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On the Controllability of Maxwell's Equations in a Class of Complex Media

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Antonis Kalliterakis (20/11/1957–19/11/2006), *in memoriam.*

Abstract

We describe some results on the problem of the controllability of the electromagnetic field in a 3–dimensional bounded body consisting of a complex medium which has specific constitutive relations.

Keywords: Maxwell's equations, nonlocal constitutive relations, controllability, chiral and complex media.

Introduction

We consider a 3–dimensional body subject to an electromagnetic field; we study the internal effect of a modification of the magnetic field on part on the boundary of the body, which is assumed to have specific behaviour with respect to the electromagnetic field inside. In subsection 1.1 we describe the different electromagnetic behaviour that we consider, and in (1.2) and (1.4) we formulate some controllability problems that we discuss in this work.

1. Position of the problem: the model and our controllability framework

1.1. *Electromagnetic relations*

Electromagnetic fields are governed by the Maxwell postulates in vacuum as well as in any material medium. At the macroscopic level they can be stated as

$$\begin{cases} \mathrm{div} B(x,t) = 0 \\ \mathrm{curl} E(x,t) = -\frac{\partial}{\partial t} B(x,t), \\ \mathrm{div} D(x,t) = \rho(x,t) \\ \mathrm{curl} H(x,t) = \frac{\partial}{\partial t} D(x,t) + \mathcal{J}(x,t), \end{cases}$$

where E, H are the electric and magnetic fields, D, B are the electric and magnetic flux densities, and ρ, $\mathcal{J}$ are the external impressed electric charge and electric current source densities, respectively.

Constitutive relations must be prescribed to relate the matter–derived fields D and H to the basic fields E, B in any material medium. The construction of these relations is primarily phenomenological, though certain epistemology mandated properties must be adhered to. We will here introduce constitutive relations for B and D which model some delayed nonlocal mean phenomena in time, localized in the close past. So, we generally consider

$$D(x,t) = \varepsilon_0 E(x,t) + \varepsilon_0 \int_0^t \chi^e(\tau) E(x, t-\tau) d\tau \tag{1.1}$$

$$H(x,t) = \tfrac{1}{\mu_0} B(x,t) - \int_0^t \chi^m(\tau) H(x, t-\tau) d\tau, \tag{1.2}$$

where ε_0, μ_0 are the permittivity and permeability of the vacuum, χ^e is the dielectric susceptibility scalar kernel, χ^m is the magnetic susceptibility scalar kernel.

These general constitutive relations are usually also used on small amplitude of time where it appears that they might be reduced to, according to phenomenological modelling,

$$D(x,t) = \varepsilon_0 (E + \beta_0 \mathrm{curl} E) \tag{1.3}$$

$$B(x,t) = \varepsilon_0 (H + \beta_0 \mathrm{curl} H), \tag{1.4}$$

and are known to be the Drude-Born-Fedorov (DBF) approximation.

A full description of the most usual and used constitutive relations can be found in [24], and some related comments further on in our work.

1.2. *The controllability problem in the non-harmonic domain*

We consider the controllability problem for the Maxwell system described above with strong restrictions. Let Ω be a bounded connected regular open set in $\mathbb{R}^3$ with a connected boundary $\Gamma = \partial\Omega$. Let Γ_0 be a part of Γ with positive measure. We want to study the effect of the tangential part of the magnetic field H (the control) on Γ_0 to the electromagnetic field (E, H) in Ω.

We consider the problem

$$\mathrm{div} B(x,t) = 0 \tag{1.5}$$

$$\mathrm{curl} E(x,t) = -\frac{\partial}{\partial t} B(x,t), \tag{1.6}$$

$$\mathrm{div} D(x,t) = 0 \tag{1.7}$$

$$\mathrm{curl} H(x,t) = \frac{\partial}{\partial t} D(x,t) \tag{1.8}$$

with the specific constitutive relations

$$D(x,t) = \varepsilon_0 E(x,t) + \varepsilon_0 \int_0^t E(x,\tau)\chi^e(t-\tau)d\tau \tag{1.9}$$

$$B(x,t) = \mu_0 H(x,t) + \mu_0 \int_0^t H(x,\tau)\chi^m(t-\tau)d\tau, \tag{1.10}$$

$$E.n = 0 \text{ on } \Sigma := \Gamma \times [0,T], \tag{1.11}$$

$$H \wedge n = J \text{ on } \Sigma_0 := \Gamma_0 \times [0,T], \tag{1.12}$$

$$H \wedge n = 0 \text{ on } \Sigma \setminus \Sigma_0, \tag{1.13}$$

and of course initial conditions

$$E(t=0) = E_0 \tag{1.14}$$

$$H(t=0) = H_0 \tag{1.15}$$

As usual n denotes the unit outward normal to Σ.

It is straightforward that this simplified model can be interpreted in the framework of a discretization in time for which one neglects the long–time effect of the convolution operator.

We assume that ε_0, μ_0, are positive constant numbers. Common physical assumptions lead to assume that $\chi^{e,m}$ are small with respect to their own dimension.

The question addressed here is the problem of the controllability of (P):

Assume that we have a function (Hilbert, in our framework) space $\mathcal{H}$ such that for $(E_0, H_0) \in \mathcal{H}$ the problem (P) consisting of (1.5)...(1.15) is well–posed in $\mathcal{H}$ for J in an adapted function space $\mathcal{C}$. Given any (E_0, H_0) and (E_1, H_1) in $\mathcal{H}$, can we find $J \in \mathcal{C}$ such that one has (1.14), (1.15), along with $E(t=T) = E_1$ and $H(t=T) = H_1$?

