $\omega(s)$ is a common factor of the elements of $P'_{h+1,n}$. If $Q^*_{h+1,n}$ is the minimal norm element of the set Σ_ω, then $\omega(s)$ is referred as an *r-order almost common factor* of $P_{h+1,n}$, and the norm of $Q^*_{h+1,n}$, denoted by $\|Q^*\|$ is defined as the *strength* of $\omega(s)$. If $\omega(s)$ is the GCD of

$$P^*_{h+1,n} = P_{h+1,n} - Q^*_{h+1,n} \tag{22}$$

then $\omega(s)$ will be called an *r-order almost GCD* of $P_{h+1,n}$ with strength $\|Q^*\|$.

Thus, any polynomial $\omega(s)$ may be considered as an "approximate GCD", as long $\deg\{\omega(s)\} \leq p$.

Proposition 4.5 *[15] Given a set $P_{h+1,n}$ with maximal degrees (n,p), $n \geq p$ and a polynomial $\omega(s) \in \mathbb{R}[s]$ with $\deg\{\omega(s)\} \leq p$. There always exists a family of (n,p)-ordered perturbations $Q_{h+1,n}$ such that for every element of this family $P'_{h+1,n} = P_{h+1,n} - Q_{h+1,n}$ has a GCD which is divisible by $\omega(s)$.*

Theorem 4.3 *[15] For $P_{h+1,n} \in \Pi(n,p;h+1)$, let $S_P \in \Psi(n,p;h+1)$ be the corresponding generalized resultant and let $\upsilon(s) \in \mathbb{R}[s]$, $\deg\{\upsilon(s)\} = r \leq p$. Then:*

(i) Any perturbation set $Q_{h+1,n} \in \Pi(n,p;h+1)$ that leads to $P'_{h+1,n} = P_{h+1,n} - Q_{h+1,n}$, which has $\upsilon(s)$ as common divisor, has a generalized resultant $S_Q \in \Psi(n,p;h+1)$ that is expressed as shown below:

(a) If $\upsilon(0) \neq 0$ then

$$S_Q = S_P - \bar{S}^{(r)}_{P^*}\,\hat{\Phi}_\upsilon = S_P - \begin{bmatrix} 0_r & \bar{S}_{P^*} \end{bmatrix} \hat{\Phi}_\upsilon \tag{23}$$

where $\hat{\Phi}_\upsilon$ is the $(n+p)\times(n+p)$ Toeplitz representation of $\upsilon(s)$ as defined by (9) and $\bar{S}_{P^*} \in \mathbb{R}^{(p+hn)\times(n+p-r)}$ is the (n,p)-expanded resultant of an arbitrary set of polynomials $P^* \in \Pi(n-r,p-r;h+1)$.

(b) If $\upsilon(s)$ has k zeros at $s=0$, then

$$S_Q = S_P - \bar{S}_{P^*}\,\Theta_\upsilon \tag{24}$$

where $\bar{S}_{P^*}$ is again the (n,p)-expanded resultant of an arbitrary set of polynomials $P^* \in \Pi(n-r,p-r;h+1)$ and Θ_υ is the $(n+p-k)\times(n+p)$

representation of $\upsilon(s)$ defined by:

$$\Theta_{\varphi} = \begin{bmatrix} \lambda_k & \lambda_{k-1} & \lambda_{k-2} & \cdots & \cdots & \lambda_0 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_k & \lambda_{k-1} & \lambda_{k-2} & \cdots & \cdots & \lambda_0 & 0 & & 0 \\ \vdots & \ddots & \ddots & & & & & \ddots & & \\ \vdots & & & \ddots & & & & & \ddots & \\ 0 & \cdots & \cdots & 0 & \lambda_k & \lambda_{k-1} & \lambda_{k-2} & \cdots & \cdots & \lambda_0 \end{bmatrix} \quad (25)$$

(ii) If the parameters of $\bar{S}_{P^*}$ are constrained such that $\bar{S}_{P^*}$ has full rank, then $\upsilon(s)$ is a GCD of the perturbed set ${P'}_{h+1,n}$.

Corollary 4.2 *Let $P_{h+1,n} \in \Pi(n,p;h+1)$ and $\upsilon(s) \in \mathbb{R}[s]$, $\deg\{\upsilon(s)\} = r \le p$. The polynomial $\upsilon(s)$ is an r-order almost common divisor of $P_{h+1,n}$ and its strength is defined as a solution of the following minimization problems:*
(a) *If $\upsilon(0) \neq 0$, then its strength is defined by the global minimum of*

$$f(P,P^*) = \min_{\forall P^*} ||S_P - \begin{bmatrix} 0_r & \bar{S}P^* \end{bmatrix} \hat{\Phi}_{\upsilon}||_F \quad (26)$$

(b) *If $\upsilon(s)$ has k zeros at $s = 0$, then its strength is defined by the global minimum of*

$$f(P,P^*) = \min_{\forall P^*} ||S_P - \bar{S}_{P^*}\Theta_{\upsilon}||_F \quad (27)$$

where P^* takes values from the set $\Pi(n,p;h+1)$. Furthermore $\upsilon(s)$ is an r-order almost GCD of $P_{h+1,n}$, if the minimal corresponds to a coprime set P^* or to full rank S_{P^*}.

The above results provide the fundamentals of a framework for the characterization of the almost GCD of a polynomial set and its strength. In [15] the notion of approximate GCD is defined as a distance problem and the quality of the approximate GCD is defined by the strength of the approximation between the given set and the given d-GCD variety. A new resultant based framework for the evaluation of the strength of a given Approximate Greatest Common Divisor of a set of polynomials is defined, as well as for addressing the problem of evaluating the best approximate GCD of a given degree. Note that almost every polynomial may be considered as an approximate GCD [15]. The key issues in these investigations are: **(i)** Define the "best" approximate GCD of a given degree. **(ii)** Qualify the "best" selection of degree for the approximate GCD.

5. Summary

The paper has considered the fundamental problem of computing nongeneric invariants and focused on the case of GCD of many polynomials. The invariance properties of the GCD have been considered and it has been shown that they provide the theoretical framework for the derivation of an efficient matrix based method, the ERES method, for computation of GCD and for the definition of 'approximate" solutions. The significance of system theory in providing a "closed form" characterization of GCD and for defining efficient algorithms for its computation has been demonstrated by the development of the "Matrix Pencil Method". The investigation of the approximate GCD for many polynomials has been introduced and the overall approach has been based on its characterization as a distance problem. This has been achieved by a combination of previous results related to the representation theory [3], the definition of the strength of the approximation [15] and the study of the optimisation properties of the defined problem. A new algorithm has been recently established, which is based on standard optimisation of functions constructed from the original sets of polynomials. New open issues may arise on this, related to the choice of the appropriate rank for the optimisation problem and the relation between the accuracy of the numerical nullity and the strength of the approximation. The theoretic characterisation of the "best" approximate GCD requires further investigation of the optimisation results of [15]. This problem is equivalent to defining the distance of the given set from the d-GCD variety. An analytic investigation of the optimisation problem established in [15] and described above is the subject of investigation at the moment.

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