BULLETIN OF THE
GREEK MATHEMATICAL SOCIETY
Volume 55, 2008 (49–58)

Growth of Entire Harmonic Functions R^n, $n \geq 2$ and Generalized Orders

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Abstract

T.B. Fugard characterized the growth of entire harmonic functions $h(x)$ in terms of the norms of $m'th$ gradient$\nabla_m h(0)$. However his results leave a big class of functions for whom the characterization can not be obtained. In this paper, we use the generalized order introduced by M.N. Seremeta to characterize the growth of such entire harmonic functions. We also consider functions of slow growth.

Keywords: Entire function, Harmonic function, order, type.

1.

Let $\phi : [a, \infty) \to R$ be a real valued function such that $\phi(x)$ is positive, strictly increasing and differentiable and tends to ∞ as $x \to \infty$. Further, for every real valued function $\gamma(x)$ such that $\gamma(x) \to 0$ as $x \to \infty$, ϕ satisfies

$$\lim_{x \to \infty} \frac{\phi\ [(1+\gamma(x)\ )\ x]}{\phi(x)} = 1. \tag{1}$$

Then ϕ is said to belong to the class L^0.The function $\phi(x) \in \Lambda$ if $\phi(x) \in L^0$ and in place of (1) satisfies the stronger condition

$$\lim_{x \to \infty} \frac{\phi\,(cx)}{\phi(x)} = 1, \tag{2}$$

for all $c, 0 < c < \infty$, where the convergence is uniform in c, $0 < c_1 \leq c \leq c_2 < \infty$. Functions$\phi$ satisfying (2) are also called slowly increasing functions (see [7]).
Let $f(z) = \sum_{n=0}^{\infty} a_n\, z^n$ be an entire function and let $M(r) = M(r, f) = \max_{|z|=r} |f(z)|$ be its maximum modulus. Using the functions of the class L^0 and Λ, Seremeta [8] obtained the following characterizations:

Theorem A. Let $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$. Set $F(x,c) = \beta^{-1}\,[c\,\alpha(x)]$. If $dF(x,c)/d\ln x$= $O(1)$as $x \to \infty$ for all c, $0 < c < \infty$, then

$$\limsup_{r\to\infty} \frac{\alpha\,(\ln M(r))}{\beta\,(\ln r)} = \limsup_{n\to\infty} \frac{\alpha\,(n)}{\beta\,\left(\ln |a_n|^{-1/n}\right)}. \tag{3}$$

Theorem B. Let $\alpha(x) \in L^0$, $\beta^{-1}(x) \in L^0$, $\gamma(x) \in L^0$. Letρ, $0 < \rho < \infty$,be a fixed number, Set $F(x,\sigma,\rho) = \gamma^{-1}\left\{\left[\beta^{-1}\,(\sigma\,\alpha(x))\right]^{1/\rho}\right\}$. Suppose that for all σ,$0 < \sigma < \infty$,

(a) If $\gamma(x) \in \Lambda$ and $\alpha(x) \in \Lambda$, then $dF(x,\sigma,\rho)/d\ln x = O(1)$ as $x \to \infty$

(b) If $\gamma(x) \in L^0 - \Lambda$ or $\alpha(x) \in L^0 - \Lambda$, then $\lim_{x\to\infty} \frac{d\,\ln\,F(x,\sigma,\rho)}{d\,\ln\,x} = \frac{1}{\rho}$.

Then we have

$$\limsup_{r\to\infty} \frac{\alpha\,(\ln M(r))}{\beta\,[(\gamma(r))^\rho]} = \limsup_{n\to\infty} \frac{\alpha\,(n/\rho)}{\beta\left\{\left[\gamma\left(e^{1/\rho}\,|a_n|^{-1/n}\right)\right]^\rho\right\}}. \tag{4}$$

It can be easily seen that by choosing $\alpha(x) = \ln x$, $\beta(x) = x$ in Theorem A and $\alpha(x) = \beta(x) = \gamma(x) = x$ in Theorem B, we obtain the coefficient characterizations for the classical order and type of the entire function $f(z)$ (see Boas[1]). Later, S.M.Shah [9] called the left hand quantity in (3) as the generalized order $\rho(\alpha,\beta,f)$ and introduced the generalized lower order $\lambda(\alpha,\beta,f)$as

$$\lambda(\alpha,\beta,f) = \liminf_{r\to\infty} \frac{\alpha\,(\ln\,M(r,f))}{\beta\,(\ln r)} \tag{5}$$