Definition : If so, we will say that (P) is exactly controllable in $\mathcal{H}$ from $t = 0$ to $t = T$.

It is now well known that when the target (E_1, H_1) is null, then the question addressed above reduces (and in fact is equivalent) to the observability of the adjoint system in $\mathcal{H}'$ (in this case one speaks about exact zero controllability). Due to the reversibility of the Maxwell's equations, even with the nonlocal terms considered here, it is clear that provided the free (i.e. when $J = 0$) evolution system is well–posed, the exact zero controllability is equivalent to the exact controllability to any final state. We will show in the present work, that we have some reversibility of our system which yields, provided that one gets an observability inequality, the exact controllability of the system.

In this paper we give partial positive answers to this question, under different assumptions. Precisely we prove that if the part of the boundary Σ_0 observes $(0,T) \times \Omega$

in the sense of the wave equation (see e.g. [35] for a definition and useful references) then the system (1.5)–(1.15) is exactly controllable. We give a complete proof through the multiplier method in the case when $\Gamma_0 = \Gamma$ as in [30], and indicate easy modifications of the result given in [35] in the case $\Gamma_0 \neq \Gamma$.

Major breakthroughs in the treatment of these type of controllability problems appeared in the linear case with the introduction of H.U.M. method (see [25]) on the one hand, and the use of micro–local analysis (see [7] and [37]) on the other hand, and in a less generalized manner the so–called extension method (see [38]). The first two deal with obtaining observability properties and inequalities for the so–called adjoint system. Roughly speaking positivity in the adjoint system is equivalent to the density of some operator, whereas coercivity in the adjoint system is equivalent to the surjectivity of this operator, namely of the map "control"→"final state".

More recent developments have been obtained by an extensive use of the Carleman inequalities to obtain these observability estimates; they have been derived in the case of Maxwell's equation in, for example, [17], but using them in our context does not seem at least relevant, and certainly not obvious. Specific works devoted to the boundary controllability of the Maxwell's equations have involved different type of models as well as for the constitutive relations, as for the type of the sought solutions. The case of classical Maxwell's equations has been essentially treated in [20], [21], [22], [28], [35], [40], and extended to non–constant coefficients in [30]. In [13] (see also [11]) it has been proved that for the time Drude–Born–Fedorov approximation the Maxwell's equations are not approximately controllable, and that for a simple bilinear medium exact controllability occurs, whereas in [14] some approximate results for the Drude–Born–Fedorov model are given in the time harmonic case, reflecting more observability determinations than controllability itself.

1.3. *The choice of the constitutive relations*

One of the most general models of Maxwell's equations, regardless of boundary conditions (which might by themselves generate great difficulties [23]), is the following

$$D = \varepsilon_0 E + \beta H + \varepsilon_0 \int_0^{+\infty} \chi^e(\tau) E(t-\tau)d\tau + \int_0^{+\infty} \chi^c(\tau) H(t-\tau)d\tau, \quad (1.16)$$

$$B = \mu_0 H + \alpha E + \mu_0 \int_0^{+\infty} \chi^m(\tau) B(t-\tau)d\tau + \int_0^{+\infty} \chi^c(\tau) E(t-\tau)d\tau. \quad (1.17)$$

When $\chi^c \neq 0$ the media are said to be chiral; they have been extensively studied when $\alpha = \beta = 0$ in the frequency domain, where a chiral parameter describing left–handed and right–handed different behaviour is employed: see e.g. [1], [2], [3], [4], [26], [31], [36]. In the time domain the related literature is less extensive, see e.g. [5], [6], [8], [11], [13], [19], [39]. The absence of bilateral symmetry in the natural

world was first recognized historically through an appreciation of the mirror–image complementarity of enantiomeric molecules and the predominance of only one of the optical isomers among the organic natural products. Initially, the appreciation of handed complementation of enantiomers was confined to their gross morphological shape in the mid 19th century, and was extended to their detailed stereochemistry only a quarter of a century later. The prevalence of only one of the two possible mirror–image enantiomers among natural organic products led Pasteur [34] to postulate, and search for, the dissymmetrical forces of nature by introducing geometrical ideas into chemistry. The notion of chirality has played a *sine qua non* role to the study of optical activity.
When $\chi^e = \chi^m = \chi^c = 0$ the medium is said to be nondispersive bilinear. In this paper we will consider the case when $\alpha = \beta = 0$ and $\chi^c = 0$. Moreover we will take into account the following approximation in (1.17): it is likely, but not obvious, that if χ^m is small, at least in the first order we can replace (1.17) by

$$B = \mu_0 H + \mu_0^2 \int_0^{+\infty} \chi^m(\tau) H(t-\tau) d\tau, \tag{1.18}$$

and it seems quite natural to consider past effects only after the initial time from which we consider the problem. This assumption leads us to consider (after rescaling the coefficient χ^m) the constitutive relation (1.10).

The case when $\alpha = \beta = 0$ in (1.16) and (1.17) has been studied in the more natural and relevant framework of existence in [19].

However in this work we only present the case when such a modification (i.e. having a linear combination of E and H in both B and D in the constitutive relations) concerns only B and not D, which we do not know if it is physically relevant. We have thus only left in our constitutive laws the terms that model a delayed effect.

Let us also mention that, historically, a coupling in the constitutive equations has been introduced in 1915 by Born, but his laws were lacking boundary conditions compatibilities, and were later modified by Fedorov (in 1959) to deal with this inconsistency.

Moreover, we are only able to treat the case when smallness and strong regularity on $\chi^{e,m}$ are assumed which is, of course, a restrictive physical assumption.

