BULLETIN OF THE
GREEK MATHEMATICAL SOCIETY
Volume 54, 2007 (47–58)

On the Image Systems for the Electric and the Magnetic Potential of the Brain

George Dassios

Received 12 December 2006 Accepted 2 September 2007

Abstract

A dipolar current of known moment and location is given within a conductive sphere modeling the human brain. For this point source excitation the interior and the exterior solutions, both for the direct Electroencephalography and for the direct Magnetoencephalography problems, are characterized as complementary distributions of singularities corresponding to appropriate images for the related potential problems. Comparable forms for all four fields are provided.

1. Introduction

The human brain is the most complicated matter known in the universe. Its mathematical description is that of an ellipsoid of approximately 1.5 liters in volume, involving 10^{11} neurons which are connected with 10^{15} synapses. This extremely complicated circuit can be electrochemically excited via the generation of ionic current within the neurons. At the physical modeling level the electromagnetic activity of this system is governed by the quasi-static approximation of Maxwell's equations [17, 22] and the generated neuronal current is usually modeled by a current dipole located at a particular point within the brain tissue. Up to this point the problem does not reflect any difficulties. Nevertheless, the complications appear when we consider the realistic case where the current dipole is embedded in the conductive environment of the brain. The primary neuronal current generates a magnetic field, which in turn causes an induction current inside the brain and both (primary and induction) currents give rise to a magnetic field which can be measured outside the head using an extremely sensitive device known as SQUID (Superconductive QUantum Interference Device). The goal of Magnetoencephalography (MEG) is to identify the primary current from measurements of the magnetic field outside the head. In doing, so we need to strip the measured magnetic field from the part that is due to the induction current and isolate the primary current alone [23]. This is the source of difficulty for the inverse

MEG problem.
To this day, a complete expression for the exterior magnetic field in closed form due to a dipole current is known only for the case of a spherical conductor [23]. In a less tractable form of series expansion some results for spheroidal [14, 19] and for ellipsoidal [4, 5, 6] conductors are also known.
Once we abandon the highly symmetric spherical model the problem of magnetoencephalography becomes much harder because of the following qualitative reason. The dipolar current, besides the magnetic field, it also generates an electric field and this electric field is directly associated with the conductivity and the generated induction current. Based on the well known Geselowitz formula [3, 10], one is easy to see that the exterior magnetic field is dependent on the values of the electric potential on the boundary of the conductor, and these values can be obtained by solving the interior Neumann problem for Poisson's equation which is satisfied by the electric potential. Therefore, we need to solve first the boundary value problem for the electric potential and then use it to evaluate the magnetic field exterior to the head. But, as Sarvas showed [23] the exterior magnetic field is independent of the electric field if and only if the conductive region (i.e. the brain) is a sphere. Hence, in this case, no need for the solution of the boundary value problem for the electric potential is necessary. For this reason almost the entire enterprise of Magnetoencephalography today is based on the spherical model of the brain.
The problem of obtaining the electric field is known as the direct problem of Electroencephalography (EEG) and the problem of recovering the current that generated a measured electric field on the surface of the head is known as the inverse EEG problem. Similarly, depending on whether we are given the current and try to obtain the exterior magnetic field, or the other way around, we have the direct or the inverse MEG problems.
In the present work we provide a thorough discussion for the electric and magnetic field inside and outside a spherical conductor, both in terms of distribution of images and in terms of closed form representations. This approach gives us a unified treatment of the direct EEG and MEG problems and reveals their similarities and differences.
A general formulation of the direct interior and the direct exterior problems of Electroencephalography and of Magnetoencephalography are presented in Section 2. The interior EEG problem for the sphere is then analyzed in Section 3 and the corresponding exterior EEG problem is investigated in Section 4. Then, Section 5 deals with the exterior and Section 6 with the interior MEG problem for the sphere. In all cases the primary current is that of a dipole source at a fixed point inside the brain and with a known dipole moment. A final concluding Section 7 gathers all fields in a comparable simple form where the Kelvin inversion is immediately observed. The interested reader can find more results for the MEG problem in [8, 24] for the case of a sphere, in [4, 11, 13, 15] for the case of the ellipsoid and further theory in [7, 12, 18, 20, 21].

