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# Opial type inequalities for vector valued functions

George A. Anastassiou

**Abstract**
Various $L_p$ form Opial type inequalities are given for functions valued in a Banach vector space. We give applications in $\mathbb{C}$.

*Keywords*: Opial inequality, Banach space valued function.

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## 1. Introduction

Our paper is greatly motivated by the article of Z. Opial [3].

**Theorem 1.** (Opial [3]) *Let $x(t) \in C^1([0,h])$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then, the following inequality holds*

$$\int_0^h |x(t)x'(t)|dt \leq \frac{h}{4}\int_0^h (x'(t))^2 dt. \tag{1}$$

In the last inequality the constant h/4 is the best possible.
Equality holds for the function

$$x(t) = t \quad \text{on } [0, h/2]$$

and

$$x(t) = h - t \quad \text{on } [h/2, h].$$

Opial type inequalities have applications in establishing uniqueness of solution to initial value problems in differential equations, see [6]. We are also motivated by [1], [2].

We need

## 2. Background

(see [5], pp. 83 - 94)

Let $f(t)$ be a function defined on $[a,b] \subseteq \mathbb{R}$ taking values in a real or complex normed linear space $(X, \|\cdot\|)$. Then $f(t)$ is said to be *differentiable at a point* $t_0 \in [a,b]$ if the limit

$$f'(t_0) := \lim_{h\to 0} \frac{f(t_0+h) - f(t_0)}{h} \tag{2}$$

exists in X, the convergence is in $\|\cdot\|$. This is called the *derivative* of $f(t)$ at $t = t_0$.

We call $f(t)$ *differentiable* on [a,b], iff there exists $f'(t) \in X$ for all $t \in [a, b]$.

Similarly and inductively are defined higher order derivatives of f, denoted $f''$, $f^{(3)}$, $\dots, f^{(k)}, k \in \mathbb{N}$, just as for numerical functions.

For all the properties of derivatives see [5], pp. 83 - 86.

Let now $(X, \|\cdot\|)$ be a Banach space, and $f : [a, b] \to X$.

We define the *vector valued Riemann integral* $\int_a^b f(t)dt \in X$ as the limit of the vector valued Riemann sums in X, convergence is in $\|\cdot\|$. The definition is as for the numerical valued functions.

If $\int_a^b f(t)dt \in X$ we call f *integrable* on [a,b].

For the properties of vector valued Riemann integrals see [5], pp. 86 - 91.

We define the space $C^n([a, b], X), n \in \mathbb{N}$, of n-times continuously differentiable functions from [a,b] into X; here continuity is with respect to $\|\cdot\|$ and defined in the usual way as for numerical functions.

Let $(X, \|\cdot\|)$ be a Banach space and $f \in C^n([a, b], X)$, then we have the *vector valued Taylor's formula* , see [5], pp. 93 - 94, and also [4], (IV, 9; 47):

It holds

$$E_n(x, y) := f(y) - f(x) - f'(x)(y - x) -$$

$$\frac{1}{2} f''(x)(y - x)^2 - \cdots - \frac{1}{(n-1)!} f^{(n-1)}(x)(y - x)^{n-1}$$

$$= \frac{1}{(n-1)!} \int_x^y (y - t)^{n-1} f^{(n)}(t)dt, \quad \forall\, x, y \in [a, b]. \tag{3}$$

In particular (3) is true when $X = \mathbb{R}^m, \mathbb{C}^m, m \in \mathbb{N}$,etc.

In case of some $x_0 \in [a, b]$ such that $f^{(k)}(x_0) = 0, k = 0, 1, ..., n - 1$, then

$$f(y) = \frac{1}{(n-1)!} \int_{x_0}^y (y - t)^{n-1} f^{(n)}(t)dt, \ \forall\, y \in [a, b]. \tag{4}$$

In that case $E_n(x_0, y) = f(y)$.

## 3. Results

Here we consider always X to be a Banach space, $n \in \mathbb{N}$, and $f \in C^n([a, b], X)$, $[a, b] \subseteq \mathbb{R}$. We fix $x_0 \in [a, b]$.

