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On the Quotient of a b-Algebra by a no Closed b-Ideal

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Résumé

We prove that an associative q-algebra, in the sense of Waelbroeck, is isomorphic to the quotient of an associative b-algebra modulo a two-sided b-ideal, and we give applications of this result.

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1. Introduction and notations

In his "Thèse d'Agrégation" [4], Waelbroeck gave a construction of the holomorphic functional calculus for elements of a unital b-algebra. The construction was bornological and used the Cauchy-Fantapiè formula. He constructed his holomorphic functional calculus with a condition about the growth of holomorphic functions. The computation was even "quotient bornological', because it is realized in a b-algebra modulo a b-ideal which is not necessarily closed. However, he did not define neither a b-subspace nor a b-ideal. He need them to prove that the spectrum of n elements of a b-algebra is not trivial. In [6], [7] and [8] Waelbroeck defined several categories of quotients spaces. A q-algebra is a quotient bornological space on which a multiplication is defined by morphism. The q-algebras which are associatives can be defined. Also, we can define what a commutative q-algebra is and define the center of a q-algebra. A morphism of q-algebras is a morphism of quotient bornological spaces which conserves the multiplication.

The above definitions are "natural" but they are difficult to understand. In his paper, we will define strict algebras as quotient of b-algebras by two-sided b-ideals, and prove that every q-algebra is isomorphic to a strict algebra. To do this, we will give necessary and sufficient conditions for which each strict algebra is associative. Next, we will prove that each associative q-algebra is isomorphic to the quotient of an associative b-algebra, and we will define the center of an associative strict algebra. Finally, we will speak on morphisms from nuclear b-algebras into q-algebras. As consequences, we obtain Theorem 4.4 and Theorem 5.2 of [1] .

To prove our results, we shall need to fix some notations and recall some definitions. Let **EV** be the category of vector spaces and linear mappings over the scalar field $\mathbb{R}$ or $\mathbb{C}$.

1- Let $(E, \|\|_E)$ be a Banach space. A Banach subspace F of E is a vector subspace that we endow with a Banach norm $\|\|_F$ such that the inclusion $(F, \|\|_F) \longrightarrow (E, \|\|_E)$ is continuous. A quotient Banach space $E \mid F$ is a vector space E/F, where E is a Banach space and F a Banach subspace. For more information about quotient Banach spaces we refer the reader to [6].

2- Let E be a real or complex vector space, and let B be an absolutely convex set of E. Let E_B be the vector space generated by B i.e. $E_B = \cup_{\lambda>0}\lambda B$. The Minkowski functional of B is a semi-norm on E_B. It is a norm, if and only if B does not contain any nonzero subspace of E. The set B is completant if its Minkowski functional is a Banach norm.

A bounded structure β on a vector space E is defined by a set of "bounded" subsets of E with the following properties :

(1) Every finite subset of E is bounded.
(2) Every union of two bounded subsets is bounded.
(3) Every subset of a bounded subset is bounded.
(4) A set homothetic to a bounded subset is bounded.
(5) Each bounded subset is contained in a completant bounded subset.

A b-space (E, β) is a vector space E with a boundedness β. A subspace F of a b-space E is bornologically closed if $F \cap E_B$ is closed in E_B for every completant bounded subset B of E.

Given two b-spaces (E, β_E) and (F, β_F), a linear mapping $u : E \longrightarrow F$ is bounded, if it maps bounded subsets of E into bounded subsets of F. The mapping $u : E \longrightarrow F$ is bornologically surjective if for every $B' \in \beta_F$, there exists $B \in \beta_E$ such that $u(B) = B'$.

We denote by $\mathbf{b}(E_1, E_2)$ the space of bounded linear mappings between the b-spaces E_1 and E_2, it is a b-space for the following boundedness : a subset B of $\mathbf{b}(E_1, E_2)$ is bounded if the set $\{u\ (x) : u \in B,\ x \in B'\}$ is bounded in E_2, for all bounded subset B' in E_1. By $\mathbf{b}$ we denote the category of b-spaces and bounded linear mappings. For more information about b-spaces the reader is referred to [2] and [5].

3- Let (E, β_E) be a b-space. A b-subspace of E is a subspace F that we endow with a boundedness β_F such that (F, β_F) is a b-space and $\beta_F \subseteq \beta_E$. A quotient bornological space $E \mid F$ is a vector space E/F, where E is a b-space and F a b-subspace of E.

If $E \mid F$ and $E' \mid F'$ are two quotient bornological spaces, a strict morphism $u : E \mid F \longrightarrow E' \mid F'$ is induced by a bounded linear mapping $u_1 : E \longrightarrow E'$ whose restriction to F is a bounded linear mapping $F \longrightarrow F'$. Two bounded linear mappings $u_1, v_1 : E \longrightarrow E'$, both inducing a strict morphism, induce the same strict morphism $E \mid F \longrightarrow E' \mid F'$ iff the linear mapping $u_1 - v_1 : F \longrightarrow F'$ is bounded. A strict morphism u is a class of equivalence of bounded linear mappings, for the equivalence just defined.

