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A Comparison Between the Classical and the Fokas Generalized Transform Method

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Abstract

Aim of the present work is to compare two techniques for solving linear Partial Differential Equations. The first method is the well-known spectral method which leads to the classical transform. The second method is the generalized transform method, introduced by Fokas in the last decade.

Keywords: Square, Laplace Equation, separation of variables, generalized transform method

1. Introduction

A given, well-posed, boundary-value problem can only be solved by means of separation of variables if there exist a coordinate system that fits the boundary of the foundamental domain and at the same time it separates the partial differential equation (PDE). Separation of variables leads to the solution of PDEs by a transform pair. The "prototype" of such a pair is the direct and inverse Fourier transform. However, for complicated problems the classical transform method fails. For example, there do not exist proper transforms for solving many boundary-value problems for elliptic equations of second order and in simple domains.

In 1997, A.S. Fokas [2] proposed a general method for solving boundary-value problems for two-dimensional linear and integrable nonlinear PDEs. An equation in two dimensions (x_1, x_2) is called integrable iff it can be expressed as the condition that a certain differential 1-form $W(x_1, x_2; k),\ k \in \mathbb{C}$, is closed, e.g. linear PDEs with constant coefficients [1]. This novel approach can be seen as a generalization of the separation of variables method, but more effective (for a review see [3]). It is based on the *simultaneous* spectral analysis of the two compatible eigenvalue equations of the Lax pair associated with the PDE, i.e. construct two scalar linear eigenvalue equations whose compatibility condition is the given PDE. It expresses the solution in terms of the solution of a matrix Riemann-Hilbert problem in the complex k-plane of the spectral parameter k. The spectral functions $\rho(k)$ determining the Riemann-Hilbert problem are expressed in terms of the boundary values of the solution. Since for a well posed boundary-value problem only one boundary condition is given, some of the boundary values appearing in $\rho(k)$ are unknown. The fact that these boundary values are in general related can be expressed in a simple way in terms of a global relation, which plays a crucial role in the analysis of boundary-value problems, satisfied by the corresponding spectral functions.

1.1. Notations

— z denotes the complex variable

$$z = x + iy$$

— An overbar denotes complex conjugation, e.g.

$$\overline{z} = x - iy$$

and $\overline{F(\overline{k})}$ denotes the Schwarz conjugate of the function $F(k)$.

— The complex numbers

$$z_1 = L + iL, \quad z_2 = \overline{z}_1, \quad z_3 = \overline{z}_4, \quad z_4 = -L + iL$$

will denote the vertices of the square, and $\mathcal{D} \subset \mathbb{C}$ denotes the interior of the square. The length of each side is 2L. The sides $(z_2, z_1), (z_3, z_2), (z_4, z_3), (z_1, z_4)$ will be referred to as sides (1),(2),(3) and (4) respectively.

— On each side we identify the positive direction as $\hat{\mathbf{T}}$ and the outward normal as $\hat{\mathbf{N}}$, as shown in figure 1. The functions

$$q_j(s), \quad \partial_N q_j(s), \quad s \in [-L, L], \quad j = 1, 2, 3, 4$$

denotes the function $q(x, y)$, as well as the derivative of $q(x, y)$, along $\hat{\mathbf{N}}$ for each side (j) respectively.

— $\langle \cdot, \cdot \rangle, \langle \cdot, \cdot \rangle_w$ denotes the inner product and the weighted inner product respectively on $L^2[-L, L]$.

1.2. Formulation of the Problem

Investigation of the Laplace equation in the interior of the square $\mathcal{D}$ (fig.1), namely

$$\partial_{xx} q(x, y) + \partial_{yy} q(x, y) = 0, \quad (x, y) \in \mathcal{D}, \tag{1}$$

where $q(x, y)$ is a real valued function, with two different approaches:
(i) The classical transform i.e. separation of variables,
(ii) The generalized transform method introduced by Fokas.

We will analyze the general Dirichlet problem

$$q_j(s) = f_j(s) \quad s \in [-L, L], \quad j = 1, 2, 3, 4\,. \tag{2}$$

We assume that the functions f_j are smooth and compatible at the corners of the square.

