BULLETIN OF THE
GREEK MATHEMATICAL SOCIETY
Volume 55, 2008 (19–27)

On The Three-Valued Monotone Connectives

George Koletsos

Abstract

We examine some questions concerning the 3-valued monotone propositional connectives, mainly as extensions of the usual 2-valued boolean connectives. We prove that the maximum monotone extensions correspond to the Quine-McCluskey minimalization. We give some results concerning adequacy of connectives.

1. Introduction

Three-valued logic was introduced by Łukasiewicz, mainly based on philosophical considerations, and by Post, based on mathematical reasons. More recently the interest shifted to the applications of this logic to theoretical computer science and artificial intelligence. A whole genealogy of three-valued logics was introduced depending on the interpretation of the third value. Among those a prominent position is held by Kleene's three-valued logic and its descendants, see [6]. In three-valued logic besides the usual truth values T (true) and F (false) there is a third truth value *. There are different interpretations of this third value. According to Kleene's this can be considered as a truth value gap, or as a state of incomplete information. Assigning to a proposition the value * means that we don't know (yet) its definite truth value (T or F); but we may eventually have the chance to find it out, that is * is the truth value of undefined. This reading suggests a natural condition: the behavior of the third truth value should be compatible with any increase in information. That is, if the value of some statement, S say, is changed from undefined to either true or false, the value of any formula with S as a component should never change from true to false or from false to true, though a change from undefined to one of false or true is allowed. Kleene referred to this as regularity; today we phrase it in terms of monotonicity in an ordering that places undefined below both false and true. In what follows we examine some questions concerning the 3-valued monotone propositional connectives, mainly as extensions of the usual 2-valued boolean connectives.

2. Monotone connectives

In what follows T is the truth value for true, F for false and $*$ is the third truth value for undefined.

Definition 2.1 Let $\mathbb{V} = \{T, F, *\}$ and $\mathbb{B} = \{T, F\}$. $\mathbb{V}^n$ and $\mathbb{B}^n$ are the set of vectors $\bar{x} = \{x_1, \ldots, x_n\}$ where each x_i belongs, correspondingly, to $\mathbb{V}$ and $\mathbb{B}$. We denote by

$\bar{T}$, $\bar{F}$ and $\bar{*}$ the vectors for which all the x_i are equal to, correspondingly, T, F and $*$. An n-ary *3-valued connective* f is any function $f : \mathbb{V}^n \to \mathbb{V}$. Any such connective f is a $\mathbb{B}$-*connective* when for each $\bar{x} \in \mathbb{B}^n$ we have that $f(\bar{x}) \in \mathbb{B}$. An *n-ary Boolean connective* is any function from $\mathbb{B}^n$ to $\mathbb{B}$.

Examples: the Kleene connectives $\neg$, $\vee$ and $\wedge$ in [6] defined by the following tables are 3-valued connectives.

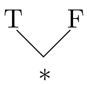
$\neg$	
T	F
F	T
*	*

$\vee$	T	F	*
T	T	T	T
F	T	F	*
*	T	*	*

$\wedge$	T	F	*
T	T	F	*
F	F	F	F
*	*	F	*

All of them are $\mathbb{B}$-connectives and they are extensions of the usual two-valued Boolean connectives $\neg$, $\vee$ and $\wedge$ for which we use the same notation.

Definition 2.2 We induce in the set $\mathbb{V}$ a partial ordering by the following diagram:



i.e. $* \leq T$ and $* \leq F$

The intuitive meaning of this is that by going from * to either T or F, that is by filling a truth value gap, we go one level higher to the order of definedness or to the amount of information we dispose.

We extend this partial ordering to $\mathbb{V}^n$ as follows:

Definition 2.3 For $\bar{x} = (x_1 \dots x_n)$ and $\bar{y} = (y_1 \dots y_n)$ in $\mathbb{V}^n$ we define $\bar{x} \leq \bar{y}$ to mean that for all $i \in \{1, ..., n\}$ we have that $x_i \leq y_i$.

