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Characterizations of Sequence Algebras

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Abstract

We show that a real semisimple Fréchet m-convex algebra A satisfiying $Ax = Ax^2$ is isomorphic to $R^{I_1} \times C^{I_2} \times H^{I_3}$, for some subsets I_1, I_2, I_3 of N. As a consequence, we obtain characterizations of sequence algebras.

Introduction. The condition $Ax = Ax^2$, has been examined by several authors ([4], [5], [6], [7], [8], [9] for example). In [5], see also [6], it is shown that a complex unitary Banach algebra A satisfiying the property $Ax = Ax^2$, for every x in A, is finite dimensional. The unitary algebra C^N, of complex sequences, is Fréchet and locally multiplicatively convex. It satisfies this property but it is not of finite dimension. In this note, we give a description of the structure of real and semisimple Fréchet multiplicatively convex algebra A which satisfy $Ax = Ax^2$, for every x in A (Theorem 2). It includes the complex case studied in [4]. We also characterize the sequence algebras R^N, C^N and H^N (Propositions 2 and 4).

Preliminaries. An algebra A is said to be noetherian if the family of its left ideals satisfies the ascending chain condition. It is equivalent to say that any left ideal, of A, is finitely generated. The Jacobson radical of A, denoted by $radA$, is the intersection of the kernels of all irreducible representations of A. The algebra A is said to be semisimple (resp. radical) if $radA = \{0\}$ (resp. $radA = A$) ([2], definition 13, p.124). A two sided ideal P, of A, is said to be prime if $P \neq A$ and the quotient algebra A/P does not contain zero divisors. If $I_1, ..., I_r$ are left ideals of A, we denote by $I_1 ... I_r$ (the so-called product of the ideals $I_1, ..., I_r$) the left ideal generated by the products $x_1 ... x_r$, with $x_i \in I_i$. If $I_i = I$, for every i, the ideal $I_1 ... I_r$ is denoted I^r. A topological algebra is said to be a Q-algebra (or has the Q-property) if the set of its quasi-invertible elements is open. An algebra topology is said to be Fréchet and m-convex if it is complete, metrizable and it is given by a family of submultiplicetive seminorms. In the sequel, all algebras considered will be real, unless a special mention. Also, R, C and H will designate, respectively, the field of real numbers, complex numbers and quaternions.

Structure results. Before stating the main result, we begin with an algebraic one which has its own interest.

Proposition 1. Let A be a semisimple and noetherian algebra. If it satisfies $Ax = Ax^2$, for every x in A, then it admits a unit.

Proof. Notice first that A is necessarily without nonzero nilpotents. Let $x, y \in A$. Since $Ax = Ax^2 = Ax^3$, there is an $a \in A$ such that $x^2 = ax^3$. One easily cheks that $(x^2 - x^2ax)^2 = 0$, hence $A(x^2 - x^2ax) = \{0\}$, by hypothesis. Now by ([2], Proposition 16, p.125), $x^2 - x^2ax \in radA = \{0\}$ Similary, there is $b \in A$ such that $y^2 = by^3 = y^2by$. Put $z = ax + by - axby$. One has $x^2z = x^2$. By the same arguments, one obtains $y^2ax - y^2axby = 0$. Hence $y^2z = y^2$. Finally, $Ax + Ay = Az$. It then follows that any left ideal, of A, is generated by a single element. We conclude by Proposition 1 of [4].

We consider first the case of Q-algebras.

Theorem 1. Let A be a semisimple Fréchet m-convex algebra which has the Q-property. If it satisfies $Ax = Ax^2$, for every x in A, then it is isomorphic to an algebra $R^r \times C^s \times H^t$, with $r, s, t \in N$.

Proof. Since A is a Q-algebra, every left regular maximal ideal, of A, is closed. Hence, by Proposition 1 of [4], any left ideal is closed. It follows, using Baire argument, that A is noetherian. So, by the previous proposition, A is unitary. Moreover, any left ideal, of A, is two-sided (cf [4], Proposition 1). Since A is noetherian, one shows, as in the commutative case, the existence of prime ideals $P_1, ..., P_r$, of A, pairwise distinct, and positive integers $m_1, ..., m_r$ such that $P_1^{m_1}...P_r^{m_r} = \{0\}$. We claim that $P_1 \cap ... \cap P_r = \{0\}$. Indeed, for $x \in P_1 \cap ... \cap P_r$ and $m = \max_{1 \leq i \leq r} m_i$, it is clear that $x^{rm} = 0$. Whence $x = 0$, due to $Ax = Ax^2$ and the fact that A is unital. Then, by Proposition 9 of [3], A is isomorphic to the product algebra $\Pi_{1 \leq i \leq r} A/P_i$. Finally, the Fréchet m-convex algebra A/P_i, being a field, is isomorphic to R or C or H.

Now, here is the general case.

Theorem 2. Let A be a semisimple Fréchet m-convex algebra. If it satisfies $Ax = Ax^2$, for every x, then there is N_1, N_2, N_3, subsets of N, such that A is isomorphic to the product algebra $R^{N_1} \times C^{N_2} \times H^{N_3}$.