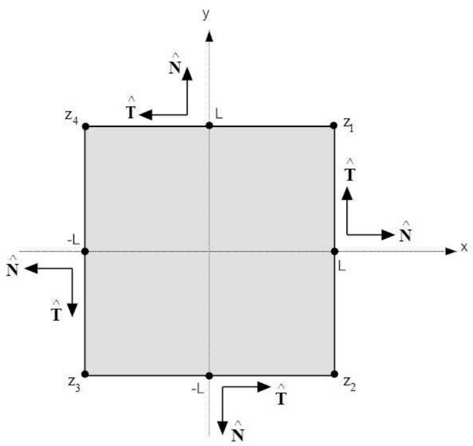

Figure 1: Fundamental Domain $\mathcal{D}$ (gray shaded)

1.3. **The global relation**

The approach presented here is based on the analysis of the global relation, which is the fundamental algebraic relation that couples the Dirichlet and the Neumann boundary values around the perimeter of the square. This equation for the case of a square is:

$$e^{-ikL}\Psi_1(k) + e^{-kL}\Psi_2(-ik) + e^{ikL}\Psi_3(-k) + e^{kL}\Psi_4(ik) = i\,G(k), \qquad k \in \mathbb{C}, \tag{3}$$

where

$$G(k) = e^{-ikL}G_1(k) + e^{-kL}G_2(-ik) + e^{ikL}G_3(-k) + e^{kL}G_4(ik), \tag{4}$$

and $\Psi_j(k)$ and $G_j(k)$ are the following transforms of the Neumann and Dirichlet boundary values

$$\Psi_j(k) = \int_{-L}^{L} e^{ks}\,\partial_N q_j(s)\,ds \tag{5}$$

$$G_j(k) = \int_{-L}^{L} e^{ks}\,\partial_T q_j(s)\,ds \tag{6}$$

for $j = 1, 2, 3, 4$ and every k.

In general, the global relation must be supplemented by its Schwarz conjugate, as well as by the six equations obtained by these two equations, replacing k with ik, $-k$, and $-ik$ in equation (3). We will refer to these eight equations as the basic algebraic relations. Due to the symmetry of the problem we will need only another set of equations.

2. The classical transform

When we apply the classical transform (i.e. separation of variables), we assume the solution expanded in a series of eigenfunctions of one of the variables, with the coefficient depending upon the other variable. Separation of variables relies upon the completeness of the eigenfunctions corresponding to one of the variables.

The solution will depend on functions which enter into the boundary conditions [5], and since the spatial domain $\mathcal{D}$ is rectangular, the yielding eigenfunctions are trigonometric.

Since every solution can be written uniquely as the sum of an even and an odd function, or in terms of Fourier expansion, every function which enters into the boundary conditions can be written as

$$f(s) \sim \sum_n \left(a_n \sin(\frac{n\pi}{L} s) + b_n \cos\left((n+1/2)\frac{\pi}{L} s \right) \right), \quad s \in [-L, L] \tag{7}$$

where the set $\mathcal{S} = \{\sin \frac{n\pi}{L} s, n \in \mathbb{N} - \{0\}\} \cup \{\cos(n+1/2)\frac{\pi}{L} s, n \in \mathbb{Z}\}$ form a complete orthogonal basis of $L^2[-L, L]$[1].

Proposition 1: Let the real valued function $q(x, y)$ satisfy the Laplace equation in the domain $\mathcal{D}$, with the boundary conditions (2), where the given function $f_j(s)$ have sufficient smoothness and are continuous at the vertices. Then

