Growth of Entire Harmonic Functions $R^n$, $n \geq 2$ and Generalized Orders

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Abstract

T.B. Fugard characterized the growth of entire harmonic functions $h(x)$ in terms of the norms of $m$th gradient $\nabla_m h(0)$. However his results leave a big class of functions for whom the characterization cannot be obtained. In this paper, we use the generalized order introduced by M.N. Seremeta to characterize the growth of such entire harmonic functions. We also consider functions of slow growth.

Keywords: Entire function, Harmonic function, order, type.

1.

Let $\phi : [a, \infty) \to \mathbb{R}$ be a real valued function such that $\phi(x)$ is positive, strictly increasing and differentiable and tends to $\infty$ as $x \to \infty$. Further, for every real valued function $\gamma(x)$ such that $\gamma(x) \to 0$ as $x \to \infty$, $\phi$ satisfies

$$\lim_{x \to \infty} \phi \left[ \frac{1 + \gamma(x)}{x} x \right] \phi(x) = 1.$$  \hspace{1cm} (1)

Then $\phi$ is said to belong to the class $L^0$. The function $\phi(x) \in \Lambda$ if $\phi(x) \in L^0$ and in place of (1) satisfies the stronger condition

$$\lim_{x \to \infty} \frac{\phi(cx)}{\phi(x)} = 1,$$  \hspace{1cm} (2)

for all $c, 0 < c < \infty$, where the convergence is uniform in $c$, $0 < c_1 \leq c \leq c_2 < \infty$. Functions $\phi$ satisfying (2) are also called slowly increasing functions (see [7]).

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function and let $M(r) = M(r, f) = \max_{|z|=r} |f(z)|$ be its maximum modulus. Using the functions of the class $L^0$ and $\Lambda$, Seremeta [8] obtained the following characterizations:
**Theorem A.** Let \( \alpha(x) \in \Lambda, \beta(x) \in L^0 \). Set \( F(x, c) = \beta^{-1} [c \alpha(x)] \). If \( dF(x, c)/d\ln x = O(1) \) as \( x \to \infty \) for all \( c, 0 < c < \infty \), then

\[
\limsup_{r \to \infty} \frac{\alpha(\ln M(r))}{\beta(\ln r)} = \limsup_{n \to \infty} \frac{\alpha(n)}{\beta \left( \ln |a_n|^{-1/n} \right)}.
\]

**Theorem B.** Let \( \alpha(x) \in L^0, \beta^{-1}(x) \in L^0, \gamma(x) \in L^0 \). Let \( \rho, 0 < \rho < \infty \), be a fixed number, Set \( F(x, \sigma, \rho) = \gamma^{-1} \left\{ \left[ \beta^{-1} (\sigma \alpha(x)) \right]^{1/\rho} \right\} \). Suppose that for all \( \sigma, 0 < \sigma < \infty \),

(a) If \( \gamma(x) \in \Lambda \) and \( \alpha(x) \in \Lambda \), then \( dF(x, \sigma, \rho)/d\ln x = O(1) \) as \( x \to \infty \)

(b) If \( \gamma(x) \in L^0 - \Lambda \) or \( \alpha(x) \in L^0 - \Lambda \), then \( \lim_{x \to \infty} \frac{d\ln F(x, \sigma, \rho)}{d\ln x} = \frac{1}{\rho} \).

Then we have

\[
\limsup_{r \to \infty} \frac{\alpha(\ln M(r))}{\beta(\ln |r\rho|)} = \limsup_{n \to \infty} \frac{\alpha(n)}{\beta \left( \ln |a_n|^{-1/n} \right)}.
\]

**Theorem C.** Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function. Set \( F(t) = \beta^{-1}(\alpha(t)) \). Let \( (6) \) For some function \( \psi(t) \) tending to \( \infty \) (howsoever slowly) as \( t \to \infty \),

\[
\frac{\beta(t \psi(t))}{\beta(e^t)} \to 0 \quad \text{as} \quad t \to \infty
\]

\[
\frac{dF(t)}{d(\log t)} = O(1) \quad \text{as} \quad t \to \infty
\]

\( |a_n/a_{n+1}| \) is ultimately a non decreasing function of \( n \).

