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A stronger version of the Sarason's type theorem for Wiener-Hopf-Hankel operators with SAP Fourier symbols

A. P. Nolasco and L. P. Castro

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Abstract

In [20], the authors obtained conditions for the Fredholm property, and the one-sided invertibility of Wiener-Hopf plus Hankel operators with semi-almost periodic Fourier symbols. This was done by using the so-called Δ–relation after extension, and therefore within a different technique from that one used by R. V. Duduchava and A. I. Saginashvili when working with Wiener-Hopf operators. The obtained conditions were anyway based on certain mean values of the representatives at infinity of the operator Fourier symbols. Considering now Wiener-Hopf plus and minus Hankel operators acting between L^2 Lebesgue spaces, and having semi-almost periodic Fourier symbols, the present paper is devoted to seek for a refinement of the conditions that lead to their Fredholm property, and one-sided invertibility. I.e., a so-called Sarason's type theorem for Wiener-Hopf plus and minus Hankel operators. In this way, following, as far as possible, a similar approach of that one used by R. V. Duduchava and A. I. Saginashvili (to Wiener-Hopf operators), it is now obtained a second version of a Sarason's type theorem for those operators. Therefore, combining the two versions, it is now possible to refine the conditions, and obtain in the present paper a stronger version of a Sarason's type theorem than that one presented in [20].

Keywords: Wiener-Hopf-Hankel operator, semi-almost periodic function, Fredholm property, one-sided invertibility

1. Introduction

In [25], Sarason developed the semi-Fredholm theory for Toeplitz operators in the Hardy space H^2, with symbols in the algebra of semi-almost periodic elements. This was in fact the origin of the semi-almost periodic functions, due to an initial suggestion of Israel Gohberg to Donald Sarason, in view of the consideration at the same time of piecewise continuous and almost periodic symbols. Later on, Duduchava and

Saginashvili [8, 23] worked out the corresponding semi-Fredholm theory for Wiener-Hopf operators with semi-almost periodic Fourier symbols, and acting between L^p spaces ($1 < p < \infty$). All this was done upon conditions on some mean numbers of certain representatives of the Fourier symbols at minus and plus infinity.

Recently, great attention has been devoted to Wiener-Hopf plus/minus Hankel operators with several different classes of Fourier symbols, cf. [3, 5, 9, 10, 14, 15, 16, 17, 19, 22, 26]. Most of interest cames directly from the Mathematical-Physics applications where those operators arise [3, 4, 18, 27]. Although we may say that the theory of Wiener-Hopf plus/minus Hankel operators is well developed for some classes of Fourier symbols like the case of continuous or piecewise continuous functions, this is not the case for the semi-almost periodic class.

Motivated by the results of Sarason, and Duduchava and Saginashvili, we established in [20] a corresponding analysis for Wiener-Hopf plus Hankel operators with semi-almost periodic Fourier symbols and acting between L^p Lebesgue spaces. There, we obtained a Duduchava-Saginashvili's type theory for these operators following a different approach of that one used by R. V. Duduchava and A. I. Saginashvili, i.e. using the Δ–relation after extension between Wiener-Hopf plus Hankel operators and Wiener-Hopf operators. This Duduchava-Saginashvili's type theory means a characterization of the Fredholm property, and one-sided invertibility of Wiener-Hopf plus Hankel operators, based on certain mean values of the representatives at infinity of their Fourier symbols. With the help of certain operator relations, we obtain a similar Duduchava-Saginashvili's type theory for Wiener-Hopf minus Hankel operators with semi-almost periodic Fourier symbols and acting between L^p Lebesgue spaces.

Considering now the case of Wiener-Hopf plus/minus Hankel operators with semi-almost periodic Fourier symbols, acting between L^2 Lebesgue spaces, we derive, through the Δ–relation after extension, a characterization of the Fredholm property, and one-sided invertibility. I.e., the so-called Sarason's type theorem for those operators. Proceeding in this way, we will obtain here a stronger version of such Sarason's type theorem. Following, as far as possible, a similar approach of that one used by R. V. Duduchava and A. I. Saginashvili, we will firstly obtain a different version of the Sarason's type theorem. This does not turn out to be an easy task since, although some parts of the approach of R. V. Duduchava and A. I. Saginashvili are possible to be carried out to Wiener-Hopf plus/minus Hankel operators, there are some other parts where we have to use a different reasoning. Combining then the two versions of the Sarason's type theorem, we can therefore refine the initial version, and obtain a stronger version of the Sarason's type theorem.

2. Preliminaries

We will consider *Wiener-Hopf plus/minus Hankel operators* of the form

$$(W \pm H)_\phi = W_\phi \pm H_\phi \; : \; L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+), \tag{1}$$

with W_ϕ and H_ϕ being *Wiener-Hopf* and *Hankel operators* defined by

$$W_\phi = r_+\mathcal{F}^{-1}\phi\cdot\mathcal{F} \;:\; L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+) \tag{2}$$

$$H_\phi = r_+\mathcal{F}^{-1}\phi\cdot\mathcal{F}J \;:\; L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+)\,, \tag{3}$$

respectively. As usual, $L^2(\mathbb{R})$ and $L^2(\mathbb{R}_+)$ denote the Banach spaces of complex-valued Lebesgue measurable functions φ, for which $|\varphi|^2$ is integrable on $\mathbb{R}$ and $\mathbb{R}_+$, respectively. $L^2_+(\mathbb{R})$ denotes the subspace of $L^2(\mathbb{R})$ formed by all the functions supported in the closure of $\mathbb{R}_+ = (0,+\infty)$, r_+ represents the *operator of restriction from* $L^2_+(\mathbb{R})$ *into* $L^2(\mathbb{R}_+)$, $\mathcal{F}$ denotes the *Fourier transformation*, J is the *reflection operator* given by the rule $J\varphi(x) = \widetilde{\varphi}(x) = \varphi(-x)$, $x \in \mathbb{R}$, and ϕ belongs to the algebra of the semi-almost periodic functions on $\mathbb{R}$.

In what follows, we will simply call *Wiener-Hopf-Hankel operators* to both Wiener-Hopf plus Hankel, and Wiener-Hopf minus Hankel operators (cf. also [17], [18], [27]).

For arriving at the definition of the algebra of the semi-almost periodic functions on $\mathbb{R}$, let us first consider the algebra of the *almost periodic functions*, usually denoted by AP, i.e. the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions e_λ $(\lambda \in \mathbb{R})$, where

$$e_\lambda(x) = e^{i\lambda x}, \quad x \in \mathbb{R}.$$

Further, let $C(\dot{\mathbb{R}})$ denote the set of all bounded continuous (complex-valued) functions on $\mathbb{R}$ for which both limits at $-\infty$ and at $+\infty$ exist and coincide, $C_0(\dot{\mathbb{R}})$ be the set of all functions in $C(\dot{\mathbb{R}})$ such that the limits at $-\infty$ and at $+\infty$ are equal to zero, and $C(\overline{\mathbb{R}})$ denote the set of all bounded continuous (complex-valued) functions on $\mathbb{R}$ with a possible jump at ∞ (i.e. the limits at $-\infty$ and at $+\infty$ exist and could be distinct).