1.4. *The controllability problem in the harmonic domain*

Here we concentrate and define our controllability problem in the case when one uses 1.3 and 1.4. This section is simply devoted to recall some known results.

In this framework the controllability problem consists in dealing with the rank of the map $J \mapsto (E, H)$ where (E, H) satisfy (1.5)...(1.8), (1.11) (1.12) (1.13) (1.14) (1.15)

Let us recall some known results in this case:

It has been proved in [13] that this problem is not controllable and not even approximately controllable (meaning that the aforementioned rank is not dense).

However it seems more natural to consider (1.3) and (1.4) in the harmonic domain, *i.e.* to consider the specific case when $E = \tilde{E}e^{i\omega t}$ with similar behavior for H and J. In deed β is in general assumed to satisfy $\beta_0\varepsilon_0\mu_0\omega^2 << 1$.

The controllability problem (1.5)...(1.8), (1.11) (1.12) (1.13) then become an elliptic controllability problem for which the map that we consider is $\tilde{J} \mapsto \text{curl}H_{/\Gamma\setminus\Gamma_0}$. It has been proved in [14] that this map has a dense image provided $\beta, \varepsilon_0, \mu_0, \omega$ belong to a generic set of positive values.

Outline of the following sections: The sequel of the paper is organized as follows: We first define in section 2 the notion of solution of (P) for which we discuss the existence and uniqueness result. This leads us to define a backward system which is of the same type as the adjoint system. Thereafter we consider the H.U.M. method by solving a backward system whose adjoint system will be proved to satisfy observability inequalities and we therefore deduce exact controllability results (theorems 4.3 and 4.4).

2. Weak formulation of (P)

Let us derive the weak formulation of (P). To this we take ϕ and ψ defined on $[0,T]\times\Omega$, and make formal computations which leads to:

$$\begin{aligned}
A := \ & \varepsilon_0 \int_\Omega E(t=T)\psi(t=T)dx + \mu_0 \int_\Omega H(t=T)\phi(t=T)dx \\
A = \ & \int_\Omega \int_0^T E\Big(-\text{curl}\phi - \varepsilon_0\chi^e(0)\psi(x,t) - \varepsilon_0 \int_t^T \chi^e_t(\tau - t)\psi(x,\tau)d\tau + \varepsilon_0\partial_t\psi\Big)dxdt \\
& + \int_\Omega \int_0^T H\Big(\text{curl}\psi - \mu_0\chi^m(0)\phi(x,t) - \mu_0 \int_t^T \chi^m_t(\tau - t)\phi(x,\tau)d\tau + \mu_0\partial_t\phi\Big)dxdt \\
& - \int_0^T \int_{\Sigma_0} J.\psi d\sigma dt + \varepsilon_0 \int_\Omega E(t=0)\psi(t=0)dx + \mu_0 \int_\Omega H(t=0)\psi(t=0)dx
\end{aligned}$$

where we use the following additional boundary conditions

$$\begin{cases} \phi \wedge n = 0 \text{ on } \Sigma \\ \psi.n = 0 \text{ on } \Sigma. \end{cases}$$

We are consequently considering the following system:

$$\partial_t\psi - \chi^e(0)\psi(x,t) - \int_t^T \chi^e_t(\tau - t)\psi(x,\tau)d\tau - \frac{\text{curl}\phi}{\varepsilon_0} = f \text{ in } \Omega \tag{2.1}$$

$$\partial_t\phi - \chi^m(0)\phi(x,t) - \int_t^T \chi^m_t(\tau - t)\phi(x,\tau)d\tau + \frac{\text{curl}\psi}{\mu_0} = g \text{ in } \Omega \tag{2.2}$$

$$\phi \wedge n = 0 \text{ on } \partial\Omega \tag{2.3}$$

$$\psi.n = 0 \text{ on } \partial\Omega \tag{2.4}$$

with final data $\psi(t=T)=\psi_1$ and $\phi(t=T)=\phi_1$.

In order to deal with this system, and others in the sequel, we introduce the following standard function spaces:

$$\begin{array}{ll} H(\mathrm{div}=0) := & \{f:\Omega\to\mathbb{R}^3,\ f\in L^2(\Omega)^3,\ \mathrm{div}f=0\} \\ H_0(\mathrm{div}=0) := & \{f:\Omega\to\mathbb{R}^3,\ f\in H(\mathrm{div}=0),\ f.n=0, \text{ in the classical weak sense, on } \Gamma\} \end{array}$$

$$\begin{array}{ll} H(\mathrm{curl}) := & \{f:\Omega\to\mathbb{R}^3,\ f\in L^2(\Omega)^3,\ \mathrm{curl}f\in L^2(\Omega)^3\} \\ H_0(\mathrm{curl}) := & \{f:\Omega\to\mathbb{R}^3,\ f\in H(\mathrm{curl}),\ f\wedge n=0, \text{ in the classical weak sense, on } \Gamma\}. \end{array}$$

Properties of these spaces are well–known; they can be found in, e.g., [27], [29].

Let us also define $\mathcal{H} := H_0(\mathrm{div}=0)\times H(\mathrm{div}=0)$.

We are first able to prove:

Theorem 2.1 *If χ^e and χ^m are in $W^{1,\infty}(0,T)$, and if $||\chi^e||_{1,\infty}$ and $||\chi^m||_{1,\infty}$ are sufficiently small, then for any $\phi_1\in H(\mathrm{div}=0)$, $\psi_1\in H_0(\mathrm{div}=0)$ and any $(f,g)\in L^1(0,T;\mathcal{H})$ there exists a unique weak solution $(\psi,\phi)\in C(0,T;\mathcal{H})\cap C([0,T];H(\mathrm{div}=0)\times H(\mathrm{div}=0))$ to (2.1)..(2.4).*

Proof: see [15]

Let $\mathcal{N}$ denote the usual space of the trace of the elements of $H(\mathrm{div})$.