$$\begin{aligned} q(x,y) = &\sum_{n=1}^{\infty} \left[a_n \sinh\left(\frac{n\pi}{L}(x+L) \right) + c_n \sinh\left(\frac{n\pi}{L}(x-L) \right) \right] \sin(\frac{n\pi}{L} y) \\ &+ \sum_{n=0}^{\infty} \left[b_n \sinh\left((n+\frac{1}{2})\frac{\pi}{L}(x+L) \right) \right. \\ &\left. + d_n \sinh\left((n+\frac{1}{2})\frac{\pi}{L}(x-L) \right) \right] \cos\left((n+\frac{1}{2})\frac{\pi}{L} y \right) \\ &+ \sum_{n=1}^{\infty} \left[e_n \sinh\left(\frac{n\pi}{L}(y-L) \right) + g_n \sinh\left(\frac{n\pi}{L}(y+L) \right) \right] \sin(\frac{n\pi}{L} x) \\ &+ \sum_{n=0}^{\infty} \left[f_n \sinh\left((n+\frac{1}{2})\frac{\pi}{L}(y-L) \right) \right. \\ &\left. + h_n \sinh\left((n+\frac{1}{2})\frac{\pi}{L}(y+L) \right) \right] \cos\left((n+\frac{1}{2})\frac{\pi}{L} x \right) \end{aligned} \tag{8}$$

Introducing a intrinsic coordinate system $(\hat{\mathbf{T}}, \hat{\mathbf{N}})$ on each side of the square, the Fourier coefficients can be expressed as

[1] The latter results upon the so-called *Rellich's principle* [6] which can be stated simplified as: The Laplacian operator with zero boundary conditions on a bounded domain $\mathcal{D} \subset \mathbb{R}^2$ has a compact resolvent $R(\lambda)$. Thus the eigenfunctions of the Laplacian form a complete orthogonal basis

$$
\begin{aligned}
a_n &= \frac{1}{L\sinh(2n\pi)}\langle f_1(s), \sin(\frac{n\pi}{L}s)\rangle && (9)\\
b_n &= \frac{1}{L\sinh[(2n+1)\pi]}\langle f_1(s), \cos\Big((n+1/2)\frac{\pi}{L}s\Big)\rangle && (10)\\
c_n &= \frac{1}{L\sinh(2n\pi)}\langle f_3(-s), \sin(\frac{n\pi}{L}s)\rangle && (11)\\
d_n &= -\frac{1}{L\sinh[(2n+1)\pi]}\langle f_3(-s), \cos\Big((n+1/2)\frac{\pi}{L}s\Big)\rangle && (12)\\
e_n &= -\frac{1}{L\sinh(2n\pi)}\langle f_2(s), \sin(\frac{n\pi}{L}s)\rangle && (13)\\
f_n &= -\frac{1}{L\sinh[(2n+1)\pi]}\langle f_2(s), \cos\Big((n+1/2)\frac{\pi}{L}s\Big)\rangle && (14)\\
g_n &= -\frac{1}{L\sinh(2n\pi)}\langle f_4(-s), \sin(\frac{n\pi}{L}s)\rangle && (15)\\
h_n &= \frac{1}{L\sinh[(2n+1)\pi]}\langle f_4(-s), \cos\Big((n+1/2)\frac{\pi}{L}s\Big)\rangle && (16)
\end{aligned}
$$

The inner product $\langle f_j, \cos\big((n+1/2)\frac{\pi}{L}s\big)\rangle$ can be considered as the difference of the weighted inner products $\langle f_j, \cos\big(\frac{n\pi}{L}s\big)\rangle_{w=\cos(\frac{\pi}{2L}s)} - \langle f_j, \sin\big(\frac{n\pi}{L}s\big)\rangle_{w=\sin(\frac{\pi}{2L}s)}$.

The Neumann boundary values are evaluated by taking the derivatives ∂_x and ∂_y respectively of equation (8).

3. The global relation

The Laplace equation by means of complex variables $(z, \bar{z})$ can be expressed as

$$\partial_{z\bar{z}}q(z, \bar{z}) = 0 \tag{17}$$

and the global relation for the interior domain $\mathcal{D}$ of the square becomes

$$\sum_{j=1}^{4}\rho_j(k) = 0 \tag{18}$$

The spectral function $\rho_j(k)$ is given by the following line integral along side (j) of the square

$$\rho_j(k) = \int_{z_{j+1}}^{z_j} e^{-ikz}\partial_z q_j(z)dz \tag{19}$$

The spectral functions (19) can be defined in terms of the functions $G_j(k)$ and $\Psi_j(k)$ by the expression

$$\rho_j(k) = \exp\Big((-i)^j kL\Big)\Big[G_j\left((-i)^{j-1}k\right) + i\Psi_j\left((-i)^{j-1}k\right)\Big] \tag{20}$$