2. Formulation of the EEG and the MEG problem

Let V^- be a bounded, convex subset of $\mathbb{R}^3$ bounded by the smooth surface S, and let $\hat{\mathbf{n}}$ denotes the outwards unit vector normal to S. V^- is a homogeneous conductive body with conductivity σ and has a current dipole at $\mathbf{r}_0$ with moment $\mathbf{Q}$. The exterior medium $V^+ = (V^- \cup S)^C$ is characterized by $\sigma = 0$ while the magnetic permeability μ_0 is constant in $\mathbb{R}^3$.
Following well known formulations [12, 18, 20, 23], the direct mathematical problems associated with the Electroencephalography and the Magnetoencephalography are the following.

Interior EEG Problem: Find the electric potential

$$u^- : V^- \to \mathbb{R}$$

which solves the interior Neumann problem

$$\sigma \Delta u^-(\mathbf{r}) = \nabla \cdot \mathbf{Q}\delta(\mathbf{r} - \mathbf{r}_0), \qquad \mathbf{r} \in V^- \tag{1}$$

$$\partial_n u^-(\mathbf{r})s = 0, \qquad \mathbf{r} \in S \tag{2}$$

where $\delta(\mathbf{r}-\mathbf{r}_0)$ denotes the Dirac measure at $\mathbf{r}_0$ and ∂_n stands for the outward normal derivative on S.

Exterior EEG Problem: Find the electric potential

$$u^+ : V^+ \to \mathbb{R}$$

which solves the exterior Dirichlet problem

$$\Delta u^+(\mathbf{r}) = 0, \qquad \mathbf{r} \in V^+ \tag{3}$$

$$u^+(\mathbf{r}) = u^-(\mathbf{r}), \qquad \mathbf{r} \in S \tag{4}$$

$$u^+(\mathbf{r}) = O\left(\frac{1}{r}\right), \qquad r \to \infty \tag{5}$$

where the Dirichlet data $u^-(\mathbf{r})$ on S come from the solution of the interior EEG problem. Once the electric potential $u^-(\mathbf{r})$ on S is known, the magnetic induction field B is obtained via the Geselowitz formula [9, 10]

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{Q} \times \frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} - \frac{\mu_0 \sigma}{4\pi} \int\limits_S u^{\pm}(\mathbf{r}')\hat{\mathbf{n}}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} ds(\mathbf{r}') \tag{6}$$

which holds for $\mathbf{r} \notin S$, and where the first term on the RHS of (6) represents the primary dipole contribution while the integral terms represents the contribution of the induction current due to $\sigma \neq 0$. The vector $\hat{\mathbf{n}}'$ is the unit outward normal on S.

Exterior MEG Problem: Find

$$\mathbf{B}^{+} : V^{+} \to \mathbb{R}^3$$

such that

$$\mathbf{B}^{+}(\mathbf{r}) = \mathbf{B}(\mathbf{r}), \quad \mathbf{r} \in V^{+}. \tag{7}$$

Interior MEG Problem: Find

$$\mathbf{B}^{-} : V^{-} \to \mathbb{R}^3$$

such that

$$\mathbf{B}^{-}(\mathbf{r}) = \mathbf{B}(\mathbf{r}), \quad \mathbf{r} \in V^{-}. \tag{8}$$

In the following sections we will analyze each one of the above four problems for the special case where the surface S is a sphere.

3. Interior Electroencephalography for the Sphere

The unique simplifying property of spherical geometry is seen immediately from Geselowitz formula (6). Indeed, since

$$\hat{\mathbf{r}}' \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \cdot \hat{\mathbf{r}} = 0 \tag{9}$$

where the hat over a vector denotes unit magnitude, it follows that the radial component of $\mathbf{B}$ is independent of the electric potential u on S. The sphere is the only shape for which this is true and therefore it is the only shape for which the EEG and the MEG problems are completely independent of each other. This is the reason why a closed form solution for EEG and MEG is known only for the spherical model of the head.
It is a straightforward procedure to solve problem (1), (2) and obtain the solution

$$u^{-}(\mathbf{r}) = \frac{1}{4\pi\sigma} \mathbf{Q} \cdot \nabla_{\mathbf{r}_0} \left[\frac{1}{|\mathbf{r} - \mathbf{r}_0|} + 4\pi \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{n+1}{n(2n+1)} \frac{r_0^n r^n}{\alpha^{2n+1}} Y_n^m(\hat{\mathbf{r}}_0) Y_n^m(\hat{\mathbf{r}}) \right] \tag{10}$$

where α is the radius of the sphere and the normalized spherical harmonics Y_n^m are defined by

$$Y_n^m(\hat{\mathbf{r}}) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}}\, P_n^{|m|}(\cos\theta)e^{im\phi} \tag{11}$$

for $n = 0, 1, 2, ...$ and $m = -n, ..., n$, and P_n^m are the associated Legendre functions of degree n and order m. Expansion (10) converges uniformly and absolutely on any compact sphere inside V^- . We further note that the scaled directional derivate $\mathbf{Q} \cdot \nabla_{\mathbf{r}_0}$ is a purely source dependent operator.