Definition 2.1 Let us take some $J\in L^2(0,T;\mathcal{N}')$ which is zero on $\Sigma\setminus\Sigma_0$, and $(E_0,H_0)\in\mathcal{H}$. We will say that $(E,H)\in L^\infty(0,T;\mathcal{H}))$ is a weak solution of (P), if there exists $(E_1,H_1)\in\mathcal{H}$ such that for any $(\psi_1,\phi_1)\in\mathcal{H}$ $(f,g)\in L^1(0,T;\mathcal{H})$ one has

$$\begin{array}{rl} & \varepsilon_0\int_\Omega \psi_1 E_1 dx + \mu_0\int_\Omega \phi_1 H_1 dx \\ = & \\ - & \varepsilon_0\int_0^T\int_\Omega f(t,x)E(t,x)dtdx - \mu_0\int_0^T\int_\Omega g(t,x)H(t,x)dtdx \\ + & \varepsilon_0\int_\Omega(\psi(t=0)E_0+\mu_0\phi(t=0)H_0)dx + \int_0^T\int_{\Sigma_0} J.\psi d\sigma dt, \end{array} \tag{2.5}$$

where (ψ,ϕ) is the solution of (2.1)– (2.4).

Let us remark that for a sufficiently regular J (basically having a lifting in $C^1(0,T;H^2(\mathrm{curl})\cap H(\mathrm{div}=0))$ a weak solution of (P) is a strong solution to (P) by setting $E_1 := E(t=T)$ and $H_1 := H(t=T)$. However in any case, we can prove (see [15])

Proposition 2.1 *A weak solution of (P) satisfies (1.5)–(1.8) in the sense of distributions.*

Now for fixed J, E_0, H_0 as in the definition 2.1 let us consider the map

$$(\psi_1,\phi_1,f,g)\mapsto \int_0^T\int_{\Sigma_0} J.\psi d\sigma dt + \varepsilon_0\int_\Omega \psi(t=0)E_0 dx + \mu_0\int_\Omega \phi(t=0)H_0 dx.$$

It is clearly a linear bounded operator from $H(\mathrm{div}=0)\times H(\mathrm{div}=0)\times L^1(0,T;\mathcal{H})$ into $\mathbb{R}$ and thus there exists a unique

$$\begin{aligned}((E_1,H_1),(E,H)) &\in (\mathcal{H}\times L^1(0,T;\mathcal{H}))' \\ &= \mathcal{H}\times L^\infty(0,T;\mathcal{H})\end{aligned}$$

such that (2.5) is satisfied.

We have thus proved

Theorem 2.2 *For any $J\in L^2(0,T;\mathcal{N}')$ which is zero on $\Sigma\setminus\Sigma_0$, and any $(E_0,H_0)\in\mathcal{H}$, there exists a unique weak solution (E,H) of (2.5).*

Let us remark that (E_1,H_1) in Definition 2.1 should be interpreted as the trace of (E,H) at $t=T$, which of course is not clear in our formulation unless one is able to prove more regularity for our weak solutions.

3. The controllability of (P)

In this section we address the question of the exact controllability of (P) for which we will use the H.U.M. introduced by J.–L. Lions (see [25]).

Let us describe this method.

We will consider the following direct problem $\mathcal{D}_{f,g}$:

$$\partial_t\psi-\chi^e(0)\psi(x,t)-\int_t^T\psi(x,\tau)\chi_t^e(\tau-t)d\tau-\frac{\mathrm{curl}\phi}{\varepsilon_0}=f \text{ in } \Omega \tag{3.1}$$

$$\partial_t\phi-\chi^m(0)\phi(x,t)-\int_t^T\chi_t^m(\tau-t)\phi(x,\tau)d\tau+\frac{\mathrm{curl}\psi}{\mu_0}=g \text{ in } \Omega \tag{3.2}$$

$$\phi\wedge n=0 \text{ on } \partial\Omega \tag{3.3}$$

$$\psi.n=0 \text{ on } \partial\Omega \tag{3.4}$$

$$\psi(t=0)=\psi_0\in H_0(\mathrm{div}=0) \tag{3.5}$$

$$\phi(t=0)=\phi_0\in H(\mathrm{div}=0) \tag{3.6}$$

A similar proof as the one of theorem 2.1 gives us

Theorem 3.1 *For any given $(f,g)\in L^1(0,T;\mathcal{H})$, there exists a unique weak solution $\psi,\phi\in C([0,T];\mathcal{H})$ to (3.1)...(3.6)*

Definition 3.1 As in definition 2.1 we will say that (E,H) is a backward solution of (1.5)...(1.8) with $E(t=T)=E_1$ and $H(t=T)=H_1$ if there exists (E_0,H_0) such that one has (3.7) for all $(f,g)\in L^1(0,T;\mathcal{H})$ where

$$\begin{aligned}&\varepsilon_0\int_\Omega\psi_1E_1dxdt+\mu_0\int_\Omega\phi_1H_1dxdt\\ =&\int_0^T\int_{\Sigma_0}J.\psi d\sigma dt-\varepsilon_0\int_0^T\int_\Omega f(t,x)E(t,x)dtdx\\ &+\varepsilon_0\int_\Omega\psi_0E_0+\mu_0\phi_0H_0dx-\mu_0\int_0^T\int_\Omega g(t,x)H(t,x)dtdx,\end{aligned}\tag{3.7}$$

with (ψ, ϕ) the solution of (3.1)... (3.6).

