BULLETIN OF THE
GREEK MATHEMATICAL SOCIETY
Volume 55, 2008 (59–63)

On Solvable Leibniz Algebras

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Abstract

A generalization of Lie's theorem for solvable Leibniz algebras to solvable Leibniz algebras leads to interesting applications concerning both of these two types of algebras.

Leibniz algebras are introduced by Loday (Loday, 1993) as nonantisymmetric generalization of Lie algebras. An algebra A over a field F is called a left Leibniz algebra if its multiplication satisfies the Leibniz identity

$$(ab)\,c = a\,(bc) - b\,(ac)$$

for all $a, b, c \in A$. A right Leibniz algebra A must satisfy the dual identity

$$a\,(bc) = (ab)\,c - (ac)\,b.$$

In this parer we are considering only left Leibniz algebras. Obviously, Lie algebras are particular case of Leibniz algebras. The theory of Leibniz algebras is developed in a closed connection with the theory of Lie algebras, which serve us as a model for suitable generalizations to Leibniz algebras.

If A is any left Leibniz algebra, then we define inductively the following three sequences of two-sided ideals of A:

$$\begin{aligned} A^1 &= A, ..., A^{n+1} = AA^n \\ A_1 &= A, ..., A_{n+1} = \sum_{p=1}^{n} A_p A_{n+1-p} \\ A^{(1)} &= A, ..., A^{(n+1)} = A^{(n)} A^{(n)}. \end{aligned}$$

Indeed, it was proved in (Patsourakos, 2007) that A^n is a two-sided ideal of A, and that $A_n = A^n$ for any integer $n \geq 1$. To verify that the subspace A^n is also a two-sided ideal of A, we proceed by induction on $n \geq 1$, the case $n = 1$ being clear. We suppose that A^n is a two-sided ideal of A, and we have

$$AA^{(n+1)} = A\left(A^{(n)}A^{(n)}\right) \subseteq \left(AA^{(n)}\right)A^{(n)} + \left(AA^{(n)}\right)A^{(n)} \subseteq A^{(n)}A^{(n)} = A^{(n+1)},$$

$$A^{(n+1)}A = \left(A^{(n)}A^{(n)}\right)A \subseteq A^{(n)}\left(A^{(n)}A\right) \subseteq A^{(n)}A^{(n)} = A^{(n+1)}.$$

As any non-associative algebra, a left Leibniz algebra is called nilpotent if there exists an integer $n \geq 1$ such that $A_n = 0$. Since $A_n = A^n$, this definition agrees with the similar one of Lie algebras. Again as in the Lie algebras case, a left Leibniz algebra is called solvable if there exists an integer $n \geq 1$ such that $A^{(n)} = 0$.

We note that a left Leibniz algebra of dimension one is an abelian Lie algebra, that is $A^2 = A_2 = A^{(2)} = 0$. We consider the two (not isomorphic) Leibniz algebras of dimension two with basis $\{e_1, e_2\}$ defined by the multiplication tables

$$e_1^2 = e_1e_2 = e_2^2 = 0, e_2e_1 = e_1$$

and

$$e_1^2 = e_1e_2 = e_2e_1 = 0, e_2^2 = e_1.$$

As in the case of right Leibniz algebras (S. Albeverio, Sh. A. Ayupov, B. A. Omirov, 2005), these are the only two left Leibniz algebras of dimension two, which are not Lie algebras. Indeed, $A_3 = 0$, $e_1e_2 \neq e_2e_1$ in the first case, and $e_2^2 \neq 0$ in the second case. Both these two left Leibniz algebras are solvable. Indeed, since $A^{(2)} = AA = Fe_1$, $A^{(3)} = 0$.