First, we observe that by using series summation techniques we can prove that the function

$$f(\rho) = \sum_{n=1}^{\infty} \frac{\rho^n}{n} P_n(\cos\theta) \tag{12}$$

where P_n is the Legendre polynomial of degree n solves the IVP

$$\rho f'(\rho) = (1 - 2\rho\cos\theta + \rho^2)^{-\frac{1}{2}} - 1 \tag{13}$$

$$f(0) = 0 \tag{14}$$

from which we arrive at

$$f(\rho) = -\ln \frac{1 - \rho\cos\theta + \sqrt{1 - 2\rho\cos\theta + \rho^2}}{2}\,. \tag{15}$$

Combining the above summation result with the addition theorem

$$P_n(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}_0) = \sum_{m=-n}^{n} \frac{4\pi}{2n+1} Y_n^m(\hat{\mathbf{r}}_0) Y_n^m(\hat{\mathbf{r}}) \tag{16}$$

we can write the interior electric field in the following closed form

$$u^-(\mathbf{r}) = \frac{1}{4\pi\sigma}\mathbf{Q}\cdot\nabla_{\mathbf{r}_0}\left[\frac{1}{|\mathbf{r}-\mathbf{r}_0|} + \frac{\frac{\alpha}{r}}{\left|\frac{\alpha^2}{r^2}\mathbf{r}-\mathbf{r}_0\right|} - \frac{1}{\alpha}\ln\frac{r}{2\alpha^2}\left(\hat{\mathbf{r}}\cdot\left(\left(\frac{\alpha^2}{r^2}\mathbf{r}-\mathbf{r}_0\right) + \left|\frac{\alpha^2}{r^2}\mathbf{r}-\mathbf{r}_0\right|\right)\right)\right]$$

$$= \frac{1}{4\pi\sigma}\mathbf{Q}\cdot\left[\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} + \frac{\alpha}{r}\frac{\frac{\alpha^2}{r^2}\mathbf{r}-\mathbf{r}_0}{\left|\frac{\alpha^2}{r^2}\mathbf{r}-\mathbf{r}_0\right|^3} + \frac{1}{\alpha\left|\frac{\alpha^2}{r^2}\mathbf{r}-\mathbf{r}_0\right|}\frac{\left|\frac{\alpha^2}{r^2}\mathbf{r}-\mathbf{r}_0\right|\hat{\mathbf{r}} + \left(\frac{\alpha^2}{r^2}\mathbf{r}-\mathbf{r}_0\right)}{\left|\frac{\alpha^2}{r^2}\mathbf{r}-\mathbf{r}_0\right| + \hat{\mathbf{r}}\cdot\left(\frac{\alpha^2}{r^2}\mathbf{r}-\mathbf{r}_0\right)}\right]. \tag{17}$$

Writing the logarithmic term as an integral we obtain the expression

$$u^-(\mathbf{r}) = \frac{1}{4\pi\sigma}\mathbf{Q}\cdot\left[\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} + \frac{\frac{\alpha}{r}\left(\frac{\alpha^2}{r^2}\mathbf{r}-\mathbf{r}_0\right)}{\left|\frac{\alpha^2}{r^2}\mathbf{r}-\mathbf{r}_0\right|^3} + \frac{2}{\alpha}\int_{\alpha}^{+\infty}\frac{\frac{\lambda}{r}\left(\frac{\lambda^2}{r^2}\mathbf{r}-\mathbf{r}_0\right)}{\left|\frac{\lambda^2}{r^2}\mathbf{r}-\mathbf{r}_0\right|^3}d\lambda\right] \tag{18}$$

which can be read as superposition of contributions from a system of images. Indeed, using Kelvin's theorem [2, 16]

$$\Delta_{\bar{\mathbf{r}}} f(\bar{\mathbf{r}}) = \left(\frac{r}{\alpha}\right)^5 \Delta_{\mathbf{r}} \frac{\alpha}{r} f\left(\frac{\alpha^2}{r^2}\mathbf{r}\right) \tag{19}$$