The same proof as the one of theorem 2.2 yields

Theorem 3.2 *For any $J \in L^2(0,T;\mathcal{N}')$ which is zero on $\Sigma \backslash \Sigma_0$ and any $(E_1, H_1) \in \mathcal{H}$ there exists a unique weak backward solution $(E,H) \in L^\infty(0,T;\mathcal{H})$ of (1.5)...(1.8) with $E(t=T) = E_1$ and $H(t=T) = H_1$.*

It is well–known that the exact (zero) controllability of (P) in a Hilbert space $\mathcal{H}$ is equivalent to the following observability inequality of $\mathcal{D}_{0,0}$; this is the cornerstone of the H.U.M. method which, relying on the Lax–Milgram theorem, consists in looking for a J being the restriction of ψ to Σ_0 :

Definition 3.2 If there exists a constant $C > 0$ such that for any (ψ_0, ϕ_0) one has

$$\int_0^T \int_{\Sigma_0} |\psi|^2 d\sigma dt \geq C||(\psi_0, \phi_0)||^2_{\mathcal{H}} \tag{3.8}$$

one says that $\mathcal{D}_{0,0}$ is observable in $\mathcal{H}$.

This inequality has been derived in [22] and [30] in the case of non chiral media, for the, respectively, constant and non–constant coefficients cases, and in $\mathcal{H}$ provided a sufficient condition on T and Σ_0 which ascertains the exact zero controllability of the wave equation.

In this paper we prove

Theorem 3.3 *The observability inequality is true in $\mathcal{H}$ for (3.1)....(3.6) provided that $||\chi^e||_{2,\infty}$ and $||\chi^m||_{2,\infty}$ are sufficiently small (with respect to T), and provided that $\Sigma_0 = \Omega$.*

4. Sketch of the proof of theorem 3.3

In order to prove (3.8) in the case of $\Sigma_0 = \Sigma$ as in [30] we will use the multiplier method, we refer to [15] for full proofs and further extensions.

4.1. *Energy estimates*

We will need energy estimates that will reveal the reversibility of the Maxwell's equations in our media.

Let (ψ, ϕ) be the solution of $\mathcal{D}_{0,0}$ and let us denote $\mathcal{E}(t) = \displaystyle\int_\Omega (\varepsilon_0 \psi^2 + \mu_0 \phi^2) dx$.

A straightforward computation allows to prove:

For sufficiently small $||\chi^{e,m}||_{1,\infty}$ there exist two constants C_1 and C_2 such that

$$\mathcal{E}(t) \leq C_1 \mathcal{E}(0) \tag{4.1}$$

$$\int_0^T \mathcal{E}(\tau) d\tau \leq C_2 \mathcal{E}(0). \tag{4.2}$$

Thus we get also

$$|\frac{d}{dt}\mathcal{E}(t)| \leq C_{\varepsilon_0,\mu_0}(||\chi^e||_{1,\infty} + ||\chi^m||_{1,\infty})(\lambda C_1\mathcal{E}(0) + \frac{T}{\lambda}C_2\mathcal{E}(0)) \tag{4.3}$$

and thus for sufficiently small $||\chi^{e,m}||_{1,\infty}$ we get

$$C_3\mathcal{E}(0) \leq \mathcal{E}(t) \leq C_4\mathcal{E}(0)$$

for some positive numbers C_3 and C_4.

We thus obtained

Theorem 4.1 *There exist two constants C_3 and C_4 such that, for sufficiently small $||\chi^{e,m}||_{1,\infty}$, we have*

$$C_3\mathcal{E}(0) \leq \mathcal{E}(t) \leq C_4\mathcal{E}(0)$$

for any $(\psi_0, \phi_0) \in \mathcal{H}$.

Now let us consider initial data $(\psi_0, \phi_0) \in \mathcal{H} \cap (H(\text{curl}) \times H_0(\text{curl}))$ and let us define

$$\tilde{\mathcal{E}}(t) = \int_\Omega (\frac{|\text{curl}\psi|^2}{\mu_0} + \frac{|\text{curl}\phi|^2}{\varepsilon_0})dx.$$

It is clear that now the solution of $\mathcal{D}_{0,0}$ is in fact a strong solution.

We have, due to the boundary condition (3.3)

$$\begin{aligned}
\frac{d}{dt}\tilde{\mathcal{E}}(t) = &\\
& 2\int_\Omega \big(-\partial_t\phi + \chi^m(0)\phi + \int_t^T \chi_t^m(\tau-t)\phi(x,\tau)d\tau\big)\text{curl}\psi_t dx \\
& +2\int_\Omega (\partial_t\psi - \chi^e(0)\psi - \int_t^T \chi_t^e(\tau-t)\psi(x,\tau)d\tau\big)\text{curl}\phi_t dx \\
= &\\
& 2\int_\Omega \big(\chi^m(0)\text{curl}\phi + \int_t^T \chi_t^m(\tau-t)\text{curl}\phi(x,\tau)d\tau\big)\psi_t dx \\
& +2\int_\Omega (-\chi^e(0)\text{curl}\psi - \int_t^T \chi_t^e(\tau-t)\text{curl}\psi(x,\tau)d\tau\big)\phi_t dx \\
= &\\
& \int_\Omega \big(\chi^m(0)\text{curl}\phi + \int_t^T \chi_t^m(\tau-t)\text{curl}\phi(x,\tau)d\tau\big)\times \\
& (-\chi^e(0)\psi - \int_t^T \chi_t^e(\tau-t)\psi(x,\tau)d\tau - \frac{\text{curl}\phi}{\varepsilon_0})dx \\
& +2\int_\Omega (-\chi^e(0)\text{curl}\psi - \int_t^T \chi_t^e(\tau-t)\text{curl}\psi(x,\tau)d\tau\big)\times \\
& (-\chi^m(0)\phi - \int_t^T \chi_t^m(\tau-t)\phi(x,\tau) + \frac{\text{curl}\psi}{\mu_0})dx
\end{aligned}$$