Further, Shah proved that [9,Theorem 2]**:**

Theorem C. Let f(z) =$\sum_{n=0}^{\infty} a_n z^n$be an entire function. Set $F(t) = \beta^{-1}(\alpha(t))$. Let (6) For some function $\psi(t)$ tending to ∞ (howsoever slowly) as $t \to \infty$,

$$\frac{\beta(t\psi(t))}{\beta(e^t)} \to 0 \qquad as \qquad t \to \infty \tag{6}$$

$$\frac{dF(t)}{d(\log t)} = O(1) \qquad as \qquad t \to \infty \tag{7}$$

$|a_n/a_{n+1}|$ is ultimately a non decreasing function of n.

Then

$$\lambda(\alpha,\beta,f) = \liminf_{n\to\infty} \frac{\alpha\,(n)}{\beta\,\left(\ln |a_n|^{-1/n}\right)} \tag{8}$$

It is clear that the functions$F(x,c)$and$F(t)$of Theorem A and Theorem C respectively , do not satisfy the required conditions when $\alpha(x) = \beta(x)$ i.e., when the entire function$f(z)$is of slow growth. To include these functions in the study, Kapoor and Nautiyal [6] considered a new class of functions and defined generalized order as follows.

Let Ω be the class of functions ϕ satisfying (8) ϕ is defined on $[a,\infty)$such that $\phi(x)$ is positive, strictly increasing and differentiable and tends to ∞ as $x \to \infty$ (9)There exists a function$\delta(x) \in \Lambda$ and numbers x_0, K_1 and K_2 such that

$$0 < K_1 \leq \frac{d(\phi(x))}{d(\delta(\log x))} \leq K_2 < \infty \quad forallx > x_0 \tag{9}$$

Let $\overline{\Omega}$ be the class of functions $\phi(x)$satisfying (8) and

$$\lim_{x\to\infty} \frac{d(\phi(x))}{d(\log x))} = K,\, 0 < K < \infty. \tag{10}$$

The generalized order $\rho(\alpha,\alpha,f)$ and the generalized lower order $\lambda(\alpha,\alpha,f)$ of the entire function $f(z)$were defined as

$$\begin{matrix} \rho(\alpha,\alpha,f) \\ \lambda(\alpha,\alpha,f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\alpha(\log M(r))}{\alpha(\log r)}, 1 \leq \lambda(\alpha,\alpha,f) \leq \rho(\alpha,\alpha,f) \leq \infty, \tag{11}$$

where $\alpha(x)$ belongs either to Ω or to $\overline{\Omega}$.
Let us put

$$\begin{cases} P_\upsilon(\zeta) = \max\{1,\zeta\} \quad if\, \alpha(x) \in \Omega, \\ \qquad = \upsilon + \zeta \qquad if\, \alpha(x) \in \overline{\Omega}, \\ P_1(\zeta) = P(\zeta). \end{cases} \tag{12}$$

The following coefficient characterizations were also obtained.

Theorem D [6, Theorem 4]. Let $f(z) = \sum_{n=0}^{\infty} a_n\, z^n$ be an entire function of generalized order$\rho(\alpha,\alpha,f)$. Then

$$\rho(\alpha,\alpha,f) = P(\overline{L}) \tag{13}$$

where $\overline{L} = \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha(\log|a_n|^{-1/n})}$.

Theorem E [6, Theorem 5]. Let $f(z) = \sum_{n=0}^{\infty} a_n\, z^n$ be an entire function of generalized lower order $\lambda(\alpha,\alpha,f)$ and $\psi(n) = |a_n/a_{n+1}|$be ultimately a non decreasing function of n . Then

$$\lambda(\alpha,\alpha,f) = P(l_0) \tag{14}$$

where $l_0 = \liminf_{n\to\infty} \frac{\alpha(n)}{\alpha(\log|a_n|^{-1/n})}$.