Using the following intrinsic parametrizations, we will show the above result.
Side 1. On side 1 the variable $z = z^{(1)}$ can be parametrized as

$$z^{(1)}(s) = L + is, \quad s \in [-L, L] \tag{21}$$

The normal and tangential derivatives are parallel to the x and y axes, respectively, so that it follows

$$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right) \Rightarrow \partial_{z^{(1)}} = \frac{1}{2}\left(\partial_N - i\partial_T\right) \tag{22}$$

Side 2.On side 2 the variable $z = z^{(2)}$ can be parametrized as

$$z^{(2)}(s) = s - iL, \quad s \in [-L, L] \tag{23}$$

and it follows that

$$\partial_{z^{(2)}} = \frac{1}{2}\left(\partial_T + i\partial_N\right) \tag{24}$$

For the remaining sides (3) and (4), we obtain:
Side 3.

$$z^{(3)}(s) = -z^{(1)}(s), \quad s \in [-L, L] \tag{25}$$

and

$$\partial_{z^{(3)}} = -\partial_{z^{(1)}} \tag{26}$$

Side 4.

$$z^{(4)}(s) = -z^{(2)}(s), \quad s \in [-L, L] \tag{27}$$

and

$$\partial_{z^{(4)}} = -\partial_{z^{(2)}} \tag{28}$$

Substituting equations (21)-(28) into expression (19), we find (20).

4. Analysis of the global relation for the general Dirichlet problem

The global relation (3) can be rewritten in a more general way

$$\sum_{j=1}^{4} e^{(-i)^j kL}\Psi_j\Big((-i)^{j-1}k\Big) = i\, G(k) \tag{29}$$

and its Schwarz conjugate

$$\sum_{j=1}^{4} e^{i^j kL}\Psi_j\Big(i^{j-1}k\Big) = -i\, \overline{G(\overline{k})} \tag{30}$$

where

$$G(k) = \sum_{j=1}^{4} e^{(-i)^j kL}\, G_j\Big((-i)^{j-1}k\Big) \tag{31}$$

We mentioned that due the symmetry we only need another set of equations, namely replacing k in equations (29) and (30) with $-k$ we find

$$\sum_{j=1}^{4} e^{-(-i)^j kL}\Psi_j\Big(-(-i)^{j-1}k\Big) \quad = \quad i\, G(-k) \tag{32}$$

$$\sum_{j=1}^{4} e^{-i^j kL}\Psi_j\Big(-i^{j-1}k\Big) \quad = \quad -i\,\overline{G(\overline{-k})} \tag{33}$$

Subtracting and adding respectively the difference of equations (29), (30) and (32), (33), we obtain

$$\begin{aligned}&-i\sin kL\Big[\Big(\Psi_1(k)+\Psi_1(-k)\Big)-\Big(\Psi_3(k)+\Psi_3(-k)\Big)\Big]\\&-\cosh kL\Big[\Big(\Psi_2(ik)-\Psi_2(-ik)\Big)-\Big(\Psi_4(ik)-\Psi_4(-ik)\Big)\Big]=\frac{i}{2}\Gamma_1(k)\end{aligned} \tag{34}$$

$$\begin{aligned}&-i\sin kL\Big[\Big(\Psi_1(k)-\Psi_1(-k)\Big)+\Big(\Psi_3(k)-\Psi_3(-k)\Big)\Big]\\&+\sinh kL\Big[\Big(\Psi_2(ik)-\Psi_2(-ik)\Big)+\Big(\Psi_4(ik)-\Psi_4(-ik)\Big)\Big]=\frac{i}{2}\Gamma_2(k)\end{aligned} \tag{35}$$