We present our first result.

**Theorem 2.** *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1,\ y \geq x_0;\ y, x_0 \in [a, b]$.

*Then*

$$I_1 := \int_{x_0}^y \|E_n(x_0, w)\| \, \|f^{(n)}(w)\| dw \leq$$

$$\frac{(y - x_0)^{n-1+\frac{2}{p}}}{2^{1/q}(n-1)![(p(n-1)+1)(p(n-1)+2)]^{1/p}} \left( \int_{x_0}^y \|f^{(n)}(w)\|^q dw \right)^{2/q}. \tag{5}$$

We give

**Corollary 3.** *For* $y \in [x_0, b]$ *we have*

$$I_1 \le \frac{(y-x_0)^n}{2(n-1)!\sqrt{n(2n-1)}} \left( \int_{x_0}^{y} \|f^{(n)}(w)\|^2 dw \right). \tag{6}$$

**Proof of Theorem 2.** We have by (3) that

$$E_n(x_0, y) = \frac{1}{(n-1)!} \int_{x_0}^{y} (y-t)^{n-1} f^{(n)}(t) dt, \ \forall\, y \ge x_0; x_0, y \in [a,b]. \tag{7}$$

By Hölder's inequality we have

$$\|E_n(x_0, y)\| \le \frac{1}{(n-1)!} \int_{x_0}^{y} (y-t)^{n-1} \|f^{(n)}(t)\| dt$$

$$\le \frac{1}{(n-1)!} \left( \int_{x_0}^{y} (y-t)^{p(n-1)} dt \right)^{1/p} \left( \int_{x_0}^{y} \|f^{(n)}(t)\|^q dt \right)^{1/q}$$

$$= \frac{1}{(n-1)!} \, \frac{(y-x_0)^{n-1+\frac{1}{p}}}{(p(n-1)+1)^{1/p}} \, (z(y))^{1/q}, \tag{8}$$

where

$$z(y) := \int_{x_0}^{y} \|f^{(n)}(t)\|^q dt, \ x_0 \le y \le b \ (z(x_0) = 0). \tag{9}$$

Thus

$$z'(y) = \|f^{(n)}(y)\|^q$$

and

$$\|f^{(n)}(y)\| = (z'(y))^{1/q}. \tag{10}$$

Therefore we get

$$\|E_n(x_0, y)\| \, \|f^{(n)}(y)\| \le \frac{1}{(n-1)!} \, \frac{(y-x_0)^{n-1+\frac{1}{p}}}{(p(n-1)+1)^{1/p}} (z(y) z'(y))^{1/q}. \tag{11}$$

Integrating the last we have

$$\int_{x_0}^{y} \|E_n(x_0, w)\| \, \|f^{(n)}(w)\| dw \le$$

$$\frac{1}{(n-1)!(p(n-1)+1)^{1/p}} \int_{x_0}^{y} (w-x_0)^{n-1+\frac{1}{p}} (z(w) z'(w))^{1/q} dw \le$$

$$\frac{1}{(n-1)!(p(n-1)+1)^{1/p}} \left( \int_{x_0}^{y} (w-x_0)^{p(n-1)+1} dw \right)^{1/p} \left( \int_{x_0}^{y} z(w) z'(w) dw \right)^{1/q}$$

$$= \frac{(y-x_0)^{n-1+\frac{2}{p}}}{2^{1/q}(n-1)![(p(n-1)+1)(p(n-1)+2)]^{1/p}}(z(y))^{2/q}, \tag{12}$$

proving the claim of the theorem. □

We continue with

**Theorem 4.** *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1,\ y \in [a, x_0]$.

*Then*

$$I_2 := \int_y^{x_0} \|E_n(x_0, w)\| \, \|f^{(n)}(w)\| dw \leq$$

$$\frac{(x_0-y)^{n-1+\frac{2}{p}}}{2^{1/q}(n-1)![(p(n-1)+1)(p(n-1)+2)]^{1/p}} \left(\int_y^{x_0} \|f^{(n)}(w)\|^q dw\right)^{2/q}. \tag{13}$$