$$\mathbf{r} \to \bar{\mathbf{r}} = \frac{\alpha^2}{r^2}\mathbf{r} \tag{20}$$

and rewriting the integral in (18) as

$$\frac{2}{\alpha}\int_{\alpha}^{\infty}\frac{\frac{\lambda}{r}\left(\frac{\lambda^2}{r^2}\mathbf{r}-\mathbf{r}_0\right)}{\left|\frac{\lambda^2}{r^2}\mathbf{r}-\mathbf{r}_0\right|^3}d\lambda = \frac{1}{\alpha}\int_{\frac{\alpha^2}{r}}^{+\infty}\frac{t\hat{\mathbf{r}}-\mathbf{r}_0}{|t\hat{\mathbf{r}}-\mathbf{r}_0|^3}dt \tag{21}$$

we observe that the field at $\mathbf{r}_0$ is a superposition of a dipole at $\mathbf{r}$ with strength one, a dipole at $(\alpha^2/r^2)\mathbf{r}$ with strength (α/r) and a continuous distribution of dipoles on the ray

$$\left\{ t\hat{\mathbf{r}} \,|\, t \in \left[\frac{\alpha^2}{r}, +\infty\right)\right\}$$

all with strength $(1/\alpha)$.

Obviously, the trace of $u^-(\mathbf{r})$ on S is given by

$$u^-(\alpha\hat{\mathbf{r}}) = \frac{1}{4\pi\sigma}\mathbf{Q}\cdot\nabla_{\mathbf{r}_0}\left[\frac{2}{|\alpha\hat{\mathbf{r}}-\mathbf{r}_0|} - \frac{1}{\alpha}\ln\frac{1}{2\alpha}\left(\hat{\mathbf{r}}\cdot(\alpha\hat{\mathbf{r}}-\mathbf{r}_0)+|\alpha\hat{\mathbf{r}}-\mathbf{r}_0|\right)\right]. \tag{22}$$

We finally remark that the function u^- , being a solution of a Neumann problem is unique only up to an additive constant. This constant assumes the value zero in the above analysis.

4. Exterior Electroencephalography for the Sphere

Working as in the previous section we obtain the solution of problem (3)-(5) in the form

$$u^+(\mathbf{r}) = \frac{1}{\sigma}\mathbf{Q}\cdot\nabla_{\mathbf{r}_0}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{r_0^n}{r^{n+1}}\frac{1}{n}Y_n^m(\hat{\mathbf{r}}_0)Y_n^m(\hat{\mathbf{r}}) \tag{23}$$

for $r > \alpha$. Using again formulae (15) and (16) we rewrite (23) as

$$u^+(\mathbf{r}) = \frac{1}{4\pi\sigma}\mathbf{Q}\cdot\nabla\mathbf{r}_0\left[\frac{2}{|\mathbf{r}-\mathbf{r}_0|} - \frac{2}{r} - \frac{1}{r}\ln\frac{1}{2r}(\hat{\mathbf{r}}\cdot(\mathbf{r}-\mathbf{r}_0) + |\mathbf{r}-\mathbf{r}_0|\,)\right]$$

$$= \frac{1}{4\pi\sigma}\mathbf{Q}\cdot\left[2\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} + \frac{1}{r|\mathbf{r}-\mathbf{r}_0|}\frac{|\mathbf{r}-\mathbf{r}_0|\hat{\mathbf{r}} + (\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0| + \hat{\mathbf{r}}\cdot(\mathbf{r}-\mathbf{r}_0)}\right]. \tag{24}$$

The term $2/r$ inside the bracket, which results from the summation process, can be omitted since it is annihilated by the differential operator $\mathbf{Q}\cdot\nabla_{\mathbf{r}_0}$. Now the image system that generates the field u^+, as it is given by (24), can be obtained from (17) by simple inspection. Indeed the transformation

$$\alpha \mapsto r \tag{25}$$

implies that

$$u^-(\mathbf{r}) \mapsto u^+(\mathbf{r}) \tag{26}$$

where u^- is given by (17) and u^+ by (23).
Consequently, the image system for the exterior problem is obtained form the image system for the interior one via the transformation (24). Hence, from (18) we obtain

$$u^+(\mathbf{r}) = \frac{1}{4\pi\sigma}\mathbf{Q}\cdot\left[2\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} + \frac{2}{r}\int_r^{+\infty}\frac{\frac{\lambda}{r}\left(\frac{\lambda^2}{r^2}\mathbf{r}-\mathbf{r}_0\right)}{\left|\frac{\lambda^2}{r^2}\mathbf{r}-\mathbf{r}_0\right|^3}d\lambda\right]. \tag{27}$$