Recently, Ganti and Srivastava [5] extended above results and defined the generalized type as follows.

For the entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of generalized order $\rho(\alpha,\alpha,f) = \rho$, $1 < \rho < \infty$ and $\alpha(x)$ belongs either to Ω or to $\overline{\Omega}$, define

$$\tau(\alpha, f) = \lim_{r\to\infty} \sup \frac{\alpha(\log M(r))}{[\alpha(\log r)]^{\rho}}. \tag{15}$$

Then for $\alpha(x)\in \overline{\Omega}$ and $1 < \rho < \infty$, we have

$$\tau = \tau(\alpha, f) = \limsup_{n\to\infty} \frac{\alpha\,(n/\rho)}{\left\{\alpha\left[\frac{\rho}{\rho-1}\log|a_n|^{-1/n}\right]\right\}^{\rho-1}} \tag{16}$$

provided $dF(x,\tau,\rho)/d\ln x = O(1)$ as $x \to \infty$ for all τ, $0 < \tau < \infty$, where we define

$$F(x,\tau,\rho) = \alpha^{-1}[\tau\{\alpha(x)\}^{1/\rho}]..$$

2.

Let $x \in R^n$, $(n \geq 2)$ be an arbitrary point where $x = (x_1, x_2, \ldots, x_n)$ and put $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. We denote by H the set of all non-constant entire harmonic functions onR^n. For each $h \in$ H and $r > 0$ we write $M_\infty(h,r) = \sup_{|x|=r} |h(x)|$. Then the order$\rho$ of h is defined as

$$\rho = \lim_{r\to\infty} \sup \frac{\ln\ln M_\infty(h,r)}{\ln r}, 0 \leq \rho \leq \infty,$$

and when $0 < \rho < \infty$, the type τ is defined as

$$\tau = \lim_{r\to\infty} \sup \frac{\ln M_\infty(h,r)}{r^{\rho}}, 0 \leq \tau \leq \infty.$$

For each n-tuple $a = (a_1, a_2, \cdots, a_n)$ of non-negative integers, we define $|a| = a_1 + a_2 + \ldots + a_n$, $a\,! = a_1!\ a_2! \cdots a_n!$ and $D^a = \frac{\partial^{|a|}}{\partial x_1^{a_1} \partial x_2^{a_2} \cdots \partial x_n^{a_n}}$.
For each $\varphi \in C^\infty(R^n)$ and non-negative integer m, following [2], we define the norm of $m - th$ gradient of φ by

$$|\nabla_m \varphi| = \left\{ m\,! \sum_{|a|=m} (D^a \varphi)^2 (a\,!)^{-1} \right\}^{1/2}.$$

Fugard [3] obtained the characterization for the order and type of an entire harmonic function in terms of the m-th gradient as defined above. From these results, it is evident that the characterization of harmonic functions of fast growth and those of slow growth can not be done. In this paper, we obtain the characterizations of generalized order and generalized type etc of the entire harmonic functions.

Let $h \in$ H.Then we define the generalized order $\rho_\circ(\alpha, \beta, h)$and generalized lower order $\lambda_0(\alpha, \beta, h)$ of h as

$$\left.\begin{array}{l} \rho_0 = \rho_0(\alpha, \beta, h) \\ \lambda_0 = \lambda_0(\alpha, \beta, h) \end{array}\right\} = \lim_{r\to\infty} \begin{array}{c} \sup \\ \inf \end{array} \frac{\alpha\ (\ln M_\infty(h, r))}{\beta\ (\ln r)}, 0 \leq \lambda_0 \leq \rho_0 \leq \infty. \tag{17}$$

Further, for $0 < \rho_\circ < \infty$, we define the generalized type $T_\circ$ of h as

$$T_\circ = \limsup_{r\to\infty} \frac{\alpha\ (\ln M_\infty(h, r))}{\beta\left\{(\gamma(r))^{\rho_\circ}\right\}}. \tag{18}$$

Here, the functionsα, β and γsatisfy the conditions stated in Theorem B. We now prove