We give

**Corollary 5.** *For* $y \in [a, x_0]$ *we have*

$$I_2 \leq \frac{(x_0-y)^n}{2(n-1)!\sqrt{n(2n-1)}} \left(\int_y^{x_0} \|f^{(n)}(w)\|^2 dw\right). \tag{14}$$

**Proof of Theorem 4.** We have by (3) that

$$\|E_n(x_0, y)\| = \frac{1}{(n-1)!} \left\|\int_y^{x_0} (y-t)^{n-1} f^{(n)}(t) dt\right\| \leq$$

$$\frac{1}{(n-1)!} \int_y^{x_0} (t-y)^{n-1} \|f^{(n)}(t)\| dt \leq$$

$$\frac{1}{(n-1)!} \left(\int_y^{x_0} (t-y)^{p(n-1)} dt\right)^{1/p} \left(\int_y^{x_0} \|f^{(n)}(t)\|^q dt\right)^{1/q} =$$

$$= \frac{(x_0-y)^{n-1+\frac{1}{p}}}{(n-1)!(p(n-1)+1)^{1/p}} (z(y))^{1/q}. \tag{15}$$

Here

$$z(y) := \int_y^{x_0} \|f^{(n)}(t)\|^q dt, \quad (z(x_0) = 0), \tag{16}$$

and

$$-z(y) = \int_{x_0}^{y} \|f^{(n)}(t)\|^q \, dt \leq 0, \tag{17}$$

$$-z'(y) = \|f^{(n)}(y)\|^q \geq 0, \tag{18}$$

and

$$\|f^{(n)}(y)\| = (-z'(y))^{1/q}, \quad a \leq y \leq x_0. \tag{19}$$

Hence

$$\|E_n(x_0, y)\| \, \|f^{(n)}(y)\| \leq$$

$$\frac{(x_0-y)^{n-1+\frac{1}{p}}}{(n-1)!(p(n-1)+1)^{1/p}} \, (z(y)(-z'(y)))^{1/q}, \quad a \leq y \leq x_0. \tag{20}$$

Therefore we find

$$\int_y^{x_0} \|E_n(x_0,w)\| \, \|f^{(n)}(w)\| dw \leq$$

$$\frac{1}{(n-1)!(p(n-1)+1)^{1/p}} \int_y^{x_0} (x_0-w)^{n-1+\frac{1}{p}} (z(w)(-z'(w)))^{1/q} dw \tag{21}$$

$$\leq \frac{1}{(n-1)!(p(n-1)+1)^{1/p}} \left( \int_y^{x_0} (x_0-w)^{p(n-1)+1} dw \right)^{1/p} \left( \int_y^{x_0} z(w)(-z'(w)) dw \right)^{1/q} \tag{22}$$

$$= \frac{(x_0-y)^{n-1+\frac{2}{p}}}{2^{1/q}(n-1)![(p(n-1)+1)(p(n-1)+2)]^{1/p}} (z(y))^{2/q}, \forall\, y \in [a,x_0], \tag{23}$$

proving the claim of the theorem. □

Combining Theorem 2 and 4 we get

**Theorem 6.** *Let $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$ and $y,x_0 \in [a,b]$.*

*Then*

$$I := \left| \int_{x_0}^{y} \|E_n(x_0,w)\| \, \|f^{(n)}(w)\| dw \right| \leq$$

$$\frac{|y-x_0|^{n-1+\frac{2}{p}}}{2^{1/q}(n-1)![(p(n-1)+1)(p(n-1)+2)]^{1/p}} \left| \int_{x_0}^{y} \|f^{(n)}(w)\|^q dw \right|^{2/q}. \tag{24}$$

Combining Corollaries 3 and 5 we have

**Corollary 7.** *For $y,x_0 \in [a,b]$ it holds*

$$I \leq \frac{|y-x_0|^n}{2(n-1)!\sqrt{n(2n-1)}} \left| \int_{x_0}^{y} \|f^{(n)}(w)\|^2 dw \right|. \tag{25}$$