The C^*–algebra of the *semi-almost periodic functions* on $\mathbb{R}$ (SAP) is by definition the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains AP and $C(\overline{\mathbb{R}})$. In a more detailed way, by a well-known characterization of SAP (cf. [25]), for any $\phi \in SAP$ there exist $\phi_l, \phi_r \in AP$ and $\phi_0 \in C_0(\dot{\mathbb{R}})$ such that (for a fixed u in $C(\overline{\mathbb{R}})$ satisfying $u(-\infty) = 0$ and $u(+\infty) = 1$) it holds

$$\phi = (1-u)\phi_l + u\phi_r + \phi_0. \tag{4}$$

The functions ϕ_l and ϕ_r are uniquely determined by ϕ and independent of the choice of the function u, and $\phi_0 \in C_0(\dot{\mathbb{R}})$. These ϕ_l and ϕ_r are usually called the *almost periodic representatives* of ϕ at $-\infty$ and $+\infty$, respectively.

Since some of our results are obtained upon certain characteristics of the almost periodic representatives of the Fourier symbols of the Wiener-Hopf-Hankel operators, let us first introduce some notions for the elements in AP. For $\phi \in AP$, the *Bohr mean value* of ϕ is defined as

$$M(\phi) = \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|}\int_{I_\alpha} \phi(x)dx,$$

where for an unbounded set $A \subset \mathbb{R}_+$, $\{I_\alpha\}_{\alpha\in A} = \{(x_\alpha, y_\alpha)\}_{\alpha\in A}$ is a family of intervals $I_\alpha \subset \mathbb{R}$ such that $|I_\alpha| = y_\alpha - x_\alpha \to \infty$, as $\alpha \to \infty$. The Bohr mean value of a function in AP exists always, is finite, and is independent of the particular choice of the family $\{I_\alpha\}$.

Let us fix the notation $\mathcal{G}B$ for the group of all invertible elements of a Banach algebra B. By Bohr's theorem, for each $\phi \in \mathcal{G}AP$ there exists a real number $\kappa(\phi)$, and a function $\psi \in AP$ such that

$$\phi = e_{\kappa(\phi)}\, e^{\psi} .$$

Since $\kappa(\phi)$ is uniquely determined, $\kappa(\phi)$ is usually called the *mean motion* of ϕ. The number

$$\mathbf{d}(\phi) = e^{M(\varphi)}$$

is called the *geometric mean value* of ϕ. In the case where $\kappa(\phi) = 0$, we may represent $\mathbf{d}(\phi)$ as

$$\mathbf{d}(\phi) = e^{M(\log \phi)}$$

where $\log\phi$ is any function in AP for which $\phi = e^{\log\phi}$.

We end up this preliminary section by recalling some basic concepts concerning certain Fredholm notions for bounded linear operators. Let $T : X \to Y$ being a bounded linear operator acting between Banach spaces. The operator T is said to be *normally solvable* if $\operatorname{Im} T$ is closed. In this case, the *cokernel of* T is defined as $\operatorname{Coker} T = Y/\operatorname{Im} T$. For a normally solvable operator T, the *deficiency numbers* of T are given by

$$n(T) = \dim \operatorname{Ker} T, \quad d(T) = \dim \operatorname{Coker} T.$$

The operator T is said to be a *Fredholm* operator if it is normally solvable and $n(T)$ and $d(T)$ are finite. In this case, the *Fredholm index* of T is defined by

$$\operatorname{Ind} T = n(T) - d(T).$$

The operator T is said to be a *semi-Fredholm* operator if it is normally solvable and at least one of the deficiency numbers $n(T)$ and $d(T)$ is finite. A semi-Fredholm operator is said to be *n-normal* if $n(T)$ is finite, and *d-normal* if $d(T)$ is finite. In the case where only one of the deficiency numbers is finite, the operator T is said to be a *properly semi-Fredholm* operator. In this case, the operator T is said to be *properly n-normal* if $n(T)$ is finite and $d(T)$ is infinite, and *properly d-normal* if $d(T)$ is finite and $n(T)$ is infinite. The Fredholm index notion is also extended to semi-Fredholm operators in the natural way.

3. Auxiliary results

From the Wiener-Hopf and Hankel operator theory, the following relations are well-known:

$$W_{\phi\varphi} = W_\phi \,\ell_0\, W_\varphi + H_\phi \,\ell_0\, H_{\widetilde{\varphi}},$$
$$H_{\phi\varphi} = W_\phi \,\ell_0\, H_\varphi + H_\phi \,\ell_0\, W_{\widetilde{\varphi}},$$

where $\ell_0 : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ is the *zero extension operator.* Additionally, from the last two identities, it follows that

$$(W+H)_{\phi\varphi} = (W+H)_\phi \,\ell_0\, (W+H)_\varphi + H_\phi \,\ell_0\, W+H)_{\widetilde{\varphi}-\varphi} \,, \tag{5}$$
$$(W-H)_{\phi\varphi} = (W-H)_\phi \,\ell_0\, (W-H)_\varphi + H_\phi \,\ell_0\, (W-H)_{\varphi-\widetilde{\varphi}} \,. \tag{6}$$

Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im m\, z > 0\}$ and $\mathbb{C}_- = \{z \in \mathbb{C} : \Im m\, z < 0\}$. As usual, let $H^\infty(\mathbb{C}_\pm)$ denote the set of all bounded and analytic functions in $\mathbb{C}_\pm$. Fatou's Theorem asserts that functions in $H^\infty(\mathbb{C}_\pm)$ have non-tangential limits on $\mathbb{R} = \partial\mathbb{C}_\pm$ almost everywhere. In this sense, let $H^\infty_\pm(\mathbb{R})$ be the set of all functions in $L^\infty(\mathbb{R})$ that are non-tangential limits of elements in $H^\infty(\mathbb{C}_\pm)$. Moreover, $H^\infty_+(\mathbb{R})$ and $H^\infty_-(\mathbb{R})$ are closed subalgebras of $L^\infty(\mathbb{R})$. Let $H^2(\mathbb{C}_\pm)$ denote the set of all functions ϕ which are analytic in $\mathbb{C}_\pm$ and such that

$$\sup_{\pm y>0} \int_{\mathbb{R}} |\phi(x+iy)|^2 \, dy < \infty \,.$$

Like in the case of $H^\infty(\mathbb{C}_\pm)$, by Fatou's theorem, it also holds that functions in $H^2(\mathbb{C}_\pm)$ have non-tangential limits almost everywhere on $\mathbb{R}$. The set of all these non-tangential functions is denoted by $H^2_\pm(\mathbb{R})$ and it is a closed subspace of $L^2(\mathbb{R})$.