Representations of left Leibniz algebras arise naturally in considering split null extensions or semidirect sums in the category of left Leibniz algebras. According to this, a representation of a left Leibniz algebra A on a vector space M over F is a pair (S, T) of linear mappings $a \rightarrow S_a$ and $a \rightarrow T_a$ of A into $End\,(M)$ such that

$$\begin{aligned} S_b \circ S_a &= S_{ab} - T_a \circ S_b \\ S_b \circ T_a &= T_a \circ S_b - S_{ab} \\ T_{ab} &= T_a \circ T_b - T_b \circ T_a \end{aligned}$$

for all $a, b \in A$. For later use, we note that the first two of these equations imply that

$$S_b \circ (S_a + T_a) = 0$$

for all $a, b \in A$.

The right multiplication R_a (resp. left multiplication L_a) of the left Leibniz algebra A determined by any element $a \in A$ is the endomorphism of A defined by $R_a\,(x) = xa$ (resp. $L_a\,(x) = ax$) for all $x \in A$. The pair (R, L) is clearly a representation of A on A itself.

In (Patsourakos, 2007) they were proved two results concerning the nilpotent properties of left Leibniz algebras.

Theorem 1. If (S, T) is a representation of a finite dimensional left Leibniz algebra A on a finite dimensional vector space M over F such that T_a is nilpotent endomorphism of M for any $a \in A$, then the endomorphism S_a of M is also nilpotent, and both T_a and S_a, for all $a \in A$, have at least one common eigenvector corresponding to their unique eigenvalue zero.

Corollary 1. A left Leibniz algebra A is nilpotent if and only if L_a is a nilpotent endomorphism of A for every $a \in A$.

The Theorem 1 can be regarded as generalization of the well known Engel's theorem for Lie algebras, and the Corollary 1 generalizes the fact that a Lie algebra $\mathfrak{g}$ is nilpotent if and only if adx is nilpotent endomorphism of $\mathfrak{g}$ for every $x \in \mathfrak{g}$.

We recall that the Engel's theorem assures us that if ρ is a representation of a finite dimensional Lie algebra $\mathfrak{g}$ on a finite dimensional vector space M such that $\rho(x)$ is a nilpotent endomorphism of M for any $x \in \mathfrak{g}$, then M contains at least one common eigenvector for all the endomorphisms $\rho(x)$ corresponding to their unique eigenvalue zero. Similar in nature is the Lie's theorem, which can be stated as follows: if the field F is algebraically closed of characteristic zero, and if $\mathfrak{g}$ is solvable, then the nonzero vector space M contains a common eigenvector for all the endomorphisms $\rho(x)$, $x \in \mathfrak{g}$. The obvious question now is what would be a suitable generalization of Lie's theorem in the case of solvable left Leibniz algebras. The answer, analogous to the above generalization of Engel's theorem, is expressed as follows:

Theorem 2. Let A be a finite dimensional solvable left Leibniz algebra over an algebraically closed field F of characteristic zero. Let (S, T) be a representation of A on a finite dimensional vector space M. If $M \neq 0$, then M contains a common eigenvector for all the endomorphisms T_a and S_b, $a, b \in A$.

Proof . The proof follows by induction on $\dim A$, the case $\dim A = 0$ being trivial. We suppose that the theorem is true for all solvable left Leibniz algebras of positive dimension less than n, and we consider a solvable left Leibniz algebra with $\dim A = n$.

First, we assert that A contains a two-sided ideal $\mathfrak{h}$ of codimension one. Indeed, given that A is a nonzero solvable left Leibniz algebra, we deduce that A properly contains AA, this implies that A contains a subspace $\mathfrak{h}$ of codimension one such that $AA \subset \mathfrak{h}$, which means that $\mathfrak{h}$ will be automatically a two-sided ideal. We have $A = \mathfrak{h} \oplus F.z$ for some nonzero $z \in A$.