Intuitively, if we have $\bar{x} \leq \bar{y}$ then $\bar{y}$ represents better information on the validity of the conditions (or propositions) corresponding to the coordinates $1, 2, \dots, n$. As usual, $\bar{x} < \bar{y}$ means $\bar{x} \leq \bar{y}$ and $\bar{x} \neq \bar{y}$. This ordering on $\mathbb{V}^n$ can be represented by a direct acyclic graph having n+1 different levels. On the lowest level, the 0-level, we have as root the vector $\bar{*}$, which is the minimum vector. By going upwards, on the k-level we put all the vectors having k coordinates belonging to $\mathbb{B}$. So on the n-level we put the maximal elements i.e. the elements of $\mathbb{B}^n$. When $\bar{x}$ is on the k-level (with $k < n$) there are $n - k$ coordinates having $*$ as value. If i is such a coordinate then by substituting x_i (which is equal to $*$) by T or by F we get two vectors lying on the $k + 1$ level, the two *successors* of $\bar{x}$ corresponding to the coordinate i. E.g. $(*, T, F)$ is on the level 2 of $\mathbb{V}^3$ and (T, T, F), (F, T, F) are the successors on the level 3 (corresponding to the coordinate 1). Of course a vector $\bar{x}$ may have many successors if it has many coordinates with value $*$. If $\bar{x}$ is on the k-level, $\bar{x}$ has $2(n-k)$ successors. If we join $\bar{x}$ and its successors by a line segment then we have that $\bar{x} \leq \bar{y}$ when the level of $\bar{x}$ is less than the level of $\bar{y}$ and there exists a path leading from $\bar{x}$ upwards to $\bar{y}$. This graph which represents the ordering $\leq$ on $\mathbb{V}^n$ is called the *level-graph* of $\mathbb{V}^n$.

e.g. Part of the level-graph of $\mathbb{V}^n$.

....(T,T,*)...................(*,T,F)................. level 2

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....(T,*,*).....(*,T,*)....(*,F,*)................. level 1
            \            /
              (*,*,*)                      level 0
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Definition 2.4 A connective $f : \mathbb{V}^n \to \mathbb{V}$ is *monotone* if whenever $\bar{x} \leq \bar{y}$ we have that $f(\bar{x}) \leq f(\bar{y})$.

Intuitively, the notion of monotonicity represents the situation in which on the basis of better information $\bar{y}$ (if $\bar{x} \leq \bar{y}$ then $\bar{y}$ is better in the sense that it has "more" information, in every coordinate, than $\bar{x}$) we get a better answer (because $f(\bar{x}) \leq f(\bar{y})$ so $f(\bar{y})$ is a better answer than $f(\bar{x})$).

Theorem 2.5 Let $f : \mathbb{V}^n \to \mathbb{V}$ be a three valued connective. The following are equivalent:

1 f is monotone

2 $f(\bar{x}) \in \mathbb{B}$ and $\bar{x} \leq \bar{y}$ imply $f(\bar{x}) = f(\bar{y})$.

Proof. Immediate.□

Definition 2.6 Let $f : \mathbb{V}^n \to \mathbb{V}$ be a monotone connective. Let $\mathbb{T}_f = \{\bar{x} \in \mathbb{V}^n : f(\bar{x}) = T\}$ and $\mathbb{F}_f = \{\bar{x} \in \mathbb{V}^n : f(\bar{x}) = F\}$. $\bar{x} \in \mathbb{V}^n$ is called *T-minimal* (*F-minimal*) if $\bar{x}$ is a minimal element of $\mathbb{T}_f$ ($\mathbb{F}_f$) with respect to $\leq$.

Theorem 2.7 $\bar{x}$ is T-minimal iff $f(\bar{x}) = T$ and for any $\bar{y} < \bar{x}$ we have that $f(\bar{y}) = *$. An analogous result holds for F-minimal.

Proof: Immediate using monotonicity and theorem 2.5.□

Theorem 2.8 If f is a monotone connective then the following *separation condition* holds:

It is not possible to have $\bar{x}, \bar{y}, \bar{z} \in \mathbb{V}^n$ *s.t.* $\bar{y} \in \mathbb{T}_f$, $\bar{z} \in \mathbb{F}_f$ *and* $\bar{y} \leq \bar{x}$, $\bar{z} \leq \bar{x}$.