Due to (5) and (6), if we consider $\phi \in H^\infty_-(\mathbb{R})$ or φ being an even function, then we obtain a factorization for Wiener-Hopf plus Hankel operators and for Wiener-Hopf minus Hankel operators, respectively:

$$(W+H)_{\phi\varphi} = (W+H)_\phi \,\ell_0\, (W+H)_\varphi \,, \tag{7}$$
$$(W-H)_{\phi\varphi} = (W-H)_\phi \,\ell_0\, (W-H)_\varphi \,. \tag{8}$$

Proposition 3.1 (cf. [21]) *If* $\phi_e \in \mathcal{G}L^\infty(\mathbb{R})$ *and* $\widetilde{\phi_e} = \phi_e$, *then* $(W\pm H)_{\phi_e}$ *is invertible and its inverse is the operator*

$$\ell_0\, (W\pm H)_{\phi_e^{-1}}\, \ell_0 : L^2(\mathbb{R}_+) \to L^2_+(\mathbb{R}).$$

Proof. On the one hand, we have

$$(W\pm H)_{\phi_e \cdot \phi_e^{-1}}\, \ell_0 = (W\pm H)_1\, \ell_0 = W_1\, \ell_0 = I_{L^2(\mathbb{R}_+)}. \tag{9}$$

On the other hand, since $\phi_e \in \mathcal{G}L^\infty(\mathbb{R})$ and $\widetilde{\phi_e} = \phi_e$, then $\widetilde{\phi_e^{-1}} = \phi_e^{-1}$ and therefore we can apply the factorizations (7) and (8) to these Wiener-Hopf plus/minus Hankel operators. So we have

$$(W \pm H)_{\phi_e \cdot \phi_e^{-1}} = (W \pm H)_{\phi_e}\, \ell_0\, (W \pm H)_{\phi_e^{-1}}. \tag{10}$$

Thus, combining (9) and (10), we get that

$$(W \pm H)_{\phi_e}\, \ell_0\, (W \pm H)_{\phi_e^{-1}}\, \ell_0 = I_{L^2(\mathbb{R}_+)}. \tag{11}$$

In the same way, we obtain that

$$\ell_0\, (W \pm H)_{\phi_e^{-1}}\, \ell_0\, (W \pm H)_{\phi_e} = I_{L^2_+(\mathbb{R})}. \tag{12}$$

Therefore, (11)–(12) show that $(W \pm H)_{\phi_e}$ is invertible and its inverse is

$$\ell_0\, (W \pm H)_{\phi_e^{-1}}\, \ell_0 : L^2(\mathbb{R}_+) \to L^2_+(\mathbb{R}). \qquad \square$$

Lemma 3.1 *Let $\phi, \varphi \in \mathcal{G}SAP$ and suppose that their almost periodic representatives $\phi_l, \varphi_l, \phi_r, \varphi_r$ are connected by*

$$\phi_l = \psi_l^- \varphi_l \psi_l^+, \ \text{with } \psi_l^\pm \in \mathcal{G}AP^\pm \text{ and } \mathbf{d}(\psi_l^\pm) = 1,$$
$$\phi_r = \psi_r^- \varphi_r \psi_r^+, \ \text{with } \psi_r^\pm \in \mathcal{G}AP^\pm \text{ and } \mathbf{d}(\psi_r^\pm) = 1.$$

Then there exist $\zeta_- \in \mathcal{G}(C(\dot{\mathbb{R}}) + H^\infty_-(\mathbb{R})) \cap \mathcal{G}SAP$ and $\zeta_e \in \mathcal{G}L^\infty(\mathbb{R})$, $\widetilde{\zeta_e} = \zeta_e$, such that

$$\phi = \zeta_- \varphi\, \zeta_e\, .$$

Proof. In [2, Lemma 3.11], it is guaranteed the existence of $\gamma_\pm \in \mathcal{G}(C(\dot{\mathbb{R}}) + H^\infty_\pm(\mathbb{R})) \cap \mathcal{G}SAP$ such that

$$\phi = \gamma_- \varphi\, \gamma_+.$$

Thus, if we consider

$$\zeta_- = \gamma_- \widetilde{\gamma_+^{-1}},$$
$$\zeta_e = \widetilde{\gamma_+} \gamma_+,$$

we obtain

$$\phi = \zeta_- \varphi\, \zeta_e$$

with $\zeta_- \in \mathcal{G}(C(\dot{\mathbb{R}}) + H^\infty_-(\mathbb{R})) \cap \mathcal{G}SAP$ and $\zeta_e \in \mathcal{G}L^\infty(\mathbb{R})$, $\widetilde{\zeta_e} = \zeta_e$.

Lemma 3.2 *Let $\phi, \varphi \in L^\infty(\mathbb{R}) \backslash \{0\}$. If there are functions $\zeta_- \in \mathcal{G}(C(\dot{\mathbb{R}}) + H^\infty_-(\mathbb{R}))$ and $\zeta_e \in \mathcal{G}L^\infty(\mathbb{R})$, $\widetilde{\zeta_e} = \zeta_e$, such that $\phi = \zeta_- \varphi\, \zeta_e$, then the operator $(W + H)_\phi$ $\big($resp. $(W - H)_\phi\big)$ is properly n-normal, properly d-normal or Fredholm operator if and only if the operator $(W + H)_\varphi$ $\big($resp. $(W - H)_\varphi\big)$ enjoys the same property.*

Proof. From the factorization of the Wiener-Hopf plus Hankel operators presented in (7), we obtain

$$(W+H)_\phi = (W+H)_{\zeta_-\varphi}\,\ell_0\,(W+H)_{\zeta_e}\,. \tag{13}$$

Applying now (5) to $(W+H)_{\zeta_-\varphi}$, it yields

$$(W+H)_{\zeta_-\varphi} = (W+H)_{\zeta_-}\,\ell_0\,(W+H)_\varphi + H_{\zeta_-}\,\ell_0\,(W+H)_{\widetilde{\varphi}-\varphi}\,. \tag{14}$$

Since $\zeta_- \in \mathcal{G}(C(\dot{\mathbb{R}}) + H^\infty_-(\mathbb{R}))$, in virtue of a theorem due to P. Hartman (see [13], [2, Theorem 2.18]), we have that H_{ζ_-} is a compact operator. Therefore we may rewrite (14) as follows

$$(W+H)_{\zeta_-\varphi} = (W+H)_{\zeta_-}\,\ell_0\,(W+H)_\varphi + C \tag{15}$$

with C being the compact operator $H_{\zeta_-}\,\ell_0\,(W+H)_{\widetilde{\varphi}-\varphi}$. Combining (13) and (15), we obtain

$$(W+H)_\phi = (W+H)_{\zeta_-}\ell_0\;\;(W+H)_\varphi\;\;\ell_0\,(W+H)_{\zeta_e} + K \tag{16}$$

where K is the compact operator $H_{\zeta_-}\,\ell_0\,(W+H)_{\widetilde{\varphi}-\varphi}\,\ell_0\,(W+H)_{\zeta_e}$. Now, due to a theorem of R. G. Douglas (cf. [6, 7], [2, Theorem 2.19]), we know that W_{ζ_-} is a Fredholm operator, and consequently, $(W+H)_{\zeta_-}$ is a Fredholm operator (as the sum of a Fredholm Wiener-Hopf operator with a compact Hankel operator). Clearly, since $(W+H)_{\zeta_-}$ is a Fredholm operator, $(W+H)_{\zeta_-}\ell_0$ will be also a Fredholm operator. Finally, from Proposition 3.1, it follows that $\ell_0(W+H)_{\zeta_e}$ is a Fredholm operator too. This, in connection with (16), means that $(W+H)_\phi$ is properly n-normal, properly d-normal or Fredholm operator if and only if $(W+H)_\varphi$ has the same property.