A special but important case follows

**Theorem 8.** *Let $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$ and $y,x_0 \in [a,b]$. Assume further that $f^{(k)}(x_0)=0,\ k=0,1,...,n-1$.*

*Then*

$$\left| \int_{x_0}^{y} \|f(w)\| \, \|f^{(n)}(w)\| dw \right| \leq$$

$$\min \left( \frac{|y-x_0|^{n-1+\frac{2}{p}}}{2^{1/q}(n-1)![(p(n-1)+1)(p(n-1)+2)]^{1/p}} \left| \int_{x_0}^{y} \|f^{(n)}(w)\|^q dw \right|^{2/q}, \right.$$

$$\left. \frac{|y-x_0|^n}{2(n-1)!\sqrt{n(2n-1)}} \left| \int_{x_0}^{y} \|f^{(n)}(w)\|^2 dw \right| \right). \tag{26}$$

**Proof.** By Theorem 6 and Corollary 7. □

We continue with

**Theorem 9.** *Let $p=1,\ q=\infty$ and $y \in [x_0,b]$.*

*Then*

$$\int_{x_0}^{y} \|E_n(x_0,w)\| \, \|f^{(n)}(w)\| dw \leq \frac{(y-x_0)^{n+1}}{(n+1)!} \|f^{(n)}\|^2_{\infty,[x_0,b]}. \tag{27}$$

**Proof.** We have by (3) that

$$\|E_n(x_0,y)\| \leq \frac{1}{(n-1)!}\int_{x_0}^{y} (y-t)^{n-1} \|f^{(n)}(t)\| dt \leq \|f^{(n)}\|_{\infty,[x_0,b]} \, \frac{(y-x_0)^n}{n!}. \tag{28}$$

Therefore it holds

$$\|E_n(x_0,y)\| \, \|f^{(n)}(y)\| \leq \|f^{(n)}\|^2_{\infty,[x_0,b]} \frac{(y-x_0)^n}{n!}, \tag{29}$$

all $x_0 \leq y \leq b$.

Hence we find

$$\int_{x_0}^{y} \|E_n(x_0,w)\| \, \|f^{(n)}(w)\| dw \leq \frac{\|f^{(n)}\|^2_{\infty,[x_0,b]}}{n!} \int_{x_0}^{y} (w-x_0)^n dw$$

$$= \frac{(y-x_0)^{n+1}}{(n+1)!} \|f^{(n)}\|^2_{\infty,[x_0,b]}. \tag{30}$$

□

The counterpart of Theorem 9 follows.

**Theorem 10.** *Let $p=1$, $q=\infty$ and $y \in [a,x_0]$.*
*Then*

$$\int_{y}^{x_0} \|E_n(x_0,w)\| \, \|f^{(n)}(w)\| dw \leq \frac{(x_0-y)^{n+1}}{(n+1)!} \|f^{(n)}\|^2_{\infty,[a,x_0]}. \tag{31}$$

**Proof.** We have by (3) that

$$\|E_n(x_0,y)\| = \frac{1}{(n-1)!} \left\| \int_{x_0}^{y} (y-t)^{n-1} f^{(n)}(t) dt \right\| =$$

$$\frac{1}{(n-1)!} \left\| \int_{y}^{x_0} (y-t)^{n-1} f^{(n)}(t) dt \right\| \leq \frac{1}{(n-1)!} \int_{y}^{x_0} (t-y)^{n-1} \|f^{(n)}(t)\| dt \tag{32}$$

$$\leq \frac{\|f^{(n)}\|_{\infty,[a,x_0]}}{(n-1)!} \int_{y}^{x_0} (t-y)^{n-1} dt = \|f^{(n)}\|_{\infty,[a,x_0]} \frac{(x_0-y)^n}{n!}. \tag{33}$$

Therefore it holds

$$\|E_n(x_0,y)\| \, \|f^{(n)}(y)\| \leq \|f^{(n)}\|^2_{\infty,[a,x_0]} \frac{(x_0-y)^n}{n!}, \tag{34}$$

all $a \leq y \leq x_0$.