As a special case we have the following:

No member of $\mathbb{T}_f$ ($\mathbb{F}_f$) is less than any member of $\mathbb{F}_f$ ($\mathbb{T}_f$).

Proof. If not then by theorem 2.5 $f(\bar{x}) = f(\bar{y})$ and $f(\bar{x}) = f(\bar{z})$. Impossible since $f(\bar{y}) = T$ and $f(\bar{z}) = F$.□

Theorem 2.9 Let $\mathbb{T}$ and $\mathbb{F} \subseteq \mathbb{V}^n$ such that if $\bar{x}$ belongs to $\mathbb{T}$ (or $\mathbb{F}$) and $\bar{x} \leq \bar{y}$ then $\bar{y} \in \mathbb{T}$ (or $\mathbb{F}$). Let also $\mathbb{T}$ and $\mathbb{F}$ satisfy the separation condition of theorem 2.8. Then there exists a unique monotone connective $f : \mathbb{V}^n \to \mathbb{V}$ s.t. $\mathbb{T}_f = \mathbb{T}$ and $\mathbb{F}_f = \mathbb{F}$.

Proof. Existence: Just put

$$f(\bar{x}) = \begin{cases} T & \text{if} \quad \bar{x} \in \mathbb{T} \\ F & \text{if} \quad \bar{x} \in \mathbb{F} \\ * & \text{otherwise} \end{cases}$$

The definition is unambiguous because $\mathbb{T} \cap \mathbb{F} = \emptyset$, from the separation condition. This connective is monotone because if $f(\bar{x}) \in \mathbb{B}$, say $f(\bar{x}) = T$, and $\bar{x} \leq \bar{y}$ then $\bar{x} \in \mathbb{T}$ and $\bar{y} \in \mathbb{T}$ from hypotheses so by definition $f(\bar{y}) = T$.

Uniqueness: If g is another connective then for each $\bar{x} \in \mathbb{V}^n$ we have either $\bar{x} \in \mathbb{T}$ so as $\mathbb{T}_f = \mathbb{T} = \mathbb{T}_g$ we take $f(\bar{x}) = g(\bar{x}) = T$ either $\bar{x} \in \mathbb{F}$, so for the same reason $f(\bar{x}) = g(\bar{x})$ or $\bar{x} \notin \mathbb{T} \cup \mathbb{F}$ so $\bar{x} \notin \mathbb{T}_f \cup \mathbb{F}_f = \mathbb{T}_g \cup \mathbb{F}_g$, so $f(\bar{x}) = g(\bar{x}) = *$.□

Every pair of $\mathbb{T}, \mathbb{F}$ satisfying the conditions of theorem 2.9 is called a *decomposition* of $\mathbb{V}^n$. Then by this theorem there is an isomorphism between the set of 3-valued monotone connectives and the set of the decompositions of $\mathbb{V}^n$.

Definition 2.10 Let $A \subseteq \mathbb{V}^n$. We say that A satisfies the *minimality* condition if there are no elements $\bar{x}, \bar{y} \in A$ s.t. $\bar{x} < \bar{y}$.

Theorem 2.11 Let $\mathbb{T}^m, \mathbb{F}^m \subseteq \mathbb{V}^n$ both satisfy the minimality condition as well as the separation property of theorem 2.8. Then there exists a unique monotone connective $f : \mathbb{V}^n \to \mathbb{V}$ having as T-minimal elements the elements of $\mathbb{T}^m$ and as F-minimal elements the elements of $\mathbb{F}^m$.