Due to (6) and (8), the proof for the Wiener-Hopf minus Hankel case runs identically.

Proposition 3.2 *Let $\phi, \varphi \in \mathcal{G}SAP$ and suppose that their almost periodic representatives $\phi_l, \varphi_l, \phi_r, \varphi_r$ are connected by*

$$\phi_l = \psi_l^-\varphi_l\psi_l^+,\ \text{with } \psi_l^\pm \in \mathcal{G}AP^\pm \text{ and } \mathbf{d}(\psi_l^\pm) = 1,$$
$$\phi_r = \psi_r^-\varphi_r\psi_r^+,\ \text{with } \psi_r^\pm \in \mathcal{G}AP^\pm \text{ and } \mathbf{d}(\psi_r^\pm) = 1.$$

Then the operator $(W+H)_\phi$ $\big($resp. $(W-H)_\phi\big)$ is properly n-normal, properly d-normal or Fredholm if and only if the operator $(W+H)_\varphi$ $\big($resp. $(W-H)_\varphi\big)$ enjoys the same property.

Proof. It runs immediately from Lemmas 3.1 and 3.2.

4. Relation between Wiener-Hopf-Hankel operators and Toeplitz-Hankel operators

Let us consider two bounded linear operators $T : X_1 \to X_2$ and $S : Y_1 \to Y_2$ acting between Banach spaces. The operators T and S are said to be *equivalent*, and we will denote this by $T \sim S$, if there are two boundedly invertible linear operators, $E : Y_2 \to X_2$ and $F : X_1 \to Y_1$, such that

$$T = E\, S\, F. \tag{17}$$

It directly follows from (17) that if two operators are equivalent, then they belong to the same regularity class. Namely, one of these operators is invertible, one-sided invertible, Fredholm, (properly) n-normal, (properly) d-normal or normally solvable, if and only if the other operator enjoys the same property.

An operator relation that generalizes the operator equivalence relation is the equivalence after extension relation. The operators T and S are said to be *equivalent after extension*, and we will denote this by $T \overset{*}{\sim} S$, if there exist two Banach spaces W and Z such that $T \oplus I_W$ and $S \oplus I_Z$ are equivalent operators, i.e.,

$$\begin{bmatrix} T & 0 \\ 0 & I_W \end{bmatrix} = E \begin{bmatrix} S & 0 \\ 0 & I_Z \end{bmatrix} F,$$

for invertible bounded linear operators $E : Y_2 \times Z \to X_2 \times W$ and $F : X_1 \times W \to Y_1 \times Z$. As we can easily see, the operator equivalence relation corresponds to the case where the extension spaces W and Z are chosen to be the trivial space (in the equivalence after extension relation). Like in the equivalence case, two equivalent after extension operators belong to the same regularity class.

Consider the *Cauchy singular integral operator S on* $L^2(\mathbb{R})$ given by

$$(S\varphi)(\xi) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{\varphi(x)}{x - \xi}\, dx, \ \xi \in \mathbb{R},$$

and the *orthogonal projection of* $L^2(\mathbb{R})$ *onto* $H^2_+(\mathbb{R})$

$$P = \frac{1}{2}\big(I_{L^2_+(\mathbb{R})} + S\big) : L^2(\mathbb{R}) \to H^2_+(\mathbb{R}).$$

With the help of Fourier transformations, it is possible to relate P and $P_+ = \ell_0 r_+$ as follows:

$$P = \mathcal{F} P_+ \mathcal{F}^{-1}.$$

Let $\mathbb{D}_+ = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{D}_- = \{z \in \mathbb{C} : |z| \geq 1\} \cup \{\infty\}$. Analogously to the definition of $H^\infty(\mathbb{C}_\pm)$ presented previously, $H^\infty(\mathbb{D}_\pm)$ denotes the set of all bounded and analytic functions in $\mathbb{D}_\pm$. Let $H^2(\mathbb{D}_+)$ denote the set of all functions ϕ which are analytic in $\mathbb{D}_+$ and such that

$$\sup_{r \in (0,1)} \int_0^{2\pi} |\phi(re^{i\theta})|^2\, d\theta < \infty,$$

and $H^2(\mathbb{D}_-)$ denote the set of all functions $\phi(z)$ $(z \in \mathbb{D}_-)$ for which $\phi(1/z)$ is a function in $H^2(\mathbb{D}_+)$. By a theorem of Fatou, we have that functions in $H^2(\mathbb{D}_\pm)$ have non-tangential limits almost everywhere on the unit circle $\mathbb{T}$ ($\mathbb{T} = \partial\mathbb{D}_\pm$). In this way, $H^2_\pm(\mathbb{T})$ represents the set of all functions on $\mathbb{T}$ that are non-tangential limits of elements in $H^2(\mathbb{D}_\pm)$. $H^2_\pm(\mathbb{T})$ are closed subspaces of $L^2(\mathbb{T})$.

Let $S_\mathbb{T}$ be the *Cauchy singular integral operator on* $L^2(\mathbb{T})$ defined by

$$(S_\mathbb{T}\varphi)(t) = \frac{1}{\pi i}\int_\mathbb{T} \frac{\varphi(\tau)}{\tau - t}\, d\tau,\ t \in \mathbb{T}.$$

Consider also the *orthogonal projection of* $L^2(\mathbb{T})$ *onto* $H^2_+(\mathbb{T})$,

$$P_\mathbb{T} = \frac{1}{2}(I_{L^2(\mathbb{T})} + S_\mathbb{T}) : L^2(\mathbb{T}) \to H^2_+(\mathbb{T}).$$

For any $\nu \in L^\infty(\mathbb{T})$, we will consider Toeplitz plus Hankel operators defined by

$$(T+H)_\nu = P_\mathbb{T}\nu P_\mathbb{T} + P_\mathbb{T}\nu J_\mathbb{T} P_\mathbb{T} : H^2_+(\mathbb{T}) \to H^2_+(\mathbb{T})$$

and Toeplitz minus Hankel operators defined by

$$(T-H)_\nu = P_\mathbb{T}\nu P_\mathbb{T} - P_\mathbb{T}\nu J_\mathbb{T} P_\mathbb{T} : H^2_+(\mathbb{T}) \to H^2_+(\mathbb{T}),$$

where $J_\mathbb{T} : \theta(t) \to \frac{1}{t}\theta\left(\frac{1}{t}\right)$, $t \in \mathbb{T}$, is the *reflection operator on* $\mathbb{T}$.