Proof. Let (p_n) be an increasing family of submultiplicative seminorms defining the topology τ of A. For every n, consider the quotient algebra $A_n = A/Kerp_n$ endowed with the norm q_n given by $q_n(\overline{x}) = p_n(x)$. Denote by τ_n the quotient topology, on A_n, induced by τ. By Proposition 5 of [4], (A_n, q_n) is a Q-algebra. But then (A_n, τ_n) is also a Q-algebra for the topology defining by q_n is coarser than τ_n. So, by Theorem 1, A_n is isomorphic to $R^{r_n} \times C^{s_n} \times H^{t_n}$. Now, as in [10], one has

$$A = \varprojlim A_n$$

. The conclusion follows from the fact that the projective limit commutes with the cartesian product operation.

In the commutative case, we have the following consequence.

Corollary 1. Let A be a commutative, semisimple, Fréchet m-convex algebra of infinite dimension. If it satisfies $Ax = Ax^2$, for every $x \in A$, then it is isomorphic to one of the following algebras: R^N, C^N, $R^N \times C^n$ for some $n \in N^*$, $R^n \times C^N$ for some $n \in N^*$, $R^N \times C^N$.

In the complex case, we find again a result of [4].

Corollary 2 ([4], Corollaire 8). Let A be a complex, semisimple, Fréchet m-convex algebra of infinite dimension. If it satisfies $Ax = Ax^2$, for every $x \in A$, then it is isomorphic to the sequence algebra C^N.

One, also, notices the following phenomenon concerning some ideals of A.

Corollary 3. Let I be a closed ideal of K^N ($K = R, C$).
(i) If I is of infinite codimension, then K^N and K^N/I are isomorphic algebras.
(ii) If I is of infinite dimension, then K^N and I are isomorphic algebras.

Proof. (i) The result is clear, in the complex case, via Corollary 1. In the real case, A/I can not be isomorphic to any algebra, of Corollary 1, else than R^N; otherwise R^N should admit a maximal ideal of codimension 2 which is impossible.

(ii) Considered as an algebra, I is a semisimple Fréchet m-convex one. Moreover it satisfies $Ix = Ix^2$, for every x in I. In the complex case, the conclusion follows by Corollary 2. In the real case, use Corollary 1, noticing that I admits no element x such that $x^4 = 1$ and $x^2 \neq 1$.

Remark. Actually, Properties (i) and (ii), together, characterize the algebras R^N and C^N. So, together, they are stronger than $Ax = Ax^2$, for every x.

Proposition 2. Let A be a unitary, commutative and Fréchet m-convex algebra of infinite dimension. If A satisfies properties (i) and (ii) of Corollary 3, then it is isomorphic to R^N or C^N.

Proof. A being unital can not be a radical algebra, so it is not isomorphic to $radA$. Hence, by (ii), $radA$ is finite dimensional. So, by (i), A is isomorphic to $A/radA$. Thus, it is semisimple. Now, for $x \in A$, if Ax, as a semisimple algebra, is of finite dimension, it must be unital and then $Ax = Ax^2$. If Ax is of infinite dimension, then, by hypothesis, A is isomorphic to the closure $\overline{Ax}$ of Ax. The latter is then unital and consequently $\overline{Ax} = \overline{Ax^2}$. Now, we claim that actually $Ax = Ax^2$. Indeed, the algebra $B = \overline{Ax}$ is a unitary and Fréchet m-convex one such that $\overline{Bx} = B$. As in ([1, p. 173, Theorem 4.2]), one shows, here is the real case, $Bx = B$. It follows that $Ax = Ax^2$. Whence the conclusion, via Corollary 1.

Now, we examine the non commutative case. We first notice a phenomenon concerning ideals of the sequence algebra H^N.

Proposition 3. Let I be a closed left ideal (necessarily two-sided) of H^N.
(i) If I is of infinite codimension, then H^N and H^N/I are isomorphic algebras.
(ii) If I is of infinite dimension, then H^N and I are isomorphic algebras.

Proof. (i) By Theorem 2, there are N_1, N_2, N_3, subsets of N, such that H^N/I is isomorphic to the product algebra $R^{N_1} \times C^{N_2} \times H^{N_3}$. If N_1 or N_2 were not void, there should exist a nonzero algebra morphism from H^N into C; which is impossible.

(ii) We argue, as in (i), taking into account that any algebra morphism from I into C extends to such morphism from H^N into C.

Proposition 4. Let A be a non commutative unitary and Fréchet m-convex algebra of infinite dimension. If it satisfies conditions (i) and (ii), then it is isomorphic to H^N.

Proof. We argue as in Proposition 2. The single point to be clarified is the semisimplicity of Ax when it is of finite dimension. Suppose Ax is not semisimple. Consider a nonzero x in $Rad(Ax)$. This element is nilpotent, due to finite dimensionality. We can suppose that $x^2 = 0$. Then $I = \{a \in A : ax = 0\}$ is a closed left ideal, of A, which is necessary of infinite dimension. Hence A is isomorphic to I. But $Ix = \{0\}$; which implies that x is in $Rad(I)$. So A should be not semisimple and this is not the case.

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