Proof. Let

$$\mathbb{T} = \{\bar{y} : \exists \bar{x} \in \mathbb{T}^m, \bar{x} \leq \bar{y}\}$$
$$\mathbb{F} = \{\bar{y} : \exists \bar{x} \in \mathbb{F}^m, \bar{x} \leq \bar{y}\}$$

Obviously $\mathbb{T}^m$ is the set of the minimal elements of $\mathbb{T}$ and $\mathbb{F}^m$ of the minimal elements of $\mathbb{F}$. $\mathbb{T}$ and $\mathbb{F}$ satisfy the conditions of theorem 2.9. The first condition is satisfied, because $\leq$ is transitive. The separation condition is satisfied, because of the transitivity and the fact that $\mathbb{T}^m$ and $\mathbb{F}^m$ satisfy this condition. The connective of the theorem 2.9 is the unique connective satisfying the conclusion of theorem 2.11.□

Every decomposition $(\mathbb{T}, \mathbb{F})$ can be uniquely determined by the pair of the sets of the minimum elements of $\mathbb{T}$ and $\mathbb{F}$ ($\mathbb{T}^n$, $\mathbb{F}^n$ correspondingly). Theorem 2.11 says that the converse is also true. So every 3-valued monotone connective can be thought of as a pair ($\mathbb{T}^n$, $\mathbb{F}^n$). We call this pair the *minimal representation* of the decomposition.

Definition 2.12 Let $\bar{x} \in \mathbb{V}$. Define $[\bar{x}] = \{\bar{y} \in \mathbb{V} : \bar{x} \leq \bar{y}\}$.

Let $g : \mathbb{B}^n \to \mathbb{V}$. We call g a *quasi-boolean* connective. Every 3-valued connective f restricted to $\mathbb{B}^n$ is a quasi boolean connective. Conversely, for every quasi boolean connective there are many 3-valued extensions. Among them there is the trivial one, which is the extension of g that takes the value * at all $\bar{x} \in \mathbb{V}^n - \mathbb{B}^n$. But interestingly enough there is also a maximum one.

Definition 2.13 We induce an ordering $\ll$ among the 3-valued n-ary connectives by the following:

If $f, g \in \mathbb{V}^n \to \mathbb{V}, f \ll g$ iff $\forall \bar{x} \in \mathbb{V}^n$ we have $f(\bar{x}) \leq g(\bar{x})$.

Theorem 2.14 Let g be a quasi-boolean connective. Consider the set M_g of the monotone connectives which are extensions of g (i.e. $\bar{x} \in \mathbb{B}^n \to f(\bar{x}) = g(\bar{x})$). Then this set has a maximum element with respect to $\ll$.

Proof. We first prove the following lemma.

Remark 2.15 Since a boolean connective is a quasi-boolean connective, theorem 2.14 holds also true for the extensions of the boolean connectives.

Lemma 2.16 Let $f_1, f_2 \in M_g$. Then there is an f, in M_g, such that $f_1 \ll f$ and $f_2 \ll f$.

Proof of the lemma: Let $\bar{x} \notin \mathbb{B}^n$. Then it is not possible to have that both $f_1(\bar{x})$ and $f_2(\bar{x})$ belong to $\mathbb{B}$ and $f_1(\bar{x}) \neq f_2(\bar{x})$. Because if not we would have, say, $f_1(\bar{x}) = T$ and $f_2(\bar{x}) = F$. Taking $\bar{y} \in \mathbb{B}^n$ with $\bar{y} \geq \bar{x}$, the monotonicity of f_1, f_2 would give $f_1(\bar{y}) = T \neq F = f_2(\bar{y})$, contradicting the fact that f_1, f_2 agree on $\mathbb{B}^n$. Define now $f(\bar{x}) = max\{f_1(\bar{x}), f_2(\bar{x})\}$, which by the above exists for every $\bar{x}$. It is clear that $f_1 \ll f$ and $f_2 \ll f$. In addition, f is monotone. Because, let us suppose (use Theorem 2.5) that $\bar{x} \leq \bar{y}$ and $f(\bar{x}) \in \mathbb{B}$ - e. g. say that $f(\bar{x}) = T$. Then one of the $f_1(\bar{x})$ and $f_2(\bar{x})$, let's say $f_1(\bar{x})$, is equal to T which gives that $f_1(\bar{y}) = T$. This gives $max\{f_1(\bar{y}), f_2(\bar{y})\} = T = f(\bar{y}) = f(\bar{x})$.

Proof of the theorem: As the number of the monotone extensions of g is finite there must exist maximal elements. But as, by the lemma, it is not possible to have two different maximal elements, there must exist a maximum one.□

As we shall see later, the maximum monotone extension of a boolean connective represents the least possible representation of g.