Theorem 1. Let $\alpha(x) \in \Lambda$, $\beta(x) \in L^0$ and set $F(x, c) = \beta^{-1}\ [c\,\alpha(x)]$. If $dF(x, c)/d\ln x$ $=O(1)$as $x \to \infty$ for all c, $0 < c < \infty$, then

$$\rho_\circ = \limsup_{m\to\infty} \frac{\alpha(m)}{\beta\left[\ln\left\{m\,!/\left|\nabla_m h(0)\right|\right\}^{1/m}\right]}. \tag{19}$$

Proof. For each $p \geq 1$ and continuous function ψ on R^n, we define

$$M_p(\psi, r) = \left[\frac{1}{c_n\ r^{n-1}} \int_{S(r)} |\psi|^p\ d\sigma\right]^{1/p},$$

where $S(r)$ is the sphere of radius r in R^n, $d\sigma$ is the surface area element on $S(r)$, and c_n the surface area of the sphere $S(1)$. Then we have for each $\lambda > 1$, (see [3, p.287])

$$M_1\,(h, r)\ \leq M_p\,(h, r)\ \leq M_\infty\,(h, r) \leq \frac{(\lambda+1)\,\lambda^{n-2}}{(\lambda-1)^{n-1}}\ M_1(h, \lambda\, r). \tag{20}$$

We also have [3, Lemma 2.4]

$$M_2\,(h, r)^2 = \Gamma(n/2) \sum_{m=0}^{\infty} \frac{\left|\nabla_m\ h(0)\right|^2}{2^m\ m\,!\ \Gamma(m+n/2)}\ r^{2m}. \tag{21}$$

We consider the complex valued function

$$g(z) = \Gamma(n/2) \sum_{m=0}^{\infty} \frac{\left|\nabla_m\ h(0)\right|}{\sqrt{2^m\ m\,!\ \Gamma(m+n/2)}}\ z^m = \sum_{m=0}^{\infty} b_m\ z^m\ \ (say). \tag{22}$$

Since $\frac{\Gamma(m+n/2)}{\Gamma(m+1)} \simeq m^{(n-1)/2}$, we get

$$b_m = \Gamma(n/2) \frac{|\nabla_m h(0)|}{\sqrt{2^m m! \,\Gamma(m+n/2)}} \cong \Gamma(n/2) \frac{|\nabla_m h(0)|}{m! \, 2^{m/2} m^{(n-1)/4}}, \qquad (23)$$

For the function h which is harmonic in the whole of R^n, we have by a result of Fugard [4, Theorem 2.2]

$$\liminf_{m\to\infty} \left[\frac{|\nabla_m h(0)|}{m!}\right]^{-1/m} = \infty. \qquad (24)$$

Hence

$$\liminf_{m\to\infty} (b_m)^{-1/m} = \infty$$

and consequently, $g(z)$ represents an entire function of the complex variable z. Further, in view of (24), we have

$$\ln (b_m)^{-1} \cong \ln \frac{m!}{|\nabla_m h(0)|} \quad as \quad m \to \infty \qquad (25)$$

On using Theorem A, we get

$$\limsup_{r\to\infty} \frac{\alpha (\ln M(g,r))}{\beta (\ln r)} = \limsup_{m\to\infty} \frac{\alpha (m)}{\beta \left(\ln[m!/\nabla_m h(0)]^{1/m}\right)} \qquad (26)$$

where α and β satisfy conditions stated as in Theorem A. Now from (20) we have

$$\log M_p(h,r) \le \log M_\infty(h,r) \le O(1) + \log M_p(h,\lambda r)$$

and since λ is fixed, we get

$$\ln M_\infty(h,r) \cong \ln M_p(h,r), p \ge 1 \qquad (27)$$

Now $|g(z)|^2 = \sum_{m=0}^{\infty} \left\{\Gamma(n/2) \frac{|\nabla_m h(0)|}{\sqrt{2^m m! \,\Gamma(m+n/2)}}\right\}^2 r^{2m}$

$$+ \sum_{m\neq k} \{\Gamma(n/2)\}^2 \frac{|\nabla_m h(0)| \; |\nabla_k h(0)|}{\sqrt{2^{m+k} k! \, m! \,\Gamma(m+n/2)\,\Gamma(k+n/2)}} z^m (\overline{z})^k$$