Similarly, subtracting and adding respectively the sum of equations (29), (30) and (32), (33), we find

$$\begin{aligned}&i\cos kL\Big[\Big(\Psi_1(k)+\Psi_1(-k)\Big)+\Big(\Psi_3(k)+\Psi_3(-k)\Big)\Big]\\&+\cosh kL\Big[\Big(\Psi_2(ik)+\Psi_2(-ik)\Big)+\Big(\Psi_4(ik)+\Psi_4(-ik)\Big)\Big]=\frac{i}{2}\Gamma_3(k)\end{aligned} \tag{36}$$

$$\begin{aligned}&i\cos kL\Big[\Big(\Psi_1(k)-\Psi_1(-k)\Big)-\Big(\Psi_3(k)-\Psi_3(-k)\Big)\Big]\\&-\sinh kL\Big[\Big(\Psi_2(ik)+\Psi_2(-ik)\Big)-\Big(\Psi_4(ik)+\Psi_4(-ik)\Big)\Big]=\frac{i}{2}\Gamma_4(k)\end{aligned} \tag{37}$$

where

$$\Gamma_1(k) \quad = \quad \Big(G(k)+\overline{G(\overline{k})}\Big)-\Big(G(-k)+\overline{G(\overline{-k})}\Big) \tag{38}$$

$$\Gamma_2(k) \quad = \quad \Big(G(k)+\overline{G(\overline{k})}\Big)+\Big(G(-k)+\overline{G(\overline{-k})}\Big) \tag{39}$$

$$\Gamma_3(k) \quad = \quad \Big(G(k)-\overline{G(\overline{k})}\Big)+\Big(G(-k)-\overline{G(\overline{-k})}\Big) \tag{40}$$

$$\Gamma_4(k) \quad = \quad \Big(G(k)-\overline{G(\overline{k})}\Big)-\Big(G(-k)-\overline{G(\overline{-k})}\Big) \tag{41}$$

The Dirichlet and Neumann problems can be solved by evaluating expressions (34)-(37) at discrete values of k. This yields the unknown boundary values in terms of infinite series. In particular, evaluating equations (34)-(37) at those values of k for which the coefficients of $\Psi_j(k) \pm \Psi_j(-k), j = 1,3$ and $\Psi_j(ik) \pm \Psi_j(-ik), j = 2,4$ vanishes, equations (34)-(37) yields

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}s\right) \partial_N q_1(s)ds = \frac{\Gamma_2(i\frac{n\pi}{L})}{8\sinh(n\pi)} + \frac{\Gamma_4(i\frac{n\pi}{L})}{8\cosh(n\pi)} \quad (42)$$

$$\int_{-L}^{L} \cos\left((n+\frac{1}{2})\frac{\pi}{L}s\right) \partial_N q_1(s)ds = i\frac{\Gamma_1(i(n+\frac{1}{2})\frac{\pi}{L})}{8\sinh((n+\frac{1}{2})\pi)} + i\frac{\Gamma_3(i(n+\frac{1}{2})\frac{\pi}{L})}{8\cosh((n+\frac{1}{2})\pi)} \quad (43)$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}s\right) \partial_N q_2(s)ds = -\frac{\Gamma_1(\frac{n\pi}{L})}{8\cosh(n\pi)} + \frac{\Gamma_2(\frac{n\pi}{L})}{8\sinh(n\pi)} \quad (44)$$

$$\int_{-L}^{L} \cos\left((n+\frac{1}{2})\frac{\pi}{L}s\right) \partial_N q_2(s)ds = i\frac{\Gamma_3((n+\frac{1}{2})\frac{\pi}{L})}{8\cosh((n+\frac{1}{2})\pi)} - i\frac{\Gamma_4((n+\frac{1}{2})\frac{\pi}{L})}{8\sinh((n+\frac{1}{2})\pi)} \quad (45)$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}s\right) \partial_N q_3(s)ds = \frac{\Gamma_2(i\frac{n\pi}{L})}{8\sinh(n\pi)} - \frac{\Gamma_4(i\frac{n\pi}{L})}{8\cosh(n\pi)} \quad (46)$$