Then

\[
\lambda(\alpha, \beta, f) = \liminf_{n \to \infty} \frac{\alpha(n)}{\beta \left( \ln |a_n|^{-1/n} \right)}
\]

It is clear that the functions \( F(x, c) \) and \( F(t) \) of Theorem A and Theorem C respectively, do not satisfy the required conditions when \( \alpha(x) = \beta(x) \) i.e., when the entire
function \( f(z) \) is of slow growth. To include these functions in the study, Kapoor and Nautiyal [6] considered a new class of functions and defined generalized order as follows.

Let \( \Omega \) be the class of functions \( \phi \) satisfying (8) \( \phi \) is defined on \([a, \infty)\) such that \( \phi(x) \) is positive, strictly increasing and differentiable and tends to \( \infty \) as \( x \to \infty \) (9) There exists a function \( \delta(x) \in \Lambda \) and numbers \( x_0, K_1 \) and \( K_2 \) such that

\[
0 < K_1 \leq \frac{d(\phi(x))}{d(\delta(\log x))} \leq K_2 < \infty \quad \text{forall} \ x > x_0 \tag{9}
\]

Let \( \overline{\Omega} \) be the class of functions \( \phi(x) \) satisfying (8) and

\[
\lim_{x \to \infty} \frac{d(\phi(x))}{d(\log x)} = K, \ 0 < K < \infty. \tag{10}
\]

The generalized order \( \rho(\alpha, \alpha, f) \) and the generalized lower order \( \lambda(\alpha, \alpha, f) \) of the entire function \( f(z) \) were defined as

\[
\rho(\alpha, \alpha, f) = \lim_{r \to \infty} \sup_{\alpha} \frac{\alpha(\log M(r))}{\alpha(\log r)}, 1 \leq \lambda(\alpha, \alpha, f) \leq \rho(\alpha, \alpha, f) \leq \infty, \tag{11}
\]

where \( \alpha(x) \) belongs either to \( \Omega \) or to \( \overline{\Omega} \).

Let us put

\[
\begin{aligned}
P_v(\zeta) &= \max \{1, \zeta\} \quad \text{if} \ \alpha(x) \in \Omega, \\
&= v + \zeta \quad \text{if} \ \alpha(x) \in \overline{\Omega}, \\
P_1(\zeta) &= P(\zeta). \tag{12}
\end{aligned}
\]

The following coefficient characterizations were also obtained.

**Theorem D** [6, Theorem 4]. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function of generalized order \( \rho(\alpha, \alpha, f) \). Then

\[
\rho(\alpha, \alpha, f) = P(\overline{L}) \tag{13}
\]

where \( \overline{L} = \lim \sup_{n \to \infty} \frac{\alpha(n)}{\alpha(\log |a_n|^{l_0})} \).

**Theorem E** [6, Theorem 5]. Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be an entire function of generalized lower order \( \lambda(\alpha, \alpha, f) \) and \( \psi(n) = |a_n/a_{n+1}| \) be ultimately a non decreasing function of \( n \). Then

\[
\lambda(\alpha, \alpha, f) = P(l_0) \tag{14}
\]

where \( l_0 = \lim \inf_{n \to \infty} \frac{\alpha(n)}{\alpha(\log |a_n|^{l_0})} \).
Recently, Ganti and Srivastava [5] extended above results and defined the generalized type as follows.