Definition 2.17 Let $\bar{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{V}^n$. Let $j_1, j_2, \ldots, j_k \in (1, 2, \ldots, n)$ be all the coordinates i s.t. $x_i \in \mathbb{B}$. That is $x_{j_1}, \ldots, x_{j_k} \in \mathbb{B}$ and if $i \neq j_1, \ldots, j_k$ we have that $x_i = *$. If $A_1, A_2, \ldots$ is an enumeration of the propositional variables we define two propositional formulae corresponding to $\bar{x}$ as follows:

The *conjunctive representation* $\Phi_{\bar{x}} = A'_{j_1} \wedge A'_{j_2} \wedge \cdots \wedge A'_{j_k}$ by:

$$A'_j = \begin{cases} A_j & if \quad x_j = T \\ \neg A_j & if \quad x_j = F \end{cases} \text{ where } j \in \{j_1, \ldots, j_k\}$$

The *disjunctive representation* $\Psi_{\bar{x}} = A'_{j_1} \vee A'_{j_2} \vee \cdots \vee A'_{j_k}$ by:

$$A'_j = \begin{cases} A_j & if \quad x_j = F \\ \neg A_j & if \quad x_j = T \end{cases}$$

Theorem 2.18 Every vector $\bar{x} = \{x_1, \ldots, x_n\}$ can be considered as an evaluation giving truth values to $A_1, \ldots, A_n$ i.e. $A_i \mapsto x_i$. Then:

1 $\Phi_{\bar{x}}$ becomes true for the evaluation $\bar{y}$ iff $\bar{x} \leq \bar{y}$.

2 $\Psi_{\bar{x}}$ becomes false for the evaluation $\bar{y}$ iff $\bar{x} \leq \bar{y}$.

Proof: (1) $\Rightarrow$: By construction all the A'_j must be true so in all the places $j_1, \ldots, j_k$ the vector $\bar{y}$ must coincide with $\bar{x}$. And because in all the other places of $\bar{x}$ we have the value * we that $\bar{x} \leq \bar{y}$.

$\Leftarrow$: As $\bar{x} \leq \bar{y}$, $\bar{y}$ coincides with $\bar{x}$ in all the places having values T or F. And as all the other places are ignored in the construction of $\Phi_{\bar{x}}$ we finally have that the evaluation $\bar{y}$ gives the same value with the evaluation $\bar{x}$ which by construction gives T.

(2) Similar proof (In fact $\Psi_{\bar{x}}$ and $\Phi_{\bar{x}}$ are dual).□

3. The representation of $\mathbb{T}_f$ and $\mathbb{F}_f$

Let f be a monotone connective. Let $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$ be the T-minimal elements of f and $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m$ the F-minimal elements. We define:

$$\Omega_T = \Phi_{\bar{x}_1} \vee \cdots \vee \Phi_{\bar{x}_k} \quad \text{and} \quad \Omega_F = \Psi_{\bar{y}_1} \wedge \cdots \wedge \Psi_{\bar{y}_k}$$

Theorem 3.1 Let $\bar{z} \in \mathbb{V}^n$ be an evaluation of $A_1, ..., A_n$. Then

1 Ω_T becomes true for $\bar{z}$ iff $\bar{z} \in \mathbb{T}_f$

2 Ω_F becomes false for $\bar{z}$ iff $\bar{z} \in \mathbb{F}_f$

Proof. (1) $\Rightarrow$: If Ω_T true then $\Phi_{\bar{x}_i}$ becomes true for $\bar{z}$ for some $\bar{x}_i$ minimal element of $\mathbb{T}_f$. But then by proposition $\bar{x}_i \leq \bar{z}$ so $\bar{z} \in \mathbb{T}_f$.
$\Leftarrow$: If $\bar{z} \in \mathbb{T}_f$ then $\bar{x}_i \leq \bar{z}$ for some T-minimal element $\bar{x}_i$. By theorem 2.18 $\Phi_{\bar{x}_i}$ becomes true for $\bar{z}$. So Ω_T also becomes true.
(2) Similar proof.□

Theorem 3.2 Let f be a $\mathbb{B}$-connective and g be its restriction to $\mathbb{B}^n$. Then Ω_T is a disjunctive form of g and Ω_F is a conjunctive form of g.