$$
\begin{aligned}
\tfrac{d}{dt}\tilde{\mathcal{E}}(t) \le \\
& |\chi^m(0)\chi^e(0)\int_\Omega(\mathrm{curl}\phi\psi - \mathrm{curl}\psi\phi)dx| \ (=0) \\
& + \max(|\chi^m(0)|, |\chi^e(0)|)\tilde{\mathcal{E}}(t) \\
& +(|\chi^e(0)|||\chi^m_t||_\infty\sqrt{T}(\int_\Omega \psi^2 dx)^{1/2}(\int_0^T\int_\Omega(\mathrm{curl}\phi)^2 dxdt)^{1/2} \\
& +(|\chi^m(0)|||\chi^e_t||_\infty\sqrt{T}(\int_\Omega \phi^2 dx)^{1/2}(\int_0^T\int_\Omega(\mathrm{curl}\psi)^2 dxdt)^{1/2} \\
& +\int_\Omega(\int_t^T \chi^m_t(\tau-t)^2 d\tau)^{1/2}(\int_0^T(\mathrm{curl}\phi)^2 d\tau)^{1/2}\frac{\mathrm{curl}\phi}{\varepsilon_0} \\
& +\int_\Omega(\int_t^T \chi^e_t(\tau-t)^2 d\tau)^{1/2}(\int_0^T(\mathrm{curl}\psi)^2 d\tau)^{1/2}\frac{\mathrm{curl}\psi}{\mu_0} \\
\le \ & C(||\chi^{m,e}||_{1,\infty},\mu_0,\varepsilon_0)(\tilde{\mathcal{E}}(t) \\
& +\sqrt{T}\mathcal{E}(t)^{1/2}(\int_0^T\tilde{\mathcal{E}}(t)dt)^{1/2}+\sqrt{T}\tilde{\mathcal{E}}(t)^{1/2}(\int_0^T\tilde{\mathcal{E}}(s)ds)^{1/2})
\end{aligned}
$$

We thus obtain through a similar computation, and using theorem 4.1, that for sufficiently small $||\chi^{e,m}||_{1,\infty}$, there exists a constant such that

$$\int_0^T\tilde{\mathcal{E}}(s)ds \le C(\mathcal{E}(0)+\tilde{\mathcal{E}}(0)).$$

Now we have

Lemma 4.1 *There exists a constant C' such that for any $(\psi_0,\phi_0)\in\mathcal{H}\cap(H(\mathrm{curl})\times H_0(\mathrm{curl}))$ one has*

$$\mathcal{E}(0)\le C'\tilde{\mathcal{E}}(0).$$

The proof is straightforward and we omit it. Let us only mention that it relies on the fact that, for example, the set $H_0(\mathrm{div}=0)\cap H(\mathrm{curl})$ is topologically equivalent to $H^1(\Omega)^3\cap H_0(\mathrm{div}=0)$ and thus compactly embedded in $H_0(\mathrm{div}=0)$ for a regular Ω (at least $C^{2+\varepsilon}$).

Now due to lemma 4.1 we have

$$\int_0^T\tilde{\mathcal{E}}(s)ds \le C''(\tilde{\mathcal{E}}(0))$$

and again we obtain

Proposition 4.1 *There exist two constants C_5 and C_6 such that for any $(\psi_0,\phi_0)\in\mathcal{H}\cap(H(\mathrm{curl})\times H_0(\mathrm{curl}))$ one has*

$$C_5\tilde{\mathcal{E}}(0)\le\tilde{\mathcal{E}}(t)\le C_6\tilde{\mathcal{E}}(0).$$

Now let us treat the observability inequality.

The usual multiplier is $m \cdot \nabla\psi$ where $m(x) = x - x_0$ for some $x_0 \in \mathbb{R}^3$.

One can easily get

$$\begin{aligned}&\int_Q \phi_t \mathrm{curl}(m \cdot \nabla\psi) dxdt = -\int_\Sigma \varepsilon_0/2 m \cdot n |\psi_t|^2 d\sigma dt \\ &+3\varepsilon_0/2 \int_Q |\psi_t|^2 dxdt + \varepsilon_0 \int_\Omega [\psi_t m \cdot \nabla\psi]_0^T dx \\ &-\varepsilon_0 \int_Q \chi^e(0)\psi_t m \cdot \nabla\psi dxdt + \\ &\int_Q \int_t^T \chi^e_{tt}(\tau - t)\psi(x,\tau) m \cdot \nabla\psi d\tau dt dx\end{aligned} \tag{4.4}$$

On the other hand, since

$$\mathrm{curl}(m \cdot \nabla\psi) = \mathrm{curl}\psi + m \cdot \nabla \mathrm{curl}\psi$$

we get

$$\begin{aligned}&\int_Q \phi_t \mathrm{curl}(m \cdot \nabla\psi) dxdt \\ = \quad &\int_Q (\chi^m(0)\phi(x,t) + \int_t^T \chi^m_t(\tau - t)\phi(x,\tau) d\tau - \frac{\mathrm{curl}\psi}{\mu_0})(\mathrm{curl}(\psi) + m \cdot \nabla \mathrm{curl}\psi) dxdt\end{aligned}$$