The aim of this subsection is to exhibit a relation between Wiener-Hopf-Hankel operators acting in L^2 Lebesgue spaces defined on the real line and Toeplitz-Hankel operators acting in H^2 Hardy spaces defined on the unit circle. To present such relation, we will need to use some isomorphisms that allow us to pass from the real line to the unit circle and vice-versa. Thus, let B_0 be an isometric isomorphism from $L^\infty(\mathbb{R})$ onto $L^\infty(\mathbb{T})$ (as well as from $H^\infty_+(\mathbb{R})$ onto $H^\infty_+(\mathbb{T})$) defined by

$$(B_0\phi)(t) = \phi\left(i\frac{1+t}{1-t}\right),\ t \in \mathbb{T}\backslash\{1\}$$

and with inverse

$$(B_0^{-1}\psi)(x) = \psi\left(\frac{x-i}{x+i}\right),\ x \in \mathbb{R}.$$

Consider also the isometric isomorphism of $L^2(\mathbb{T})$ onto $L^2(\mathbb{R})$ (as well as of $H^2_+(\mathbb{T})$ onto $H^2_+(\mathbb{R})$) given by

$$(B\varphi)(x) = \frac{\sqrt{2}}{x+i}\varphi\left(\frac{x-i}{x+i}\right),\ x \in \mathbb{R}.$$

The inverse of B is given by

$$(B^{-1}\psi)(t) = \frac{i\sqrt{2}}{1-t}\psi\left(i\frac{1+t}{1-t}\right),\ t \in \mathbb{T}\backslash\{1\}.$$

Additionally, it is also useful to observe that for any $\phi \in L^{\infty}(\mathbb{R})$ it holds

$$B^{-1}\phi \cdot B = (B_0\phi) I_{L^2(\mathbb{R})}. \tag{18}$$

Using now the isomorphism B and its inverse, we see how orthogonal projections $P_{\mathbb{T}}$ and P, and reflection operators $J_{\mathbb{T}}$ and J can be related:

$$P_{\mathbb{T}} = B^{-1} P B, \tag{19}$$

$$JB = -BJ_{\mathbb{T}}. \tag{20}$$

After all this background on spaces and operators, we are now in a position to present some relations between Wiener-Hopf-Hankel operators and Toeplitz-Hankel operators.

Lemma 4.1 *Let $\phi \in L^{\infty}(\mathbb{R})$. The Wiener-Hopf plus Hankel operator*

$$(W+H)_{\phi} \ : \ L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+)$$

and the Toeplitz minus Hankel operator

$$(T-H)_{B_0\phi} \ : \ H^2_+(\mathbb{T}) \to H^2_+(\mathbb{T})$$

are equivalent operators.

Proof. According to (1), (2) and (3), we have that

$$(W+H)_{\phi} = r_+ \mathcal{F}^{-1}\phi \cdot \mathcal{F}(I_{L^2_+(\mathbb{R})} + J) : L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+).$$

From this it follows that

$$\begin{aligned}(W+H)_{\phi} \sim \ell_0\,(W+H)_{\phi} &= \ell_0 r_+ \mathcal{F}^{-1}\phi \cdot \mathcal{F}(I_{L^2_+(\mathbb{R})} + J) \\ &= P_+ \mathcal{F}^{-1}\phi \cdot \mathcal{F}(I_{L^2_+(\mathbb{R})} + J) P_+ : L^2_+(\mathbb{R}) \to L^2_+(\mathbb{R}).\end{aligned}$$

More than this, we have

$$\begin{aligned}P_+ \mathcal{F}^{-1}\phi \cdot \mathcal{F}(I_{L^2_+(\mathbb{R})} + J) P_+ \sim \mathcal{F} P_+ \mathcal{F}^{-1}\phi \cdot \mathcal{F}(I_{L^2_+(\mathbb{R})} + J) P_+ \mathcal{F}^{-1} \\ : \mathcal{F}L^2_+(\mathbb{R}) \to \mathcal{F}L^2_+(\mathbb{R}),\end{aligned}$$

which, due to the fact that $\mathcal{F}L^2_+(\mathbb{R}) = H^2_+(\mathbb{R})$, leads to

$$(W+H)_{\phi} \sim \mathcal{F} P_+ \mathcal{F}^{-1}\phi \cdot \mathcal{F}(I_{L^2_+(\mathbb{R})} + J) P_+ \mathcal{F}^{-1} : H^2_+(\mathbb{R}) \to H^2_+(\mathbb{R}). \tag{21}$$

Thus, taking into account that $J\mathcal{F} = \mathcal{F}J$, (21) yields that

$$(W+H)_{\phi} \sim P\phi \cdot (I_{L^2_+(\mathbb{R})} + J) P : H^2_+(\mathbb{R}) \to H^2_+(\mathbb{R}).$$

Consequently, one obtains

$$
\begin{aligned}
(W+H)_\phi \sim B^{-1}P\phi \cdot (I_{L^2_+(\mathbb{R})} + J)PB = \\
P_{\mathbb{T}}(B^{-1}\phi \cdot B)P_{\mathbb{T}} + P_{\mathbb{T}}(B^{-1}\phi \cdot JB)P_{\mathbb{T}} : H^2_+(\mathbb{T}) \to H^2_+(\mathbb{T}), \qquad (22)
\end{aligned}
$$

having in consideration (19). Now, we are able to rewrite the last operator in (22) as

$$
(T-H)_{B_0\phi} = P_{\mathbb{T}}(B_0\phi)P_{\mathbb{T}} - P_{\mathbb{T}}(B_0\phi)J_{\mathbb{T}}P_{\mathbb{T}} : H^2_+(\mathbb{T}) \to H^2_+(\mathbb{T}),
$$

just by observing that it holds (18) and (20).

Corollary 4.1 *The operators $(W+H)_\phi$ and $(T-H)_{B_0\phi}$ have the same regularity properties.*

Proof. The statement is a direct consequence of the equivalence relation presented in Lemma 4.1.

Similarly, one can also relate Wiener-Hopf minus Hankel operators with Toeplitz plus Hankel operators.

Lemma 4.2 *Let $\phi \in L^\infty(\mathbb{R})$. The Wiener-Hopf minus Hankel operator*

$$
(W-H)_\phi \ : \ L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+)
$$

and the Toeplitz plus Hankel operator

$$
(T+H)_{B_0\phi} \ : \ H^2_+(\mathbb{T}) \to H^2_+(\mathbb{T})
$$

are equivalent operators.

Proof. Similar to the proof of Theorem 4.1.

Corollary 4.2 *The operators $(W-H)_\phi$ and $(T+H)_{B_0\phi}$ have the same regularity properties.*

Proof. The result follows directly from the equivalence relation stated in Lemma 4.2.

5. A stronger version of the Sarason's type theorem for Wiener-Hopf-Hankel operators

From [20], and with the help of certain operator relations, we achieve the following Sarason's type theorem for Wiener-Hopf-Hankel operators with semi-almost periodic Fourier symbols, and acting between L^2 Lebesgue spaces.

Theorem 5.1 *Let $\phi \in \mathcal{G}SAP$.*

(a) *If $\kappa(\phi_l)+\kappa(\phi_r) < 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are right-invertible. Moreover, at least one of these operators is properly d-normal.*

(b) *If $\kappa(\phi_l)+\kappa(\phi_r) > 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are left-invertible. In addition, at least one of these operators is properly n-normal.*

(c) *If $\kappa(\phi_l)+\kappa(\phi_r) = 0$ and $\Re e\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \neq 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are Fredholm operators.*

(d) *If $\kappa(\phi_l)+\kappa(\phi_r) = 0$ and $\Re e\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) = 0$, then at least one of the operators $(W+H)_\phi$ and $(W-H)_\phi$ is not normally solvable.*

Following now the main idea of the approach of R. V. Duduchava and A. I. Saginashvili (cf. [8]), we reach to a different version of the Sarason's type theorem presented before. In what follows, let $PC(\mathbb{T})$ denote the C^*-algebra of all *piecewise continuous functions on* $\mathbb{T}$, i.e., functions $\theta \in L^\infty(\mathbb{T})$ for which the one-sided limits

$$\theta(\tau - 0) = \lim_{\varepsilon \to 0^-} \theta(\tau e^{i\varepsilon}), \quad \theta(\tau + 0) = \lim_{\varepsilon \to 0^+} \theta(\tau e^{i\varepsilon})$$

exist for each $\tau \in \mathbb{T}$.