For the entire function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) of generalized order \( \rho(\alpha, \alpha, f) = \rho, \ 1 < \rho < \infty \) and \( \alpha(x) \) belongs either to \( \Omega \) or to \( \overline{\Omega} \), define

\[
\tau(\alpha, f) = \lim_{r \to \infty} \sup \frac{\alpha(\log M(r))}{[\alpha(\log r)]^\rho} \tag{15}
\]

Then for \( \alpha(x) \in \overline{\Omega} \) and \( 1 < \rho < \infty \), we have

\[
\tau = \tau(\alpha, f) = \lim_{n \to \infty} \frac{\alpha(n/\rho)}{\left\{ \alpha\left[ \frac{\rho}{n-1} \log |a_n| \right] \right\}^{\rho-1}} \tag{16}
\]

provided \( dF(x, \tau, \rho)/d \ln x = O(1) \) as \( x \to \infty \) for all \( \tau, 0 < \tau < \infty \), where we define

\[
F(x, \tau, \rho) = \alpha^{-1}[\tau \{\alpha(x)\}^{1/\rho}]..
\]

2.

Let \( x \in \mathbb{R}^n, \ (n \geq 2) \) be an arbitrary point where \( x = (x_1, x_2, \ldots, x_n) \) and put \( |x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} \). We denote by \( H \) the set of all non-constant entire harmonic functions on \( \mathbb{R}^n \). For each \( h \in H \) and \( r > 0 \) we write \( M_\infty(h, r) = \sup_{|x|=r} |h(x)| \).

Then the order \( \rho \) of \( h \) is defined as

\[
\rho = \limsup_{r \to \infty} \frac{\ln M_\infty(h, r)}{\ln r}, \ 0 \leq \rho \leq \infty,
\]

and when \( 0 < \rho < \infty \), the type \( \tau \) is defined as

\[
\tau = \limsup_{r \to \infty} \frac{\ln M_\infty(h, r)}{r^\rho}, \ 0 \leq \tau \leq \infty.
\]

For each \( n \)-tuple \( a = (a_1, a_2, \cdots, a_n) \) of non-negative integers, we define \( |a| = a_1 + a_2 + \cdots + a_n \), \( a! = a_1! a_2! \cdots a_n! \) and \( D^a = \frac{\partial^{|a|}}{\partial x_1^{a_1} \partial x_2^{a_2} \cdots \partial x_n^{a_n}} \).

For each \( \varphi \in C^\infty(\mathbb{R}^n) \) and non-negative integer \( m \), following [2], we define the norm of \( m-th \) gradient of \( \varphi \) by

\[
|\nabla_m \varphi| = \left\{ \frac{m!}{|a|=m} \left( D^a \varphi \right)^2 (a!)^{-1} \right\}^{1/2}.
\]

Fugard [3] obtained the characterization for the order and type of an entire harmonic function in terms of the \( m-th \) gradient as defined above. From these results, it is evident that the characterization of harmonic functions of fast growth and those of slow
growth can not be done. In this paper, we obtain the characterizations of generalized order and generalized type etc of the entire harmonic functions.

Let $h \in H$. Then we define the generalized order $\rho_{\circ}(\alpha, \beta, h)$ and generalized lower order $\lambda_{0}(\alpha, \beta, h)$ of $h$ as

$$\rho_{\circ} = \rho_{0}(\alpha, \beta, h), \quad \lambda_{0} = \lambda_{0}(\alpha, \beta, h) = \lim_{r \to \infty} \sup_{r_{1} > r} \frac{\alpha(\ln M_{\infty}(h, r_{1}))}{\beta(\ln r)}, 0 \leq \lambda_{0} \leq \rho_{0} \leq \infty. \quad (17)$$

Further, for $0 < \rho_{\circ} < \infty$, we define the generalized type $T_{\circ}$ of $h$ as

$$T_{\circ} = \lim_{r \to \infty} \sup_{r_{1} > r} \frac{\alpha(\ln M_{\infty}(h, r_{1}))}{\beta((\gamma(r))^{\rho_{\circ}})}. \quad (18)$$

Here, the functions $\alpha$, $\beta$, and $\gamma$ satisfy the conditions stated in Theorem B. We now prove

**Theorem 1.** Let $\alpha(x) \in \Lambda$, $\beta(x) \in L^{0}$ and set $F(x, c) = \beta^{-1} [c \alpha(x)]$. If $dF(x, c)/d \ln x = O(1)$ as $x \to \infty$ for all $c$, $0 < c < \infty$, then

$$\rho_{\circ} = \lim_{m \to \infty} \sup_{r_{1} > r} \frac{\alpha(m)}{\ln \{m!/|\nabla m h(0)|\}^{1/m}}, \quad (19)$$