Since

$$\frac{2}{r}\int_r^{+\infty}\frac{\frac{\lambda}{r}\left(\frac{\lambda^2}{r^2}\mathbf{r}-\mathbf{r}_0\right)}{\left|\frac{\lambda^2}{r^2}\mathbf{r}-\mathbf{r}_0\right|^3}d\lambda = \frac{1}{r}\int_r^{+\infty}\frac{t\hat{\mathbf{r}}-\mathbf{r}_0}{|t\hat{\mathbf{r}}-\mathbf{r}_0|^3}dt \tag{28}$$

we conclude that the field at $\mathbf{r}_0$ is due to a dipole of strength 2 at $\mathbf{r}$ and a continuous distribution of dipoles with strength $(1/r)$ along the ray

$$\{\,t\hat{\mathbf{r}}\,|\,t\in[r,+\infty)\}\,.$$

Note that for both, the interior and the exterior electric potentials, the interpretation concerns the field at the point $\mathbf{r}_0$ due to images associated with the variable $\mathbf{r}$. In that sense, the above interpretation is more appropriate to the lead field approach of Electroencephalography.
A second observation is that the exterior electric potential is independent of the radius α of the conductive sphere, a property that is not shared by the interior field.

5. Exterior Magnetoencephalography for the Sphere

The exterior magnetic field can be obtained via long but straightforward calculations in the form

$$\mathbf{B}^+(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{Q} \times \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} + \frac{\mu_0}{4\pi}(\mathbf{Q}\cdot\nabla_{\mathbf{r}_0})(\mathbf{r}\times\nabla_{\mathbf{r}})\sum_{n=1}^{\infty}\frac{r_0^n}{r^{n+1}}\frac{1}{n}P_n(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}_0) \tag{29}$$

and the summation process (12), (15) gives

$$\sum_{n=1}^{\infty}\frac{r_0^n}{r^{n+1}}\frac{1}{n}P_n(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}_0) = -\frac{1}{r}\ln\frac{F(\mathbf{r};\mathbf{r}_0)}{2r^2|\mathbf{r}-\mathbf{r}_0|}, \quad r > r_0 \tag{30}$$

where

$$F(\mathbf{r};\mathbf{r}_0) = |\mathbf{r}-\mathbf{r}_0|[r|\mathbf{r}-\mathbf{r}_0| + \mathbf{r}\cdot(\mathbf{r}-\mathbf{r}_0)]. \tag{31}$$

Utilizing the fact that $\mathbf{B}^+$ is both irrotational and solenoidal for $|\mathbf{r}| > \alpha$ [23] we can write

$$\mathbf{B}^+(\mathbf{r}) = \frac{\mu_0}{4\pi}\nabla_{\mathbf{r}}U(\mathbf{r};\mathbf{r}_0) \tag{32}$$

where the scalar magnetic potential U is given by

$$U(\mathbf{r};\mathbf{r}_0) = \frac{\mathbf{Q}\times\mathbf{r}_0\cdot\mathbf{r}}{F(\mathbf{r};\mathbf{r}_0)}. \tag{33}$$

The representation (32), (33) was first established by Bronzan [1] but it was successfully utilized by Sarvas [23] , on the work of which almost all literature of Magnetoencephalography has been founded.
The singularities of the Sarvas function U lie on the segment

$$\{\, t\hat{\mathbf{r}}_0 \,|\, t \in [0, r_0]\}$$

which therefore forms the image system for the exterior magnetic field at the point $\mathbf{r}$, $|\mathbf{r}| > \alpha$. This is so, since F vanishes for $\mathbf{r} = \mathbf{r}_0$ and for any $\mathbf{r}$ which lies between the origin $\mathbf{0}$ and $\mathbf{r}_0$. In fact, F has a double root at $\mathbf{r}_0$ and a single root on the $\mathbf{0}$ to $\mathbf{r}_0$ segment.
The field $\mathbf{B}^+$, being solenoidal, has also the representation