Proof. It suffices to prove that Ω_T becomes true for the same $\bar{z} \in \mathbb{B}^n$ for which we have that $g(\bar{z}) = T$. But according to theorem 3.1 Ω_T becomes true only for $\bar{z} \in \mathbb{T}_f$, so for such a $\bar{z} \in \mathbb{B}^n$ we have that $g(\bar{z}) = T$, and conversely if $g(\bar{z}) = T$ then $\bar{z} \in \mathbb{T}_f$ so Ω_T becomes true for $\bar{z}$. Now as $\mathbb{T}_f$ and $\mathbb{F}_f$ do not have any elements in common (theorem 2.8) we have by theorem 3.1 that for every $\bar{y} \in \mathbb{B}^n, \Omega_T(\bar{y}) = T$ iff $\bar{y} \in \mathbb{T}_f$ iff $\bar{y} \notin \mathbb{F}_f$ iff $\Omega_F(\bar{y}) \neq F$ iff $\Omega_F(\bar{y}) = T$.□

Remark 3.3 Instead of taking all the $\bar{x}_1, \ldots, \bar{x}_k$ for the construction of Ω_T it suffices to take any subset $\bar{x}_{j_1}, \ldots, \bar{x}_{j_p}$ of them for which $[\bar{x}_{j_1}] \cup \cdots [\bar{x}_{j_p}] = \{\bar{x} \in \mathbb{B}^n : f(\bar{x}) = T\}$. Call this a *T-cover* of f. Then the Ω_T, so chosen, is still a disjunctive form of g. So we can choose Ω_T to be minimal in the sense of having the least possible number of occurrences of the variables in the disjunctive representation. Similar considerations apply for Ω_F.

If we have a Boolean connective g then according to theorem 3.2 in order to find a disjunctive (conjunctive) form of it it suffices to take the Ω_T (Ω_F) of any monotone extension of g. What happens if we choose the maximum of its monotone extensions? We claim that by doing so we in fact choose the best possible minimization in the sense of Quine (see [5] and [9]).

Theorem 3.4 If $P \equiv P_1 \vee \cdots \vee P_k$ is a disjunctive form then P is equivalent to some Ω_T, for some f which is a monotone extension of the boolean connective g defined by P. Furthermore Ω_T is a disjunction of some of the components $P_1, \ldots, P_n$.

Proof. Let $\bar{x}_i$ be the vector of which P_i is the conjunctive representation. Let $\mathbb{T}^m$ be the minimal elements of $\bar{x}_1, \ldots, \bar{x}_k$. Let $\mathbb{F}^m = \{\bar{y}_1, \ldots, \bar{y}_m\}$ be the vectors belonging to $\mathbb{B}^n$ and corresponding to the evaluations refuting P. Obviously the sets $\mathbb{T}^m, \mathbb{F}^m$ satisfy the minimality and the separation condition so, by theorem 2.11, they give a monotone connective f. It is clear that g is the restriction of f to $\mathbb{B}^n$.

By theorem 3.2 Ω_T (of the extension f) is a disjunctive form of g which in addition can be considered as a first simplification of P. A further simplification can be obtained by taking any of the T-covers. In the sequel, taking an extension h of f we have that every component of the Ω_T (of h) contains only a subset of literals of some component of Ω_T (of f). That is a further essential simplification. It is clear that the maximum monotone extension of f will give the best possible simplification (or minimization) of f or equivalently of the disjunctive form P realizing f.□

How can we construct the maximum extension, given a boolean connective g or a disjunctive form P? One method is to work on the level graph of $\mathbb{V}^n$ starting from the top and going downwards, each time grouping all the elements which must take the value T in the maximum monotone extension. Starting from the top means equivalently that we take the (full) disjunctive normal form of g, that is the disjunctive form which contains all the literals $A_1, \dots, A_n$ in every component of the disjunction.