**Proof.** For each $p \geq 1$ and continuous function $\psi$ on $R^{n}$, we define

$$M_{p}(\psi, r) = \left[ \frac{1}{c_{n}} \int_{S(r)} |\psi|^{p} d\sigma \right]^{1/p}, \quad (20)$$

where $S(r)$ is the sphere of radius $r$ in $R^{n}$, $d\sigma$ is the surface area element on $S(r)$, and $c_{n}$ the surface area of the sphere $S(1)$. Then we have for each $\lambda > 1$, (see [3, p.287] )

$$M_{1}(h, r) \leq M_{p}(h, r) \leq M_{\infty}(h, r) \leq \frac{(\lambda + 1) \lambda^{n-2}}{(\lambda - 1)^{n-1}} M_{1}(h, \lambda r). \quad (21)$$

We also have [3, Lemma 2.4]

$$M_{2}(h, r)^{2} = \Gamma(n/2) \sum_{m=0}^{\infty} \frac{|\nabla m h(0)|^{2}}{2^{m} m! \Gamma(m + n/2)} r^{2m}. \quad (22)$$

We consider the complex valued function

$$g(z) = \Gamma(n/2) \sum_{m=0}^{\infty} \frac{|\nabla m h(0)|}{\sqrt{2^{m} m! \Gamma(m + n/2)}} z^{m} = \sum_{m=0}^{\infty} b_{m} z^{m} \quad (say). \quad (23)$$


Since \( \Gamma(m+n/2) / \Gamma(m+1) \approx m^{(n-1)/2} \), we get

\[
 b_m = \Gamma(n/2) \frac{\nabla_m h(0)}{\sqrt{2^m m! \Gamma(m+n/2)}} \approx \Gamma(n/2) \frac{\nabla_m h(0)}{m! 2^{m/2} m^{(n-1)/4}}, \quad (23)
\]

For the function \( h \) which is harmonic in the whole of \( \mathbb{R}^n \), we have by a result of Fugard [4, Theorem 2.2]

\[
 \liminf_{m \to \infty} \left[ \frac{\nabla_m h(0)}{m!} \right]^{-1/m} = \infty. \quad (24)
\]

Hence

\[
 \liminf_{m \to \infty} (b_m)^{-1/m} = \infty
\]

and consequently, \( g(z) \) represents an entire function of the complex variable \( z \). Further, in view of (24), we have

\[
 \ln (b_m)^{-1} \approx \ln \frac{m!}{\nabla_m h(0)} \quad \text{as} \quad m \to \infty \quad (25)
\]

On using Theorem A, we get

\[
 \limsup_{r \to \infty} \frac{\alpha}{{\beta} (\ln r)} = \limsup_{m \to \infty} \frac{\alpha (m)}{{\beta} \left( \ln[m! / \nabla_m h(0)]^{1/m} \right)} \quad (26)
\]

where \( \alpha \) and \( \beta \) satisfy conditions stated as in Theorem A. Now from (20) we have

\[
 \log M_p(h, r) \leq \log M_\infty(h, r) \leq O(1) + \log M_p(h, \lambda r) \quad (27)
\]

and since \( \lambda \) is fixed, we get

\[
 \ln M_\infty(h, r) \equiv \ln M_p(h, r), p \geq 1 \quad (27)
\]

Now \( |g(z)|^2 = \sum_{m=0}^{\infty} \left\{ \Gamma(n/2) \frac{\nabla_m h(0)}{\sqrt{2^m m! \Gamma(m+n/2)}} \right\}^2 r^{2m} + \sum_{m \neq k} \left\{ \Gamma(n/2) \right\}^2 \frac{\nabla_m h(0) \nabla_k h(0)}{\sqrt{2^{m+k} m! k! \Gamma(m+n/2) \Gamma(k+n/2)}} z^m (z)^k
\]