Consequently

$$\int_y^{x_0} \|E_n(x_0,w)\| \|f^{(n)}(w)\| dw \leq \frac{\|f^{(n)}\|^2_{\infty,[a,x_0]}}{n!} \int_y^{x_0} (x_0-w)^n dw$$

$$= \|f^{(n)}\|^2_{\infty,[a,x_0]} \frac{(x_0-y)^{n+1}}{(n+1)!}. \tag{35}$$

□

Combining Theorems 9 and 10 we get

**Proposition 11.** *Let* $p=1,\ q=\infty,\ y,x_0 \in [a,b]$.

*Then*

$$\left| \int_{x_0}^{y} \|E_n(x_0,w)\| \, \|f^{(n)}(w)\| dw \right| \leq \frac{|y-x_0|^{n+1}}{(n+1)!} \|f^{(n)}\|^2_{\infty}. \tag{36}$$

In particular we obtain

**Proposition 12.** *Let* $p=1$, $q=\infty$, $y,x_0 \in [a,b]$. *Further assume that* $f^{(k)}(x_0)=0$, $k=0,1,...,n-1$.

*Then*

$$\left| \int_{x_0}^{y} \|f(w)\| \, \|f^{(n)}(w)\| dw \right| \leq \frac{|y-x_0|^{n+1}}{(n+1)!} \|f^{(n)}\|^2_{\infty}. \tag{37}$$

## 4. Applications

Here $X=\mathbb{C}$, $f \in C^n([a,b],\mathbb{C})$, $n \in \mathbb{N}$, $x_0,y \in [a,b]$. Furthermore assume that $f^{(k)}(x_0)=0$, $k=0,1,...,n-1$.

Applying Theorem 8 we get

**Theorem 13.** *Let* $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$.

*Then*

$$\left| \int_{x_0}^{y} |f(w)| \, |f^{(n)}(w)| dw \right| \leq$$

$$\min\left( \frac{|y-x_0|^{n-1+\frac{2}{p}}}{2^{1/q}(n-1)![(p(n-1)+1)(p(n-1)+2)]^{1/p}} \left| \int_{x_0}^{y} |f^{(n)}(w)|^q dw \right|^{2/q}, \right.$$

$$\left. \frac{|y-x_0|^n}{2(n-1)!\sqrt{n(2n-1)}} \left| \int_{x_0}^{y} |f^{(n)}(w)|^2 dw \right| \right). \tag{38}$$

By Proposition 12 we find

**Proposition 14.** *Let* $p=1,\ q=\infty$.

*Then*

$$\left| \int_{x_0}^{y} |f(w)| \, |f^{(n)}(w)| dw \right| \leq \frac{|y-x_0|^{n+1}}{(n+1)!} \|f^{(n)}\|^2_{\infty}. \tag{39}$$

Let now $f \in C^1([a,b],\mathbb{C}),\ x_0, y \in [a,b]$ with $f(x_0) = 0$. Applying Theorem 13 we obtain

**Theorem 15.** *Let* $p,\ q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

*Then*

$$\left|\int_{x_0}^{y} |f(w)|\,|f'(w)|dw\right| \leq$$

$$\frac{1}{2}\min\left(|y-x_0|^{2/p}\left|\int_{x_0}^{y}|f'(w)|^q dw\right|^{2/q}, |y-x_0|\left|\int_{x_0}^{y}|f'(w)|^2 dw\right|\right). \tag{40}$$

Finally by applying Proposition 14 we get

**Proposition 16.** *Let* $p = 1,\ q = \infty$.

*Then*

$$\left|\int_{x_0}^{y} |f(w)|\,|f'(w)|dw\right| \leq \frac{(y-x_0)^2}{2}\|f'\|_\infty^2. \tag{41}$$

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◇ George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152
USA
ganastss@memphis.edu