$$\mathbf{B}^+(\mathbf{r}) = \frac{\mu_0}{4\pi}\nabla_{\mathbf{r}}\times\mathbf{A}(\mathbf{r};\mathbf{r}_0) \tag{34}$$

where the vector potential $\mathbf{A}$ can easily be evaluated from (29) and (30) as

$$\mathbf{A}(\mathbf{r};\mathbf{r}_0) = \frac{\mathbf{Q}}{|\mathbf{r}-\mathbf{r}_0|} + \hat{\mathbf{r}}(\mathbf{Q}\cdot\nabla_{\mathbf{r}_0})\ln\frac{F(\mathbf{r};\mathbf{r}_0)}{2r^2|\mathbf{r}-\mathbf{r}_0|}. \tag{35}$$

The identity

$$\nabla_{\mathbf{r}} U(\mathbf{r};\mathbf{r}_0) = \nabla_{\mathbf{r}} \times \mathbf{A}(\mathbf{r};\mathbf{r}_0) \tag{36}$$

for $|\mathbf{r}| > \alpha$ is theoretically obvious but its straightforward justification is involved. The closed form expression for $\mathbf{B}^+$ is therefore given by (32), or by (34), or finally by

$$\mathbf{B}^+(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{Q} \times \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} - \frac{\mu_0}{4\pi}\frac{1}{r}(\mathbf{Q}\cdot\nabla_{\mathbf{r}_0})(\mathbf{r}\times\nabla_{\mathbf{r}})\ln\frac{F(\mathbf{r};\mathbf{r}_0)}{2r^2|\mathbf{r}-\mathbf{r}_0|} \tag{37}$$

for $|\mathbf{r}| > \alpha$.

6. Interior Magnetoencephalography for the Sphere

Following the steps of the previous section we obtain the interior magnetic induction field as

$$\mathbf{B}^-(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{Q} \times \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} + \frac{\mu_0}{4\pi}(\mathbf{Q}\cdot\nabla_{\mathbf{r}_0})(\mathbf{r}\times\nabla_{\mathbf{r}})\sum_{n=1}^{\infty}\frac{(r_0 r)^n}{\alpha^{2n+1}}\frac{1}{n}P_n(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}_0) \tag{38}$$

which by virtue of (12), (15) is also written as

$$\mathbf{B}^-(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{Q} \times \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} - \frac{\mu_0}{4\pi}(\mathbf{Q}\cdot\nabla_{\mathbf{r}_0})(\mathbf{r}\times\nabla_{\mathbf{r}})\frac{1}{\alpha}\ln\frac{F\left(\frac{\alpha^2}{r^2}\mathbf{r};\mathbf{r}_0\right)}{2\frac{\alpha^4}{r^2}\left|\frac{\alpha^2}{r^2}\mathbf{r}-\mathbf{r}_0\right|} \tag{39}$$

for $|\mathbf{r}| < \alpha$.
In view of the Kelvin inversion (20) and the relations

$$\nabla_{\mathbf{r}} = \left(\frac{\bar{r}}{\alpha}\right)^2(\widetilde{\mathbf{I}} - 2\hat{\bar{\mathbf{r}}}\otimes\hat{\bar{\mathbf{r}}})\cdot\nabla_{\bar{\mathbf{r}}} \tag{40}$$

$$\mathbf{r}\times\nabla_{\mathbf{r}} = \bar{\mathbf{r}}\times\nabla_{\bar{\mathbf{r}}} \tag{41}$$

where $\widetilde{\mathbf{I}}$ stands for the identity dyadic, $\bar{\mathbf{r}}$ denotes the Kelvin image of $\mathbf{r}$, (39) is written as

$$\mathbf{B}^-(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{Q} \times \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|^3} - \frac{\mu_0}{4\pi}\frac{1}{\alpha}(\mathbf{Q}\cdot\nabla_{\mathbf{r}_0})(\bar{\mathbf{r}}\times\nabla_{\mathbf{r}})\ln\frac{F(\bar{\mathbf{r}};\mathbf{r}_0)}{2\bar{r}^2|\bar{\mathbf{r}}-\mathbf{r}_0|} \tag{42}$$

for $|\mathbf{r}| < \alpha$.
It is easy to see that the inversion (20), which essentially identifies $\mathbf{B}^-$ as the Kelvin inverted field of $\mathbf{B}^+$ (with the exception of the actual dipole field), maps the interior images over the $\mathbf{0}$ to $\mathbf{r}_0$ segment to the exterior ray

$$\{\, t\hat{\mathbf{r}}_0 \,|\, t \in [\bar{r}_0, +\infty)\}$$

which supports the singularities of $\mathbf{B}^-$ that come from the conductivity current. Indeed, the singularities $t\hat{\mathbf{r}}_0$ with $t \in [0, r_0]$ are mapped in to the singularities $\bar{t}\hat{\mathbf{r}}_0$ with $\bar{t} \in \left[\frac{\alpha^2}{r_0}, \infty\right)$.