Theorem 5.2 *Let $\phi \in \mathcal{G}SAP$.*

(a) *If $\kappa(\phi_l) > 0$ and $\kappa(\phi_r) > 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are properly n-normal and left-invertible.*

(b) *If $\kappa(\phi_l) < 0$ and $\kappa(\phi_r) < 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are properly d-normal and right-invertible.*

(c) *Let $\kappa(\phi_l) = \kappa(\phi_r) = 0$.*

  (i) *$(W+H)_\phi$ is a Fredholm operator if and only if $\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \notin \mathbb{Z} + \frac{1}{4}$.*

  (ii) *$(W-H)_\phi$ is a Fredholm operator if and only if $\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \notin \mathbb{Z} - \frac{1}{4}$.*

Proof. By the well-known characterization of SAP (cf. (4)), considering $u \in C(\mathbb{R})$ for which $u(-\infty) = 0$ and $u(+\infty) = 1$, there exists $\phi_l, \phi_r \in AP$ and $\phi_0 \in C_0(\dot{\mathbb{R}})$ such that

$$\phi = (1-u)\phi_l + u\phi_r + \phi_0.$$

Because $\phi \in \mathcal{G}SAP$, it results that $\phi_l, \phi_r \in \mathcal{G}AP$. Consequently, taking into consideration Bohr's Theorem and the definition of the geometric mean value, it follows that

$$\phi_l = e_{\kappa(\phi_l)}\mathbf{d}(\phi_l)e^{\omega_l}$$

and

$$\phi_r = e_{\kappa(\phi_r)}\mathbf{d}(\phi_r)e^{\omega_r}$$

with $\omega_l, \omega_r \in AP$ and $M(\omega_l) = M(\omega_r) = 0$ (and obviously $\mathbf{d}(\phi_l)\mathbf{d}(\phi_r) \neq 0$). Thus

$$\phi = (1-u)\mathbf{d}(\phi_l)e_{\kappa(\phi_l)}e^{\omega_l} + u\mathbf{d}(\phi_r)e_{\kappa(\phi_r)}e^{\omega_r} + \phi_0.$$

Consider now

$$\varphi = (1-u)\,\mathbf{d}(\phi_l)e_{\kappa(\phi_l)} + u\,\mathbf{d}(\phi_r)e_{\kappa(\phi_r)} + \varphi_0,$$

where $\varphi_0 \in C_0(\dot{\mathbb{R}})$ is chosen in order to $\varphi \in \mathcal{G}SAP$.

In the first part of this proof, we will prove that $(W+H)_\phi$ is properly n-normal, properly d-normal or Fredholm operator if and only if $(W+H)_\varphi$ enjoys the same property.

Suppose that $(W+H)_\phi$ is properly n-normal. Then, by a well-known property concerning to the index of semi-Fredholm operators (cf. [12, §6.7]), there exists an $\delta > 0$ such that

$$\mathrm{Ind}\,(W+H)_\varrho = \mathrm{Ind}\,(W+H)_\phi,$$

for all operators $(W+H)_\varrho$ with Fourier symbols ϱ satisfying the condition $\|\phi-\varrho\|_\infty < \delta$. This means that there exists an $\delta > 0$ such that $(W+H)_\varrho$ is properly n-normal for ϱ satisfying $\|\phi - \varrho\|_\infty < \delta$. Let $p_l^\pm, p_r^\pm \in AP^\pm$ be almost periodic polynomials such that

$$M(p_l^\pm) = M(p_l^\pm) = 0, \tag{23}$$

$$\|\omega_l - p_l^- - p_l^+\|_\infty < \varepsilon, \tag{24}$$

$$\|\omega_r - p_r^- - p_r^+\|_\infty < \varepsilon, \tag{25}$$

for $\varepsilon > 0$. Consider now

$$\zeta = (1-u)\,\mathbf{d}(\phi_l)e^{p_l^-}e_{\kappa(\phi_l)}e^{p_l^+} + u\,\mathbf{d}(\phi_r)e^{p_r^-}e_{\kappa(\phi_r)}e^{p_r^+} + \phi_0.$$

If, in (24) and (25), we chose a sufficiently small ε, then we obtain $\|\phi-\zeta\|_\infty < \delta$, which assures that $(W+H)_\zeta$ is properly n-normal. In virtue of this, it holds that $\zeta \in \mathcal{G}L^\infty(\mathbb{R})$. Since $\zeta \in SAP$ and being SAP an inverse closed subalgebra in $L^\infty(\mathbb{R})$, it follows that $\zeta \in \mathcal{G}SAP$. Moreover, due to (23), we have the guarantee that $\mathbf{d}\big(e^{p_l^\pm}\big) = \mathbf{d}\big(e^{p_r^\pm}\big) = 1$. So, ζ and φ are in the conditions of Proposition 3.2, and therefore, since $(W+H)_\zeta$ is properly n-normal, we conclude that $(W+H)_\varphi$ is properly n-normal.

Assume now that $(W+H)_\varphi$ is properly n-normal. Consider

$$\gamma = (1-u)\,\mathbf{d}(\phi_l)e_{\kappa(\phi_l)}e^{\sigma_l} + u\,\mathbf{d}(\phi_r)e_{\kappa(\phi_r)}e^{\sigma_r} + \varphi_0, \tag{26}$$

with $\sigma_l, \sigma_r \in AP$. Once again, by a well-known property concerning to the index of semi-Fredholm operators (cf. [12, §6.7]), there exists an $\delta > 0$ such that $(W+H)_\gamma$ is properly n-normal if in (26) we choose $\|\sigma_l\|_\infty < \delta$ and $\|\sigma_r\|_\infty < \delta$. In the same way as before, if $(W+H)_\gamma$ is properly n-normal, then $\gamma \in \mathcal{G}SAP$. Define now

$$\eta = (1-u)\,\mathbf{d}(\phi_l) e^{q_l^-} e_{\kappa(\phi_l)} e^{\sigma_l} e^{q_l^+} + u\,\mathbf{d}(\phi_r) e^{q_r^-} e_{\kappa(\phi_r)} e^{\sigma_r} e^{q_r^+} + \eta_0, \qquad (27)$$

where $q_l^\pm, q_r^\pm \in AP^\pm$ are almost periodic polynomials such that $M(q_l^\pm) = M(q_l^\pm) = 0$ (consequently $\mathbf{d}(e^{q_l^\pm}) = \mathbf{d}(e^{q_r^\pm}) = 1$) and $\eta_0 \in C_0(\dot{\mathbb{R}})$ is so that $\eta \in \mathcal{G}SAP$. According to Proposition 3.2, we have that $(W+H)_\eta$ is properly n-normal. If, in (27), we choose $\eta_0 = \phi_0$ and $q_l^\pm, q_r^\pm$ such that

$$\omega_l = \sigma_l + q_l^- + q_l^+ \text{ and } \omega_r = \sigma_r + q_r^- + q_r^+,$$

we obtain $\eta = \phi$, and therefore, we get that $(W+H)_\phi$ is properly n-normal (note that the last identities put additional conditions to σ_l and σ_r). At this moment, we proved that $(W+H)_\phi$ is properly n-normal if and only if $(W+H)_\varphi$ is also properly n-normal. Analogously, it can be shown that $(W+H)_\phi$ is properly d-normal (resp. Fredholm) if and only if $(W+H)_\varphi$ is properly d-normal (resp. Fredholm). Using the same reasoning, we obtain that $(W-H)_\phi$ is properly n-normal, properly d-normal or Fredholm operator if and only if $(W-H)_\varphi$ enjoys the same property.