Hence using above relation and the estimate [3,p.290] of $M_2(h,r)$,we get

$$M(g,r) \ge \{M_2(h,r)\} \ge \frac{Ar^m}{2^{m/2}} \left(\frac{\nabla_m h(0)}{m!}\right),$$

where A is a finite constant. If $\mu(g,r)$ denotes the maximum term of $g(z)$ then by a result of Valiron [10, p.34], we have $\log M(g,r) \simeq \log \mu(g,r)$ as $r \to \infty$.Hence using last inequalities and (27),we get

$$\log M(g,r) \simeq \log M_\infty(h,r) \quad as \quad r \to \infty \qquad (28)$$

Now combining (26) and (17), we get (19). This proves Theorem 1. Next we prove

Theorem 2. Let h be harmonic in the entire R^n, $(n \geq 2)$ and of generalized order $\rho_\circ$, $0 < \rho_\circ < \infty$. Let the functionsα,β and γ satisfy the conditions of Theorem B (with $\rho_\circ$ in place of ρ). Then the generalized type $T_\circ$ of h is given by

$$T_\circ = \limsup_{m\to\infty} \frac{\alpha\left(m/\rho_\circ\right)}{\beta\left[\left\{\gamma\left(e^{1/\rho_\circ}\left(m!\,2^{m/2}\,|\nabla_m\,h(0)|^{-1}\right)^{1/m}\right)\right\}^{\rho_\circ}\right]} \tag{29}$$

Proof. Using (28) ,since $\alpha \in L^0$, we get

$$\alpha\left(\ln\, M_\infty(h,r)\right) \cong \alpha\left(\ln\, M(g,r)\right). \tag{30}$$

From Theorem 1, $g(z)$ is also an entire function of generalized order $\rho_\circ$ and hence by Theorem B, we have

$$\limsup_{r\to\infty} \frac{\alpha\left[\ln\, M(g,r)\right]}{\beta\left[(\gamma(r))^{\rho_\circ}\right]} = \limsup_{m\to\infty} \frac{\alpha\left(m/\rho_\circ\right)}{\beta\left[\left\{\,\gamma\,\left(e^{1/\rho_\circ}\,|b_m|^{-1/m}\right)\,\right\}^{\rho_\circ}\right]} \tag{31}$$

where b_m is defined as above. Since $\gamma(x) \in L^\circ$, we have

$$\begin{aligned} &\limsup_{m\to\infty} \frac{\alpha\left(m/\rho_\circ\right)}{\beta\left[\left\{\,\gamma\,\left(e^{1/\rho_\circ}\,|b_m|^{-1/m}\right)\,\right\}^{\rho_\circ}\right]} \\ &= \limsup_{m\to\infty} \frac{\alpha\left(m/\rho_\circ\right)}{\beta\left[\left\{\,\gamma\,\left(e^{1/\rho_\circ}(\,2^{m/2}\;m\,!/\,|\nabla_m\,h(0)|\,)^{1/m}\right)\,\right\}^{\rho_\circ}\right]} \end{aligned}$$

The above relations (30) and (31) give us (29) and Theorem 2 follows.

Remarks:
(i) In Theorem 1, if we choose $\alpha(x) = \ln\, x$ and $\beta(x) = x$ then we obtain Theorem 2.1 of Fugard [3]
(ii) In Theorem 2, if we choose $\alpha(x) = \beta(x) = \gamma(x) = x$, then we obtain Theorem 2.6 of Fugard [3].
In the next result, we characterize the lower order of h. We have

Theorem 3. Let the functions $\alpha(x)$,$\beta(x)$and $F(x,c)$be as defined in Theorem C and satisfy conditions (6) and (7). Further, let the function

$$\mu(m) = \{|\,\nabla_m h(0)|/|\nabla_{m+1} h(0)|\}\sqrt{(m+1)\,(m+n/2)}$$

be a non decreasing function of $m$for all large values of m. Then the generalized lower order λ_0 is given by

$$\lambda_\circ = \liminf_{m\to\infty} \frac{\alpha(m)}{\beta\{\ln[m!/|\nabla_m h(0)|]^{1/m}\}}. \tag{32}$$