Now, if (S, T) is a representation of A on a finite dimensional vector space M, then its restriction to $\mathfrak{h}$ is a representation of the Leibniz algebra $\mathfrak{h}$ on M. Since $\mathfrak{h}$ is of course solvable with $\dim \mathfrak{h} = n - 1$, by the induction hypothesis there exists a common eigenvector $m_0 \in M$ for all T_a and S_b, $a, b \in \mathfrak{h}$. Thus, for some linear functions $\lambda : T(\mathfrak{h}) \to F$, $\mu : S(\mathfrak{h}) \to F$ and for all $a, b \in \mathfrak{h}$, we have the two relations

$$T_a(m_0) = \lambda(T_a).m_0 \text{ and } S_b(m_0) = \mu(S_b).m_0,$$

and we denote by W the nonzero subspace of M consisted of all vectors $w \in M$ satisfying these two relations.

Next, we will show that $\lambda(T_{az}) = 0$ for all $a \in \mathfrak{h}$. To verify this, we introduce the subspace $U \subset M$ of all vectors $u \in M$ such that $T_a(u) = \lambda(T_a).u$ for all $a \in \mathfrak{h}$. Clearly, $W \subset U$; so $U \neq 0$. Since $T(\mathfrak{h})$ is an ideal in the Lie algebra $T(A)$, by a standard argument (f. e. as in Lemma 9.13 of (Fulton, Harris, 1991)), we obtain that $T_z(U) \subset U$. From this and the commutation relation

$$T_a(T_z(u)) = \lambda(T_{az}).u + \lambda(T_a).T_z(u),$$

where u is any nonzero vector in U, we conclude that $T_z(u)$ lies in U if and only if $\lambda(T_{az}).u = 0$. Since $charF = 0$, this forces $\lambda(T_{az}) = 0$, as required.

As a consequence, we obtain the relation $\lambda(T_{za}) = -\lambda(T_{az}) = 0$ for all $a \in \mathfrak{h}$. Moreover, we have the two obvious relations $\lambda(T_{ab}) = 0$ for all $a, b \in \mathfrak{h}$ and $\lambda(T_{z^2}) = 0$. These three relations imply that the linear function $\lambda : T(\mathfrak{h}) \to F$ must be zero on the subspace $T(AA) \subset T(\mathfrak{h})$.

Now, we come to the linear function $\mu : S(\mathfrak{h}) \to F$. We show that if $\lambda(T_a) = 0$ for some element $a \in \mathfrak{h}$, then $\mu(S_a) = 0$. Indeed, from the relation $S_a \circ (S_a + T_a) = 0$, we obtain the relation $\mu(S_a)(\mu(S_a) + \lambda(T_a)).m_0 = 0$, which implies that $\mu(S_a) = 0$, since $charF = 0$. Thus, the linear function $\mu : S(\mathfrak{h}) \to F$ must be zero on the subspace $S(AA) \subset S(\mathfrak{h})$.

It follows that the endomorphism T_z carries W to itself. Indeed, if w is any vector in W, then

$$\begin{aligned} T_a(T_z(w)) &= T_{az}(w) + T_z(T_a(w)) \\ &= \lambda(T_{az}(w)).w + \lambda(T_a).T_z(w) \\ &= \lambda(T_a).T_z(w) \end{aligned}$$

and

$$\begin{aligned} S_b(T_z(w)) &= T_z(S_b(w)) - S_{zb}(w) \\ &= \mu(S_b).T_z(w) - \mu(S_{zb}).w \\ &= \mu(S_b).T_z(w) \end{aligned}$$

for all $a, b \in \mathfrak{h}$.

As a consequence, using the fact that F is algebraically closed, we find an eigenvector $w_0 \in W$ of T_z (for some eigenvalue of T_z). Clearly, w_0 is a common eigenvector for all T_x and S_b, $x \in A,\ b \in \mathfrak{h}$. Denoting the eigenvalue of T_z corresponding to w_0 by $\lambda(T_z)$, we extend λ to a linear function on $T(A)$ such that $T_x(w_0) = \lambda(T_x).w_0$ for all $x \in A$. Let $V \subset W$ be the nonzero subspace of all $u \in W$ satisfying the relations