Due to this, and in a second part, we will prove the theorem for the Wiener-Hopf-Hankel operators $(W\pm H)_\varphi$.

Suppose that $\kappa(\phi_l) < 0$ and $\kappa(\phi_r) < 0$. Due to a result of Sarason (cf. [24, 25]), it follows $\varphi \in C(\dot{\mathbb{R}}) + H^\infty_-(\mathbb{R})$, and consequently H_φ is a compact operator. From the proof of Theorem 3.9 in [2], we have that W_φ is properly d-normal. Therefore, since $(W\pm H)_\varphi$ is the sum/difference of a properly d-normal Wiener-Hopf operator with a compact Hankel operator, it results that $(W\pm H)_\varphi$ is a properly d-normal operator (cf. [11, Theorem 15.3]), which completes the proof of part (b) of the theorem. Part (a) derives from part (b) by passage to adjoint operators, and in this way we prove that $(W\pm H)_\varphi$ is properly n-normal if $\kappa(\phi_l) > 0$ and $\kappa(\phi_r) > 0$.

Finally, consider $\kappa(\phi_l) = \kappa(\phi_r) = 0$. In this case, $\varphi \in \mathcal{G}C(\mathbb{R})$ seeing that

$$\varphi = (1-u)\,\mathbf{d}(\phi_l) + u\,\mathbf{d}(\phi_r) + \varphi_0,$$

with $u \in C(\mathbb{R})$ and $\varphi_0 \in C_0(\dot{\mathbb{R}})$. Thus φ can be viewed as a piecewise continuous function with a single jump at ∞. Using the results of E. L. Basor and T. Ehrhardt on Toeplitz plus Hankel operators with $PC(\mathbb{T})$ symbols, we will see how one can prove part (c). We will start by proving the assertion for the Wiener-Hopf minus Hankel operator $(W-H)_\varphi$, and then we will proceed by proving the corresponding one for the case of the Wiener-Hopf plus Hankel operator $(W+H)_\varphi$. Recall (from Lemma 4.2) that the Wiener-Hopf minus Hankel operator $(W-H)_\varphi$ and the Toeplitz plus Hankel operator $(T+H)_{B_0\varphi}$ are equivalent operators. In [1, Corollary 3.2], we find the following result concerning to the Fredholm property of Toeplitz plus Hankel operators with $PC(\mathbb{T})$ symbols: for $\theta \in PC(\mathbb{T})$, the Toeplitz plus Hankel operator $(T+H)_\theta$ is a Fredholm operator if and only if $\theta(\tau \pm 0) \neq 0$ for each $\tau \in \mathbb{T}$ and

$$\frac{1}{2\pi} \arg \left(\frac{\theta(\tau - 0)\theta(\bar{\tau} - 0)}{\theta(\tau + 0)\theta(\bar{\tau} + 0)} \right) \notin \mathbb{Z} + \frac{1}{2} \quad \text{for each } \tau \in \mathbb{T}_+, \tag{28}$$

$$\frac{1}{2\pi} \arg \left(\frac{\theta(\tau - 0)}{\theta(\tau + 0)} \right) \notin \mathbb{Z} + \frac{\tau}{4} \quad \text{for each } \tau \in \{-1, 1\}. \tag{29}$$

Here, and in what follows, $\mathbb{T}_+ = \{t \in \mathbb{T} : \Im m\, t > 0\}$. In our case,

$$\begin{aligned} \theta(t) &= B_0\varphi(t) \\ &= \left(1 - u\left(i\frac{1+t}{1-t}\right)\right) \mathbf{d}(\phi_l) + u\left(i\frac{1+t}{1-t}\right) \mathbf{d}(\phi_r) + \varphi_0\left(i\frac{1+t}{1-t}\right), \ t \in \mathbb{T}\backslash\{1\}. \end{aligned}$$

Since $\varphi \in \mathcal{G}C(\mathbb{R})$, it follows that $\theta(\tau \pm 0) \neq 0$ for each $\tau \in \mathbb{T}$. Taking into account that φ is a piecewise continuous function with a single jump at ∞, it yields that $\theta(\tau - 0) = \theta(\tau + 0)$ and $\theta(\bar{\tau} - 0) = \theta(\bar{\tau} + 0)$ for all $\tau \in \mathbb{T}_+$. Consequently, it holds that

$$\frac{1}{2\pi} \arg \left(\frac{\theta(\tau - 0)\theta(\bar{\tau} - 0)}{\theta(\tau + 0)\theta(\bar{\tau} + 0)} \right) \in \mathbb{Z} \quad \text{for each } \tau \in \mathbb{T}_+,$$

which implies that

$$\frac{1}{2\pi} \arg \left(\frac{\theta(\tau - 0)\theta(\bar{\tau} - 0)}{\theta(\tau + 0)\theta(\bar{\tau} + 0)} \right) \notin \mathbb{Z} + \frac{1}{2} \quad \text{for each } \tau \in \mathbb{T}_+. \tag{30}$$

Moreover, due to the equality $\theta(-1 - 0) = \theta(-1 + 0)$, it also holds that

$$\frac{1}{2\pi} \arg \left(\frac{\theta(-1 - 0)}{\theta(-1 + 0)} \right) \in \mathbb{Z}, \tag{31}$$

and therefore

$$\frac{1}{2\pi} \arg \left(\frac{\theta(-1 - 0)}{\theta(-1 + 0)} \right) \notin \mathbb{Z} - \frac{1}{4}.$$

Then, since

$$\begin{aligned} \theta(1 - 0) &= (1 - u(+\infty))\, \mathbf{d}(\phi_l) + u(+\infty)\, \mathbf{d}(\phi_r) + \varphi_0(+\infty) = \mathbf{d}(\phi_r), \\ \theta(1 + 0) &= (1 - u(-\infty))\, \mathbf{d}(\phi_l) + u(-\infty)\, \mathbf{d}(\phi_r) + \varphi_0(-\infty) = \mathbf{d}(\phi_l), \end{aligned}$$

it follows that

$$\frac{1}{2\pi} \arg \left(\frac{\theta(1 - 0)}{\theta(1 + 0)} \right) = \frac{1}{2\pi} \arg \left(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_l)} \right). \tag{32}$$