$$\int_{-L}^{L} \cos\left((n+\frac{1}{2})\frac{\pi}{L}s\right) \partial_N q_3(s)ds = -i\frac{\Gamma_1(i(n+\frac{1}{2})\frac{\pi}{L})}{8\sinh((n+\frac{1}{2})\pi)} + i\frac{\Gamma_3(i(n+\frac{1}{2})\frac{\pi}{L})}{8\cosh((n+\frac{1}{2})\pi)} \quad (47)$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}s\right) \partial_N q_4(s)ds = \frac{\Gamma_1(\frac{n\pi}{L})}{8\cosh(n\pi)} + \frac{\Gamma_2(\frac{n\pi}{L})}{8\sinh(n\pi)} \quad (48)$$

$$\int_{-L}^{L} \cos\left((n+\frac{1}{2})\frac{\pi}{L}s\right) \partial_N q_4(s)ds = i\frac{\Gamma_3((n+\frac{1}{2})\frac{\pi}{L})}{8\cosh((n+\frac{1}{2})\pi)} + i\frac{\Gamma_4((n+\frac{1}{2})\frac{\pi}{L})}{8\sinh((n+\frac{1}{2})\pi)} \quad (49)$$

Proposition 2: Let the real valued function $q(x,y)$ satisfy the Laplace equation (1), with the boundary conditions (2), where the given function $f_j(s)$ have sufficient smoothness and are continuous at the vertices. Then the Neumann data $\partial_N q_j(s)$, $j = 1,2,3,4$ can be expressed in terms of the given Dirichlet data by the Fourier series

$$\partial_N q_j(s) = \sum_{n=0}^{\infty} \left(\mathfrak{A}_{j,n} \sin\left(\frac{n\pi}{L}s\right) + \mathfrak{B}_{j,n} \cos\left((n+1/2)\frac{\pi}{L}s\right) \right) \quad (50)$$

where the coefficients $\mathfrak{A}_{j,n}$ and $\mathfrak{B}_{j,n}$ are given by equations (42)-(49). The coefficients $\mathfrak{A}_{j,n}$ and $\mathfrak{B}_{j,n}$ can be correlated with the Fourier coefficients (9)-(16) through the known functions $\Gamma_j(k)$ e.g. for $j=1$ and $k_n = i(n+1/2)\frac{\pi}{L}$ we obtain

$$\Gamma_1\left(i(n+1/2)\frac{\pi}{L}\right) = -4i\pi(n+1/2)\cosh[(n+1/2)\pi]\sinh[(2n+1)\pi](b_n+d_n) + 8i(n+1/2)(-1)^n\sinh[(n+1/2)\pi]\sum_{m=1}^{\infty}(-1)^m\frac{m}{m^2+(n+1/2)^2}\sinh(2m\pi)(e_m-g_m) \tag{51}$$

The solution is then given by the expression [4]

$$q(x,y) = \frac{1}{2\pi}\mathrm{Re}\left(\sum_{j=1}^{4}(-i)^{2j+1}\int_0^{\infty}\frac{\exp[\left(-i^{j+1}z-L\right)k]}{k} \times\left[G_j\left(-ik\right)+i\Psi_j\left(-ik\right)\right]dk + const.\right), \tag{52}$$

After long and tedious calculations, Eq.(52) yields Eq.(8).

5. Conclusions

The main differences between the two methods solving PDE's i.e. the classical transform (separation of variables) and the generalized transform method introduced by Fokas, can be described briefly as:

(1) Applying the classical transform, we assume that the solution to a given boundary-value problem can be expanded in a series of eigenfunctions. On the other hand, the generalized transform method constructs the solutions without the need of using eigenfunction expansions. Actually, the latter method can be seen as a generalization of the first one.

(2) In the latter method we are forced to evaluate *first* the unknown boundary conditions and *then* compute the solution. The exact opposite applies for the classical transform method.

(3) The method introduced by Fokas does't depend on the geometry of the Domain at hand, but on the linearity of the PDE's. In contradiction with the method of separation of variables which is strongly based on the geometry of the fundamental domain.

(4) The Fokas method performs simultaneous spectral analysis, but separation of variables is based on independent spectral analysis.

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