The procedure: We generate the sequences $\Omega_n, \Omega_{n-1}, \dots$ where each Ω_k is a set of elements at the k-level, as follows. For each Ω_k we examine whether there exist two elements $\bar{x}, \bar{y} \in \Omega_k$ s.t. $\bar{x}, \bar{y}$ are the successors of some element $\bar{z}$. Then we put $\bar{z}$ in Ω_{k-1}. All the elements in Ω_k which are not bigger than some element in Ω_{k-1} constitute the set Ω'_k. Then the set Ω of the minimal elements of the maximum extension is the union $\Omega'_n \cup \Omega'_{n-1} \cup \dots \cup \Omega'_p$ (the procedure terminates at the level p where we put $\Omega'_p = \Omega_p$). If we start by putting $\Omega_n = \{\bar{x} \in \mathbb{B}^n : g(\bar{x}) = T\}$ then it is clear that the set Ω are the T-minimal elements of a monotone extension of g. These must be the T-minimal elements of the maximum extension as any other extension of Ω would not satisfy the separation condition with the set $\mathbb{B}^n - \Omega_n$. Because if we take any proper extension f' by putting $f'(\bar{z}) = T$, where before $f(\bar{z}) = *$, we must have that $\bar{z}$ is a successor of some element in Ω'_k $(n > k \geq p)$ [because otherwise we would already have put $f(\bar{z}) = T$]. By construction, there must exist a path starting from $\bar{z}$ and leading to some element $\bar{w} \notin \Omega_n$ i.e. $f(\bar{w}) = F$. So $\bar{w}$ cannot take the value T. A similar extension procedure applies to the F-side.

But this procedure of finding the maximum monotone extension is exactly the Quine-McCluskey procedure of the minimization of a Boolean connective! (see [9], [3]).

4. Adequacy of the connectives

We saw in theorem 2.11 that a monotone connective can be identified with the pair $(\mathbb{T}^m, \mathbb{F}^m)$ of its T-minimal and F-minimal elements. All these elements are necessary since the addition or omission of any of them would change the connective. These elements, in their turn, are represented, correspondingly, with a disjunctive form Ω_T and a conjunctive form Ω_F which use only the connectives $\neg, \vee, \wedge$ in their construction. In what follows we see that these connectives are adequate for the class of $\mathbb{B}$-connectives. We consider $\mathbb{B}$-connectives which are not constant on $\mathbb{B}^n$. We prove first the following sharpening of theorem 3.1.

Theorem 4.1 let f be a monotone connective. We use the symbols defined in 2.17 and theorems 2.18 and 3.1. Let $\bar{z} \in \mathbb{V}^n$. Then

1 $\Omega_T(\bar{z}) = T$ iff $\bar{x}_i \leq \bar{z}$ for some T-minimal element $\bar{x}_i$.

2 $\Omega_T(\bar{z}) = F$ iff the sets $\mathbb{T}^m$ and $\{\bar{z}\}$ satisfy the separation condition.

3 $\Omega_T(\bar{z}) = *$ iff either $\exists\ \bar{x}_i$ s.t. $\bar{z} \leq \bar{x}_i$ or $\bar{z}$ is not related with any of the $\bar{x}_i$ and in addition there exists $\bar{x}_i$ s.t. $\bar{z}$ agrees with $\bar{x}_i$ in all the places j where $x_j \in \mathbb{B}$ except at some of them (at least one) where it takes the value $*$.

All the above holds true for the dual case, that is when T is replaced by F and F by T.

Proof: (1) is just theorem 2.18.

(2) $\Omega_T(\bar{z}) = F$ iff for every $\bar{x}_i$ of the disjunction Ω_T we have that $\Phi_{\bar{x}_i}(\bar{z}) = F$, so every conjunction $\Phi_{\bar{x}_i} = x_{j_1} \wedge \cdots \wedge x_{j_k}$ must be false, which means, by the very construction of $\Phi_{\bar{x}_i}$, that at least in one place j (where $x_j \in \mathbb{B}$) we have that $\bar{x}_i$ and $\bar{z}$ have different values, both belonging to $\mathbb{B}$, so impossible to have that $\bar{x}_i \leq \bar{w}$ and $\bar{z} \leq \bar{w}$ for some $\bar{w}$.