Therefore, by [1, Corollary 3.2] stated above (cf. (28) and (29)), we conclude that the Toeplitz plus Hankel operator $(T+H)_{B_0\varphi}$ is a Fredholm operator if and only if

$$\frac{1}{2\pi} \arg \left(\frac{\mathbf{d}(\phi_r)}{\mathbf{d}(\phi_l)} \right) \notin \mathbb{Z} + \frac{1}{4}.$$

In this way, the Wiener-Hopf minus Hankel operator $(W-H)_\varphi$ is also a Fredholm operator (since $(W-H)_\varphi$ and $(T+H)_{B_0\varphi}$ are equivalent operators) if and only if

$$\frac{1}{2\pi}\arg\begin{pmatrix}\mathbf{d}(\phi_l)\\ \mathbf{d}(\phi_r)\end{pmatrix} = -\frac{1}{2\pi}\arg\begin{pmatrix}\mathbf{d}(\phi_r)\\ \mathbf{d}(\phi_l)\end{pmatrix} \notin \mathbb{Z}-\frac{1}{4}.$$

Now that we have already proved the Fredholm characterization of the Wiener-Hopf minus Hankel operator $(W-H)_\varphi$, we will prove the Fredholm characterization of the Wiener-Hopf plus Hankel operator $(W+H)_\varphi$. From Lemma 4.1, we have that the Wiener-Hopf plus Hankel operator $(W+H)_\varphi$ and the Toeplitz minus Hankel operator $(T-H)_{B_0\varphi}$ are equivalent operators. Using [10, Theorem A.4], we obtain a similar result of [1, Corollary 3.2] for Toeplitz minus Hankel operators. In this case, the Fredholm characterization is the following: for $\theta\in PC(\mathbb{T})$, the Toeplitz minus Hankel operator $(T-H)_\theta$ is a Fredholm operator if and only if $\theta(\tau\pm 0)\neq 0$ for each $\tau\in\mathbb{T}$ and

$$\frac{1}{2\pi}\arg\begin{pmatrix}\theta(\tau-0)\theta(\bar\tau-0)\\ \theta(\tau+0)\theta(\bar\tau+0)\end{pmatrix}\notin\mathbb{Z}+\frac{1}{2}\quad\text{for each } \tau\in\mathbb{T}_+,$$
$$\frac{1}{2\pi}\arg\begin{pmatrix}\theta(\tau-0)\\ \theta(\tau+0)\end{pmatrix}\notin\mathbb{Z}-\frac{\tau}{4}\quad\text{for each } \tau\in\{-1,1\}.$$

From (30) and (31), we already know that

$$\frac{1}{2\pi}\arg\begin{pmatrix}\theta(\tau-0)\theta(\bar\tau-0)\\ \theta(\tau+0)\theta(\bar\tau+0)\end{pmatrix}\notin\mathbb{Z}+\frac{1}{2}\quad\text{for each } \tau\in\mathbb{T}_+,$$
$$\frac{1}{2\pi}\arg\begin{pmatrix}\theta(-1-0)\\ \theta(-1+0)\end{pmatrix}\notin\mathbb{Z}+\frac{1}{4}.$$

Therefore, from (32), we conclude that the Toeplitz minus Hankel operator $(T–H)_{B_0\varphi}$ is a Fredholm operator if and only if

$$\frac{1}{2\pi}\arg\begin{pmatrix}\mathbf{d}(\phi_r)\\ \mathbf{d}(\phi_l)\end{pmatrix}\notin\mathbb{Z}-\frac{1}{4},$$

i.e., if and only if

$$\frac{1}{2\pi}\arg\begin{pmatrix}\mathbf{d}(\phi_l)\\ \mathbf{d}(\phi_r)\end{pmatrix} = -\frac{1}{2\pi}\arg\begin{pmatrix}\mathbf{d}(\phi_r)\\ \mathbf{d}(\phi_l)\end{pmatrix} \notin \mathbb{Z}+\frac{1}{4}. \tag{33}$$

Finally, due to the equivalence between the Wiener-Hopf plus Hankel operator $(W+H)_\varphi$ and the Toeplitz minus Hankel operator $(T-H)_{B_0\varphi}$, we have that the Wiener-Hopf plus Hankel operator $(W+H)_\varphi$ is a Fredholm operator if and only if (33) holds true.

After having proved the second version of the Sarason's type theorem, we are now in conditions to present the main result of this paper, that is, a stronger version of the Sarason's type theorems provided in Theorems 5.1 and 5.2.

Theorem 5.3 *Let $\phi \in \mathcal{G}SAP$.*

(a) *If $\kappa(\phi_l) + \kappa(\phi_r) < 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are right-invertible, and at least one of these operators is properly d-normal. Moreover, if $\kappa(\phi_l) < 0$ and $\kappa(\phi_r) < 0$, then both operators are properly d-normal.*

(b) *If $\kappa(\phi_l) + \kappa(\phi_r) > 0$, then $(W+H)_\phi$ and $(W-H)_\phi$ are left-invertible, and at least one of these operators is properly n-normal. Moreover, if $\kappa(\phi_l) > 0$ and $\kappa(\phi_r) > 0$, then both operators are properly n-normal.*

(c) *If $\kappa(\phi_l)+\kappa(\phi_r) = 0$ and $\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \notin \mathbb{Z}\pm\frac{1}{4}$, then $(W+H)_\phi$ and $(W-H)_\phi$ are Fredholm operators.*

(d) *If $\kappa(\phi_l)+\kappa(\phi_r) = 0$ and $\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \in \mathbb{Z}+\frac{1}{4}$ or $\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \in \mathbb{Z}-\frac{1}{4}$, then at least one of the operators $(W+H)_\phi$ and $(W-H)_\phi$ is not normally solvable.*

(e) *Let $\kappa(\phi_l) = \kappa(\phi_r) = 0$.*

  (i) *$(W+H)_\phi$ is a Fredholm operator if and only if $\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \notin \mathbb{Z}+\frac{1}{4}$.*

  (ii) *$(W-H)_\phi$ is a Fredholm operator if and only if $\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \notin \mathbb{Z}-\frac{1}{4}$.*

  (iii) *$(W+H)_\phi$ is a Fredholm operator and $(W-H)_\phi$ is not a normally solvable operator if and only if $\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \in \mathbb{Z}-\frac{1}{4}$.*

  (iv) *$(W-H)_\phi$ is a Fredholm operator and $(W+H)_\phi$ is not a normally solvable operator if and only if $\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \in \mathbb{Z}+\frac{1}{4}$.*

Proof. All the assertions follow immediately from Theorems 5.1 and 5.2, if noticing that the condition

$$\Re e\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \neq 0$$

which appears in Theorem 5.1 is equivalent to the condition

$$\frac{1}{2\pi}\arg\left(\frac{\mathbf{d}(\phi_l)}{\mathbf{d}(\phi_r)}\right) \notin \mathbb{Z}\pm\frac{1}{4}.$$ □

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◇ A. P. Nolasco
Research Unit *Mathematics and Applications*
Department of Mathematics
University of Aveiro
3810-193 Aveiro, PORTUGAL
anolasco@mat.ua.pt

◇ L. P. Castro
Research Unit *Mathematics and Applications*
Department of Mathematics
University of Aveiro
3810-193 Aveiro, PORTUGAL
castro@ua.pt