Further,

$$\mathrm{div}(m|\mathrm{curl}\psi|^2) = 3|\mathrm{curl}\psi|^2 + 2\mathrm{curl}\psi m \cdot \nabla \mathrm{curl}\psi,$$

one has

$$\begin{aligned}A := \quad &\int_Q \mathrm{curl}\psi m \cdot \nabla \mathrm{curl}\psi dxdt \\ = \quad &\tfrac{1}{2}\int_Q \mathrm{div}(m|\mathrm{curl}\psi|^2) dxdt - \frac{3}{2}\int_Q |\mathrm{curl}\psi|^2 dxdt \\ = \quad &\tfrac{1}{2}\int_\Sigma m \cdot n |\mathrm{curl}\psi|^2 d\sigma dt - \frac{3}{2}\int_Q |\mathrm{curl}\psi|^2 dxdt\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_Q \phi_t \mathrm{curl}(m \cdot \nabla\psi) dxdt \\ = \quad &\int_Q (\chi^m(0)\phi(x,t) + \int_t^T \chi^m_t(\tau - t)\phi(x,\tau) d\tau) \mathrm{curl}(m \cdot \nabla\psi) dxdt \\ &-\tfrac{1}{2\mu_0}\int_\Sigma m \cdot n |\mathrm{curl}\psi|^2 d\sigma dt + \frac{1}{2\mu_0}\int_Q |\mathrm{curl}\psi|^2 dxdt \\ = \quad &\int_Q (\chi^m(0)\mathrm{curl}\phi(x,t) + \int_t^T \chi^m_t(\tau - t)\mathrm{curl}\phi(x,\tau) d\tau)(m \cdot \nabla\psi) dxdt \\ &-\tfrac{1}{2\mu_0}\int_\Sigma m \cdot n |\mathrm{curl}\psi|^2 d\sigma dt + \frac{1}{2\mu_0}\int_Q |\mathrm{curl}\psi|^2 dxdt.\end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5) one gets

$$
\begin{aligned}
&\int_\Sigma m\cdot n(\frac{\varepsilon_0}{2}|\psi_t|^2-\frac{1}{2\mu_0}|\mathrm{curl}\psi|^2)d\sigma dt+\int_Q\frac{1}{2\mu_0}|\mathrm{curl}\psi|^2-\frac{3\varepsilon_0}{2}|\psi_t|^2dxdt\\
=\ &-\int_Q\chi^m(0)\mathrm{curl}\phi.(m\cdot\nabla\psi)dxdt\\
&-\int_Q\int_t^T\chi_t^m(\tau-t)\mathrm{curl}\phi(x,\tau)m\cdot\nabla\psi(x,t)dxd\tau dt\\
&+\varepsilon_0\int_\Omega[\psi_t m\cdot\nabla\psi]_0^T-\varepsilon_0\int_Q\chi^e(0)\psi_t m\cdot\nabla\psi dxdt\\
&+\int_Q\int_t^T\chi_{tt}^e(\tau-t)\psi(x,\tau)m\cdot\nabla\psi d\tau dtdx.
\end{aligned}
\tag{4.6}
$$

Substituting (3.1) and (3.2) in (4.6) one gets

$$
\begin{aligned}
&\int_\Sigma m\cdot n(\frac{\varepsilon_0}{2}|\chi^e(0)\psi(x,t)+\int_t^T\psi(x,\tau)\chi_t^e(\tau-t)d\tau+\frac{\mathrm{curl}\phi}{\varepsilon_0}|^2)d\sigma dt\\
&-\int_\Sigma m\cdot n\frac{1}{2\mu_0}|\mathrm{curl}\psi|^2dxdt+\int_Q\frac{1}{2\mu_0}|\mathrm{curl}\psi|^2dxdt\\
&-\int_Q\frac{3\varepsilon_0}{2}|\chi^e(0)\psi(x,t)+\int_t^T\psi(x,\tau)\chi_t^e(\tau-t)d\tau+\frac{\mathrm{curl}\phi}{\varepsilon_0}|^2dxdt\\
=\ &-\int_Q\chi^m(0)\mathrm{curl}\phi.(m\cdot\nabla\psi)dxdt\\
&-\int_Q\int_t^T\chi_t^m(\tau-t)\mathrm{curl}\phi(x,\tau)m\cdot\nabla\psi(x,t)dxd\tau dt\\
&+\varepsilon_0\int_\Omega[\psi_t m\cdot\nabla\psi]_0^T-\varepsilon_0\int_Q\chi^e(0)\psi_t m\cdot\nabla\psi dxdt\\
&+\int_Q\int_t^T\chi_{tt}^e(\tau-t)\psi(x,\tau)m\cdot\nabla\psi d\tau dtdx.
\end{aligned}
\tag{4.7}
$$