Note that as with the case of EEG, the exterior magnetic field $\mathbf{B}^+$ is independent of the radius of the sphere while the interior field $\mathbf{B}^-$ does depend on the radius α.

7. Conclusions

In this section we collect all the results obtained in the present work in a simple list where comparison comes via simple observation. We set

$$\bar{\mathbf{r}} = \frac{\alpha^2}{r^2}\mathbf{r} \tag{43}$$

$$\mathbf{P} = \mathbf{r} - \mathbf{r}_0 \tag{44}$$

$$\mathbf{R} = \frac{\alpha^2}{r^2}\mathbf{r} - \mathbf{r}_0 \tag{45}$$

and $\hat{\mathbf{r}}$, $\hat{\bar{\mathbf{r}}}$, $\hat{\mathbf{P}}$, $\hat{\mathbf{R}}$ denote the corresponding unit vectors.

Then

$$u^-(\mathbf{r}) = \frac{1}{4\pi\sigma}\mathbf{Q} \cdot \left[\frac{\mathbf{P}}{P^3} + \frac{\bar{r}}{\alpha}\frac{\mathbf{R}}{R^3} + \frac{1}{\alpha R}\frac{\hat{\mathbf{r}} + \hat{\mathbf{R}}}{1 + \hat{\mathbf{r}} \cdot \hat{\mathbf{R}}}\right], \ r < \alpha \tag{46}$$

$$u^+(\mathbf{r}) = \frac{1}{4\pi\sigma}\mathbf{Q} \cdot \left[2\frac{\mathbf{P}}{P^3} + \frac{1}{rP}\frac{\hat{\mathbf{r}} + \hat{\mathbf{P}}}{1 + \hat{\mathbf{r}} \cdot \hat{\mathbf{P}}}\right], \ r > \alpha \tag{47}$$

$$\mathbf{B}^-(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{Q} \times \frac{\mathbf{P}}{P^3} - \frac{\mu_0}{4\pi}\frac{1}{\alpha}(\mathbf{Q} \cdot \nabla_{\mathbf{r}_0})(\bar{\mathbf{r}} \times \nabla_{\bar{\mathbf{r}}}) \ln \frac{F(\bar{\mathbf{r}}; \mathbf{r}_0)}{2\bar{r}^2 R}, \ r < \alpha \tag{48}$$

$$\mathbf{B}^+(\mathbf{r}) = \frac{\mu_0}{4\pi}\mathbf{Q} \times \frac{\mathbf{P}}{P^3} - \frac{\mu_0}{4\pi}\frac{1}{r}(\mathbf{Q} \cdot \nabla_{\mathbf{r}_0})(\mathbf{r} \times \nabla_{\mathbf{r}}) \ln \frac{F(\mathbf{r}; \mathbf{r}_0)}{2r^2 R}, \ r > \alpha. \tag{49}$$

In each one of these expressions the first term on the RHS provides the field due to the primary dipolar current, and this is the reason why it is the same for the interior as well as the exterior fields. All other terms on the RHS of (46)-(49) represent the contribution of the conductive medium to the corresponding fields. The interior electric potential (46) is associated with the exterior electric potential (47) via Kelvin inversion. Similarly, the interior magnetic field (48) and the exterior magnetic field (49) are connected in the same way.

Acknowledgement: We thank the European Social Fund (ESF), Operational Program for Educational and Vocational Training II (EPEAEK II), and particularly the Program PYTHAGORAS II, for funding the above work.

References

[1] Bronzan J.B., *The Magnetic Scalar Potential*, American Journal of Physics, **39**, 1971, pp. 1357–1359.

[2] Dassios G. and Kleinman R., *On Kelvin Inversion and Low-Frequency Scattering*, SIAM Review, **31**, 1989, pp. 565–585.

[3] G. Dassios and F. Kariotou, *On the Geselowitz Formula in Biomagnetics*, Quart. Appl. Math., **61**, 2003, pp. 387–400.