Hence using above relation and the estimate [3, p.290] of \( M_2(h, r) \), we get

\[
 M(g, r) \geq \{ M_2(h, r) \} \geq \frac{Ar}{2^{m/2}} \left( \frac{\nabla_m h(0)}{m!} \right), \quad \text{where} \quad A \text{ is a finite constant.}
\]

If \( \mu(g, r) \) denotes the maximum term of \( g(z) \) then by a result of Valiron [10, p.34], we have \( \log M(g, r) \approx \log \mu(g, r) \) as \( r \to \infty \). Hence using last inequalities and (27), we get

\[
 \log M(g, r) \approx \log M_\infty(h, r) \quad \text{as} \quad r \to \infty \quad (28)
\]
Now combining (26) and (17), we get (19). This proves Theorem 1. Next we prove

**Theorem 2.** Let \( h \) be harmonic in the entire \( \mathbb{R}^n \), \( n \geq 2 \) and of generalized order \( \rho_\circ \), \( 0 < \rho_\circ < \infty \). Let the functions \( \alpha, \beta \) and \( \gamma \) satisfy the conditions of Theorem B (with \( \rho_\circ \) in place of \( \rho \)). Then the generalized type \( T_\circ \) of \( h \) is given by

\[
T_\circ = \limsup_{m \to \infty} \frac{\alpha(m/\rho_\circ)}{\beta \left\lbrace \gamma \left( e^{1/\rho_\circ} \left( m! 2^{m/2} |\nabla_m h(0)|^{-1}/m^m \right) \right) \right\rbrace^\rho_\circ}
\]

(29)

**Proof.** Using (28), since \( \alpha \in L^0 \), we get

\[
\alpha (\ln M_\infty(h, r)) \cong \alpha (\ln M(g, r)) .
\]

(30)

From Theorem 1, \( g(z) \) is also an entire function of generalized order \( \rho_\circ \) and hence by Theorem B, we have

\[
\limsup_{r \to \infty} \frac{\alpha [\ln M(g, r)]}{\beta [(\gamma(r))^{\rho_\circ}]} = \limsup_{m \to \infty} \frac{\alpha(m/\rho_\circ)}{\beta \left\lbrace \gamma \left( e^{1/\rho_\circ} [b_m]^{-1/m} \right) \right\rbrace^\rho_\circ}
\]

(31)

where \( b_m \) is defined as above. Since \( \gamma(x) \in L^0 \), we have

\[
\limsup_{m \to \infty} \frac{\alpha(m/\rho_\circ)}{\beta \left\lbrace \gamma \left( e^{1/\rho_\circ} [b_m]^{-1/m} \right) \right\rbrace^\rho_\circ} = \limsup_{m \to \infty} \frac{\alpha(m/\rho_\circ)}{\beta \left\lbrace \gamma \left( e^{1/\rho_\circ} (2m/2m^{1/2}) |\nabla_m h(0)|^{1/m} \right) \right\rbrace^\rho_\circ}
\]

The above relations (30) and (31) give us (29) and Theorem 2 follows.

**Remarks:**
(i) In Theorem 1, if we choose \( \alpha(x) = \ln x \) and \( \beta(x) = x \) then we obtain Theorem 2.1 of Fugard [3].
(ii) In Theorem 2, if we choose \( \alpha(x) = \beta(x) = \gamma(x) = x \), then we obtain Theorem 2.6 of Fugard [3].