Proof. As proved in Theorem 1, $g(z)$ defined by (22) is an entire function and

$$\ln M_\infty(h,r) \cong \ln M(g,r) \quad as \quad r\to\infty$$

Hence $g(z)$ is also of generalized lower order λ_0 . Now, under the assumption,

$$b_m/b_{m+1} = \{j\,\nabla_m h(0)|/|\nabla_{m+1}h(0)|\}\sqrt{2(m+1)\,(m+n/2)}$$

is a non decreasing function of m. Now using (24) and applying Theorem C to the function $g(z)$, we get the relation

$$\lambda_\circ = \liminf_{m\to\infty} \frac{\alpha(m)}{\beta\{\log|b_m|^{-1/m}\}} = \liminf_{m\to\infty} \frac{\alpha(m)}{\beta\{\ln[m!/|\nabla_m h(0)|]^{1/m}\}}.$$

3.

In this section, we consider entire harmonic functions of slow growth. Following (11) we define the generalized order and generalized lower order for harmonic function h by

$$\left.\begin{array}{rl} \rho_1 = & \rho(\alpha,\alpha,h) \\ \lambda_1 = & \lambda(\alpha,\alpha,h) \end{array}\right\} = \lim_{r\to\infty} \begin{array}{c}\sup\\ \inf\end{array} \frac{\alpha(\log M_\infty(h,r))}{\alpha(\log r)}, 1\le\lambda_1\le\rho_1\le\infty \tag{33}$$

Further, for $1<\rho_1<\infty$,we define the generalized type of h by

$$\tau_1 = \tau(\alpha,h) = \lim_{r\to\infty}\ \sup \frac{\alpha(\log\ M_\infty(h,r))}{[\alpha(\log r)]^\rho}. \tag{34}$$

Here the function $\alpha(x)\in\Omega$ or $\overline{\Omega}$ as defined before. Now we prove

Theorem 4. Let h be entire harmonic function of generalized order ρ_1.Then

$$\rho_1 = \ P(\overline{L_1}) \tag{35}$$

where

$$\overline{L_1} = \lim_{m\to\infty} \sup \frac{\alpha(m)}{\alpha(\ln[m!/|\nabla_m h(0)|]^{1/m})}$$

The result follows on the lines of Theorem 1 on using Theorem D.

Theorem 5. Let the function $\mu(m)=\{|\,\nabla_m h(0)|/|\nabla_{m+1}h(0)|\}\sqrt{(m+1)\,(m+n/2)}$ be a non decreasing function of $m$for all large values of m. Then the generalized lower order λ_1 is given by

$$\lambda_1 = \begin{cases} \max\{1, l_1\} \; if \, \alpha(x) \in \Omega \\ 1 + l_1 \qquad if \, \alpha(x) \in \overline{\Omega} \end{cases} \tag{36}$$

where

$$l_1 = \liminf_{m\to\infty} \frac{\alpha(m)}{\alpha(\ln[m!/|\nabla_m h(0)|]^{1/m})}$$

The result follows on the lines of Theorem 3 above on combining with Theorem E. It should be noted that condition (ii) of [5, Theorem 5,p.74] holds in our case since $\lambda_m = m$.

Lastly we characterize the generalized type defined by (34). We have

Theorem 6. Let the non constant entire harmonic function $h \in$ H, be of generalized order ρ_1,$0 < \rho_1 < \infty$, and generalized type $\tau_1$and $\alpha(x) \in \overline{\Omega}$. Then

$$\tau_1 = \limsup_{m\to\infty} \frac{\alpha\left(m/\rho_1\right)}{\left\{\alpha\left[\frac{\rho_1}{\rho_1-1}\log\{|\nabla_m h(0)|/m!\}^{-1/m}\right]\right\}^{\rho_1-1}}. \tag{37}$$

The result follows on using (16) for the entire function

$$g(z) = \Gamma(n/2) \sum_{m=0}^{\infty} \frac{|\nabla_m \; h(0)|}{\sqrt{2^m \; m! \; \Gamma(m+n/2)}} \; z^m$$

and the relations (23) and (24).

4. Acknowledgement

The author is very much thankful to the learned referee for his valuable comments and suggestions.

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