[4] G. Dassios and F. Kariotou, *Magnetoencephalography in Ellipsoidal Geometry*, J. Math. Phys., **44**, 2003, pp. 220–241.

[5] G. Dassios and F. Kariotou, *On the Exterior Magnetic Field and Silent Sources in Magnetoencephalography*, Abstract and Applied Analysis, **2004:4**, 2004, pp. 307–314.

[6] G. Dassios and F. Kariotou, *The Direct MEG Problem in the Presence of an Ellipsoidal Shell Inhomogeneity*, Quart. Appl. Math., **63**, 2005, pp. 601–618.

[7] G. Dassios, A. S. Fokas and F. Kariotou, *On the Non-Uniqueness of the Inverse MEG Problem*, Inverse Problems, **21**, 2005, L1–L5.

[8] A. S. Fokas, Y. Kurylev and V. Marinakis, *The Unique Determination of Neuronal Currents in the Brain via Magnetoencephalography*, Inverse Problems, **20**, 2004, pp. 1067–1082.

[9] D. B. Geselowitz, *On Bioelectric Potentials in an Inhomogeneous Volume Conductor*, Biophysical Journal, **7**, 1967, pp. 1–11.

[10] D. B. Geselowitz, *On the Magnetic Field Generated Outside an Inhomogeneous Volume Conductor by Internal Current Sources*, IEEE Trans. Magn., **6**, 1970, pp. 346–347.

[11] S. N. Giapalaki and F. Kariotou, *The Complete Ellipsoidal Shell-Model in EEG Imaging*, Abstract and Applied Analysis, **2006**, 2006 pp. 1–18.

[12] M. Hamalainen, R. Hari, R. J. Ilmoniemi, J. Knuutila and O. V. Lounasmaa, *Magnetoencephalography-Theory, Instrumentation and Applications to Noninvasive Studies of the Working Human Brain*, Rev. Mod. Phys., **65**, 1993, pp. 413–497.

[13] G. Kamvyssas and F. Kariotou, *Confocal Ellipsoidal Boundaries in EEG Modeling*, Bulletin of the Greek Mathematical Society, **50**, 2005, pp. 119–133.

[14] F. Kariotou, *On the Mathematics of EEG and MEG in Spheroidal Geometry*, Bulletin of the Greek Mathematical Society, **47**, 2003, pp. 117–135.

[15] F. Kariotou, *Electroencephalography in Ellipsoidal Geometry*, J. Math. Analysis and Applications, **290**, 2004, pp. 324–342.

[16] Kellogg, O.D., *Foundations of Potential Theory*, Dover, New York, 1953.

[17] L.D. Landau and E.M. Lifshitz, *Electrodynamis of Continuous Media*, Pergamon Press, Oxford 1960.

[18] J.Malmivuo and R.Plonsey, *Bioelectromagnetism*, Oxford University Press, New York, 1995.

[19] de Munck J.C., *The Potential Distribution in a Layered Anisotropic Spheroidal Volume Conductor*, Journal of Applied Physics, **64**, 1988, pp. 464–470.

[20] G. Nolte, *The Magnetic Lead Field Theorem in the Quasi-Static Approximation and its Use for MEG Forward Calculation in Realistic Volume Conductors*, Phys. Med. Biol., **48**, 2003, pp. 3637–3652.

[21] G. Nolte and G. Dassios, *Analytic Expansion of the EEG Lead Field for Realistic Volume Conductors*, Phys. Med. Biol., **50**, 2005, pp. 3807–3823.

[22] R. Plonsey and D.B. Heppner, *Considerations of Quasi-Stationarity in Electrophysiological Systems*, Bulletin of Mathematical Biophysics, **29**, 1967, pp. 657–664.

[23] J. Sarvas, *Basic Mathematical and Electromagnetic Concepts of the Biomagnetic Inverse*

Problem, Phys. Med. Biol., **32**, 1987, pp. 11–22.

[24] W.X. Wang, *The Potential for a Homogeneous Spheroid in a Spheroidal Coordinate System: I. At an Exterior Point*, Journal of Physics A. Mathematical and General, **21**, 1988, pp. 4245–4250.

◇ George Dassios
Department of Applied Mathematics and Theoretical Physics
University of Cambridge
Cambridge CB3 0WA, U.K.
G.Dassios@damtp.cam.ac.uk