In the next result, we characterize the lower order of \( h \). We have

**Theorem 3.** Let the functions \( \alpha(x), \beta(x) \) and \( F(x, c) \) be as defined in Theorem C and satisfy conditions (6) and (7). Further, let the function

\[
\mu(m) = \{ |\nabla_m h(0)|/|\nabla_{m+1} h(0)| \} \sqrt{(m + 1) (m + n/2)}
\]

be a non-decreasing function of \( m \) for all large values of \( m \). Then the generalized lower order \( \lambda_0 \) is given by
\[ \lambda_\circ = \liminf_{m \to \infty} \frac{\alpha(m)}{\beta[\ln[m!]/|\nabla_m h(0)|^{1/m}].} \] (32)

**Proof.** As proved in Theorem 1, \( g(z) \) defined by (22) is an entire function and
\[ \ln M_\infty(h, r) \cong \ln M(g, r) \quad \text{as} \quad r \to \infty \]
Hence \( g(z) \) is also of generalized lower order \( \lambda_0 \). Now, under the assumption,
\[ \frac{b_m}{b_{m+1}} = \left\{ \frac{j \nabla_m h(0)}{|\nabla_{m+1} h(0)|} \right\} \sqrt{2(m+1)(m+n/2)} \]
is a non decreasing function of \( m \). Now using (24) and applying Theorem C to the
function \( g(z) \), we get the relation
\[ \lambda_\circ = \liminf_{m \to \infty} \frac{\alpha(m)}{\beta[\ln[m!]/|\nabla_m h(0)|^{1/m}].} \]

3.
In this section, we consider entire harmonic functions of slow growth. Following (11)
we define the generalized order and generalized lower order for harmonic function \( h \) by
\[ \rho_1 = \rho(\alpha, \alpha, h) \quad \lambda_1 = \lambda(\alpha, \alpha, h) \quad \text{where} \quad 1 \leq \lambda_1 \leq \rho_1 \leq \infty \] (33)

Further, for \( 1 < \rho_1 < \infty \), we define the generalized type of \( h \) by
\[ \tau_1 = \tau(\alpha, h) = \lim_{r \to \infty} \sup \frac{\alpha(\log M_\infty(h, r))}{\alpha(\log r)} \] (34)
Here the function \( \alpha(x) \in \Omega \) or \( \Omega \) as defined before. Now we prove

**Theorem 4.** Let \( h \) be entire harmonic function of generalized order \( \rho_1 \). Then
\[ \rho_1 = P(L_1) \] (35)
where
\[ L_1 = \limsup_{m \to \infty} \frac{\alpha(m)}{\alpha(\ln[m!]/|\nabla_m h(0)|^{1/m})}. \]
The result follows on the lines of Theorem 1 on using Theorem D.

**Theorem 5.** Let the function \( \mu(m) = \left\{ \frac{j \nabla_m h(0)}{|\nabla_{m+1} h(0)|}\right\} \sqrt{(m+1)(m+n/2)} \)
be a non decreasing function of \( m \) for all large values of \( m \). Then the generalized lower
order \( \lambda_1 \) is given by
\[ \lambda_1 = \begin{cases} \max \{1, l_1\} & \text{if } \alpha(x) \in \Omega \\ 1 + l_1 & \text{if } \alpha(x) \in \overline{\Omega} \end{cases} \]  

(36)

where

\[ l_1 = \liminf_{m \to \infty} \frac{\alpha(m)}{\alpha(m!/[\nabla^m h(0)]^{1/m})} \]

The result follows on the lines of Theorem 3 above on combining with Theorem E. It should be noted that condition (ii) of [5, Theorem 5, p.74] holds in our case since \( \lambda_m = m \).

Lastly we characterize the generalized type defined by (34). We have

**Theorem 6.** Let the non constant entire harmonic function \( h \in H \), be of generalized order \( \rho_1, 0 < \rho_1 < \infty \), and generalized type \( \tau_1 \) and \( \alpha(x) \in \Omega \). Then

\[ \tau_1 = \limsup_{m \to \infty} \frac{\alpha(m/\rho_1)}{\left\{ \alpha\left(\frac{\rho_1}{\rho_1-1}\log\left(\frac{1}{m!}\right)^{-1/m}\right)\right\}^{\rho_1-1}} \]  

(37)

The result follows on using (16) for the entire function

\[ g(z) = \Gamma(n/2) \sum_{m=0}^{\infty} \frac{|\nabla_m h(0)|}{\sqrt{2^n m! \Gamma(m+n/2)}} z^m \]

and the relations (23) and (24).

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References


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