Now let us remark that for $(\psi_0,\phi_0)\in\mathcal{H}\cap(H(\mathrm{curl})\times H_0(\mathrm{curl}))$ the same is true for the solution of $(\psi(t),\phi(t))$, which ensures that $\psi(t)\in H^1(\Omega)$, and thus that there exists a constant $C>0$ such that $||\psi||_{L^1(0,T;L^2(\Sigma))}\le CT\tilde{\mathcal{E}}(0)$. Moreover, each term except $\varepsilon_0\int_\Omega[\psi_t m\cdot\nabla\psi]_0^T$ in the right hand side of (4.7), is controlled by $\max(1,T^2)\times O(||\chi^{e,m}||_{2,\infty})\tilde{\mathcal{E}}(0)$ using lemma 4.1 and proposition 4.1. We thus get

$$
\begin{aligned}
&\Big|(1+O(\max(1,T^2)||\chi^{e,m}||_{1,\infty}))\int_\Sigma m\cdot n(\frac{\varepsilon_0}{2}|\frac{\mathrm{curl}\phi}{\varepsilon_0}|^2)d\sigma dt\\
&-\int_\Sigma m\cdot n\frac{1}{2\mu_0}|\mathrm{curl}\psi|^2dxdt\\
&+\int_Q\frac{1}{2\mu_0}|\mathrm{curl}\psi|^2dxdt-(1+O(||\max(1,T^2)\chi^{e,m}||_{1,\infty}))\int_Q\frac{|\mathrm{curl}\phi|^2}{2\varepsilon_0}dxdt\\
&-\varepsilon_0\int_\Omega[\psi_t m\cdot\nabla\psi]_0^T\Big|\\
&\le O(\max(1,T^2)||\chi^{e,m}||_{2,\infty})\tilde{\mathcal{E}}(0)
\end{aligned}
\tag{4.8}
$$

Estimating again ψ_t in the left hand side and as in [30] we get

$$(1+O(\max(1,T^2)||\chi^{e,m}||_{1,\infty}))\Big(\int_\Sigma m\cdot n(\frac{\varepsilon_0}{2}|\frac{\text{curl}\phi}{\varepsilon_0}|^2)d\sigma dt - \int_\Sigma m\cdot n\frac{1}{2\mu_0}|\text{curl}\psi|^2 dxdt\Big) \geq (CT+O(\max(1,T^2)||\chi^{e,m}||_{2,\infty})-C_0)\tilde{\mathcal{E}}(0) \tag{4.9}$$

Now if we let $||\chi^{e,m}||_{2,\infty}$ to tend to 0, for a fixed T the factor in the right hand side tends to (see [30]) $(T-T_0)$ for some $T_0>0$. Fixing then a $T>T_0$, and choosing $||\chi^{e,m}||_{2,\infty}$ small enough with respect to T, one gets

Theorem 4.2 *There exists a constant $K>0$, and T_0 such that if $T>T_0$ and $\chi^{e,m}\in W^{2,\infty}(\mathbb{R}^+)$, for sufficiently small $||\chi^{e,m}||_{2,\infty}$, then for any initial data $(\phi_0,\psi_0)\in \mathcal{H}\cap(H(\text{curl})\times H_0(\text{curl}))$ one has*

$$\Big(\int_\Sigma m\cdot n\frac{|\text{curl}\phi|^2}{2\varepsilon_0}d\sigma dt - \int_\Sigma m\cdot n\frac{|\text{curl}\psi|^2}{2\mu_0}d\sigma dt\Big)\geq K(T-T_0)\tilde{\mathcal{E}}(0) \tag{4.10}$$

An easy consequence of this result is the following

Corollary 4.1 *If Ω is star–shaped for some x_0, then, under the assumption of theorem 4.2, one has*

$$\int_\Sigma m\cdot n\frac{|\text{curl}\phi|^2}{2\varepsilon_0}d\sigma dt\geq K(T-T_0)\tilde{\mathcal{E}}(0) \tag{4.11}$$

Furthermore, we can easily generalize this result for solutions of similar problems

Proposition 4.2 *Let (g,h) be in $C(0,T,\mathcal{H}\cap(H(\text{curl})\times H_0(\text{curl})))$ such that*

$$\begin{cases} \partial_t g - a_0 g(x,t) - A(g)(x,t) - \frac{\text{curl}h}{\varepsilon_0} = 0 \text{ in } \Omega\\ \partial_t h - b_0 h(x,t) - B(h)(x,t) + \frac{\text{curl}g}{\mu_0} = 0 \text{ in } \Omega\\ h\wedge n = 0 \text{ on } \partial\Omega\\ g.n = 0 \text{ on } \partial\Omega \end{cases}$$

where A (resp B) are bounded linear operators in $C(0,T,H_0(\text{div}=0))$ (resp. $C(0,T,H(\text{div}=0))$) which satisfy

- $\forall\phi\in C(0,T,H(\text{div}=0))$, $\text{curl}(A(\phi))=A(\text{curl}(\phi))$,
- $\forall\phi\in C(0,T,H(\text{div}=0))$, $\text{curl}(B(\phi))=B(\text{curl}(\phi))$,
- $A':\phi\longmapsto\partial_t(A(\phi))$ is a bounded linear operator in $C(0,T,H_0(\text{div}=0))$,
- $B':\phi\longmapsto\partial_t(B(\phi))$ is a bounded linear operator in $C(0,T,H(\text{div}=0))$.

Then for Max$\{\|A\|,\|A'\|,\|B\|,\|B'\|,|a_0|,|b_0|\}$ sufficiently small, there exists a constant $K>0$ such that

$$\int_\Sigma m\cdot n\frac{|\text{curl}h|^2}{2\varepsilon_0}d\sigma dt\geq K\tilde{\mathcal{E}}(0) \tag{4.12}$$

As in [30] we are able to prove:

Theorem 4.3 *If Ω is star–shaped with respect to one point, then the system (1.5)–(1.15) is exactly controllable in $\mathcal{H}$.*

Proof: see again [15]

When dealing with controllability on one part of the boundary, the multiplier argument which allows to have explicit constants cannot be applied directly, and we believe that the results given in [35] gives

Theorem 4.4 *If Σ_0 observes Ω, then, if $\chi^m = 0$ and $\chi^e_t(0) = 0$, the problem $\mathcal{D}_{0,0}$ is observable.*

can be modified to fit to our problem (see [15]).

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