$$T_x(u) = \lambda(T_x).u \quad \text{and} \quad S_b(u) = \mu(S_b).u$$

for all $x \in A$ and $b \in \mathfrak{h}$. Then, we have

$$\begin{aligned} T_x(S_z(u)) &= S_{xz}(u) + S_z(T_x(u)) \\ &= \mu(S_{xz}).u + \lambda(T_x).S_z(u) \\ &= \lambda(T_x).S_z(u) \end{aligned}$$

and

$$S_b(S_z(u)) = [S_b, S_z](u) + \mu(S_b).S_z(u).$$

Thus, for a nonzero element $u \in V$, $S_z(u)$ lies in V if and only if $[S_b, S_z](u) = 0$ for all $b \in \mathfrak{h}$. Hence, if we assume that the restriction of the commutator $[S_b, S_z]$ to V is zero for any $b \in \mathfrak{h}$, then we find an eigenvector $u_0 \in V$ of S_z (for some eigenvalue denoted by $\mu(S_z)$). This time, u_0 is obviously a common eigenvector for all T_x and S_y, $x, y \in A$, such that

$$T_x(u_0) = \lambda(T_x).u_0 \quad \text{and} \quad S_y(u_0) = \mu(S_y).u_0.$$

It remains to prove the theorem when there exists $b_0 \in \mathfrak{h}$ such that the restriction of $[S_{b_0}, S_z]$ to V is different from zero. According to this, calculating

$$\begin{aligned} [S_{b_0}, S_z](u) &= S_{b_0}(S_z(u)) - S_z(S_{b_0}(u)) \\ &= -S_{b_0}(T_z(u)) - S_z(S_{b_0}(u)) \\ &= -\mu(S_{b_0}).(T_z + S_z)(u), \end{aligned}$$

we conclude that there exists a nonzero vector $u_1 \in V$ such that its image $m_1 = (T_z + S_z)(u_1) \neq 0$ in M. Since $S_y \circ (T_z + S_z) = 0$ for any $y \in A$, the vector m_1 is a common eigenvector for any S_y corresponding to its eigenvalue zero. Finally, for any $x \in A$, we have

$$\begin{aligned} T_x(m_1) &= T_x((T_z + S_z)(u_1)) \\ &= T_x(T_z(u_1)) + T_x(S_z(u_1)) \\ &= \lambda(T_x).T_z(u_1) + \lambda(T_x).S_z(u_1) \\ &= \lambda(T_x).(T_z + S_z)(u_1) \\ &= \lambda(T_x).m_1, \end{aligned}$$

which means that the vector $m_1$is also a common eigenvector for any T_x corresponding to its eigenvalue $\lambda(T_x)$, as required.

Corollary 2. For any finite dimensional representation (S, T) of a finite dimensional solvable left Leibniz algebra A over an algebraically closed field F of characteristic zero on a vector space $M \neq 0$, there exists a basis of M relative to which all the endomorphisms $T_a, S_b,\ a, b \in A$, represented by upper triangular matrices.

Proof. We use the Theorem 2, along with induction on $\dim M$.

Corollary 3. Let A be a finite dimensional solvable left Leibniz algebra A over an algebraically closed field F of characteristic zero. If $x \in A^2$, then the endomorphism L_x of A is nilpotent. In particular, A^2 is nilpotent.

Proof. Relative to a suitable basis of A constructed according to Corollary 2 the matrix of L_a, for any $a \in A$, is upper triangular. Therefore, the matrix of $L_{ab} = [L_a, L_b]$, for any $a, b \in A$, is strictly upper triangular. We deduce that L_x is nilpotent for any $x \in A^2$, thus, by Corollary 1, A^2 is nilpotent.

In the particular case of Lie algebras, the above two corollaries imply the well-known results that any representation of a solvable Lie algebra $\mathfrak{g}$ is upper triangular, and that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

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Keywords: Leibniz algebras, Lie algebras, solvable, representation, Lie theorem.
2007 MSC: 17A32 Leibniz algebras

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