(3) $\Omega_T(\bar{z}) = *$ iff some of the $\Phi_{\bar{x}_i}$ become $*$ and none becomes T iff some of the $A'_{j_1} \wedge \cdots A'_{j_k}$ become $*$ (so because of the impossibility of the conjunction taking the value F, we have coincidence – except for some $*$ – at all the places) and as the $\Phi_{\bar{x}_i}(\bar{z})$ become true only if $\bar{x}_i \leq \bar{z}$ we have the second clause.

As the condition (1) implies the condition (2) but not vice versa we have:

$$\Omega_T(\bar{z}) = T \;\Rightarrow\; \Omega_F(\bar{z}) = T$$
$$\Omega_F(\bar{z}) = F \;\Rightarrow\; \Omega_T(\bar{z}) = F$$

Let $\star$ be a n-ary connective which takes the value $*$ when at least one of its arguments takes the value $*$ e.g. $\star = (A_1 \vee \neg A_1) \wedge \cdots \wedge (A_n \vee \neg A_n)$. Let f be a monotone connective. Then we have that

Theorem 4.2 The sentence $\Pi \equiv (\star \wedge \Omega_F) \vee \Omega_T$ represents the connective f.

Proof:

1 $f(\bar{x}) = T \;\Rightarrow\; \Omega_T(\bar{x}) = T \;\Rightarrow\; \Pi(\bar{x}) = T$.

2 $f(\bar{x}) = F \;\Rightarrow\; \Omega_F(\bar{x}) = F \;\Rightarrow\; \Pi(\bar{x}) = F$.

3 $f(\bar{x}) = * \;\Rightarrow \Omega_T(\bar{x}) \neq T$ and $\Omega_F(\bar{x}) \neq F$ i.e. the possibilities we have are $\Omega_T(\bar{x}) = F$ or $*$ and $\Omega_F(\bar{x}) = T$ or $*$. In any case $\Pi(\bar{x}) = *$.□

Remark 4.3 The dual sentence $(\star \vee \Omega_T) \wedge \Omega_F$ does the same job. This theorem, in a slightly different version was first proved in [7] (see also [2]). For a complete treatment of adequacy see [1].

Remark 4.4 All the combinations of the possibilities in clause 3 are possible. So the presence of $\star$ in Π seems to be necessary. But there is a case in which this is redundant.

Theorem 4.5 Suppose $f(\bar{z}) = *$ in a $\mathbb{B}$-connective f. Suppose also that $\exists\ \bar{y}$ s.t. $\bar{z} \leq \bar{y}$ and $f(\bar{y}) = T$. Then $\Omega_T(\bar{z}) = *$. (Similarly for the dual case).

Proof: Either $\bar{z}$ is less or equal with some T-minimal element or $\exists$ T-minimal $\bar{x}$ s.t. $\bar{x} \leq \bar{y}$. The vectors $\bar{z}$ and $\bar{x}$ agree on the places taking values in $\mathbb{B}$ except in at least one since otherwise we would have $\bar{x} \leq \bar{z}$ (and of course $f(\bar{z}) = T$). And $\bar{z}$ is not related with any of the other T-elements $\bar{x}_i$ because we would have $\bar{x}_i \leq \bar{z}$. By theorem 4.2 (3) we have that $\Omega_T(\bar{z}) = *$.□

So in order to have $\Omega_T(\bar{x}) = \Omega_F(\bar{x}) = *$ it suffices to have that $[\bar{x}]$ contains both T and F. This happens in the maximum extension! Because if $f(\bar{x}) = *$ and $[\bar{x}] = \{T\}$ then by putting $f(\bar{y}) = T$ to every $\bar{y} \geq \bar{x}$ we obtain a proper monotone extension of f. So we have proved that

Theorem 4.6 Let f be a maximum monotone extension (of a boolean connective g). Then $\Omega_T \vee \Omega_F$ (or equivalently $\Omega_T \wedge \Omega_F$) represents the 3-valued connective f.

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◇ George Koletsos
National Technical University of Athens
Department of Computer Science
Zografou Campus - 157 80 Athens - Greece
koletsos@math.ntua.gr