The class of quotient bornological spaces and strict morphisms is a category, that we call $\widetilde{\mathbf{q}}$. A pseudo-isomorphism $u : E \mid F \longrightarrow E' \mid F'$ is a strict morphism induced by a bounded linear mapping $u_1 : E \longrightarrow E'$ which is bornologically surjective and such that $u_1^{-1}(F') = F$ as b-spaces (i.e. $B \in \beta_F$ if $B \in \beta_E$ and $u_1(B) \in \beta_{F'}$).

There are pseudo-isomorphisms which do not have strict inverses. In fact, if E is a Banach space and F a closed subspace of E, it would be very nice if the quotient bornological space $E \mid F$ where isomorphic to the quotient $(E/F) \mid \{0\}$. This is not the case in $\tilde{\mathbf{q}}$ unless F is complemented in E. Waelbroeck [8] constructed an abelian category $\mathbf{q}$ that contains $\widetilde{\mathbf{q}}$ and in which all pseudo-isomorphisms of $\widetilde{\mathbf{q}}$ are isomorphisms.

4- A b-algebra $\mathcal{A}$ is an algebra on $\mathbb{C}$, with a boundedness of b-space $\beta_{sl\mathcal{A}}$ such that the product of two elements of $\beta_{sl\mathcal{A}}$ is an element of $\beta_{sl\mathcal{A}}$ i.e. if B_1 and B_2 are two completant bounded subsets of $calA$, there exists a completant bounded subset

B_3 of $\mathcal{A}$ such that the bilinear mapping $\mathcal{A}_{B_1}\times\mathcal{A}_{B_2}\longrightarrow\mathcal{A}_{B_3}$ is bounded. A left b-ideal of a b-algebra $(\mathcal{A},\, \beta_{sl\mathcal{A}})$ is a left ideal α of $\mathcal{A}$, with a boundedness of b-space β_α such that the mapping

$$(\mathcal{A},\, \beta_{sl\mathcal{A}}) \times (\alpha,\, \beta_\alpha) \longrightarrow (\alpha,\, \beta_\alpha),\ (a, x) \longmapsto a.x$$

is bounded. In the same way we define right b-ideal. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are b-algebras, a homomorphism $u : \mathcal{A}_1 \longrightarrow\mathcal{A}_2$ is a bounded linear mapping such that

$$u(a.b) = u(a).u(b) \text{ for all } a,\, b \in\mathcal{A}_1.$$

If $\mathcal{A}$ is a b-algebra, a b-subspace α of $\mathcal{A}$ is a left b-ideal (resp. right b-ideal) if $B.C$ (resp. $C.B$) is a bounded subset in α whenever B is bounded in $\mathcal{A}$ and C is bounded in α. The b-subspace α is a two-sided b-ideal of $\mathcal{A}$ if it is a left b-ideal and a right b-ideal. For more information about b-algebras and b-ideals we refer the reader to [3].

2. Strict q-algebras

We start with the definition of q-algebras. If $E \mid F$ and $E_1 \mid F_1$ are two quotient bornological spaces, we design by $\mathbf{q}(E \mid F,\, E_1 \mid F_1)$ the quotient bornological space $\mathbf{q}^1(E \mid F,\, E_1 \mid F_1) \mid \mathbf{q}^0(E \mid F,\, E_1 \mid F_1)$, where $\mathbf{q}^1(E \mid F,\, E_1 \mid F_1)$ is the space of $f \in \mathbf{b}(E, E_1)$ such that the restriction $f_{|_F} \in \mathbf{b}(F, F_1)$, that we endow with the boundedness for which a subset B is bounded if it is equibounded in $\mathbf{b}(E, E_1)$ and $B_{|_F} = \{f_{|_F} : f \in B\}$ is equibounded in $\mathbf{b}(F, F_1)$, also we denote by $\mathbf{q}^0(E \mid F, E_1 \mid F_1) = \mathbf{b}(E, F_1)$.
If $E \mid F, \ldots\ldots, E_n \mid F_n$ are quotient bornological spaces, we define by induction :

$$\mathbf{q}_1(E \mid F, E_1 \mid F_1) = \mathbf{q}(E \mid F, E_1 \mid F_1)$$

and

$$\mathbf{q}_n(E \mid F, ..., E_{n-1} \mid F_{n-1}; E_n \mid F_n) = \mathbf{q}(E \mid F, \mathbf{q}_{n-1}(E_1 \mid F_1, ..., E_{n-1} \mid F_{n-1}; E_n \mid F_n)).$$

A q-algebra is a quotient bornological space $E \mid F$ on which a multiplication is defined by an element $m \in \sigma\mathbf{q}_2(E \mid F, E \mid F; E \mid F)$. We observe that $m\,(Id, m)$ and $m\,(m, Id)$ are two elements of $\sigma\mathbf{q}_3(E \mid F, E \mid F, E \mid F; E \mid F)$. The q-algebra is said to be associative if $m\,(Id, m) = m\,(m, Id)$.

For a nonempty set X, we denote by $l^1(X)$ the Banach space whose elements are mappings $u : X \longrightarrow \mathbb{C}$ such that

$$\|u\|_1 = \textstyle\sum_{x\in X} |u(x)| < +\infty.$$

If $u \in l^1(X)$, at most a countable number of values $u(x)$ can not be null. Clearly the mapping $u \longrightarrow \|u\|_1$ is the norm of $l^1(X)$.

Now, a Banach space E is free if E is isometrically isomorphic to $l^1(X)$ for some set X, and a b-space E is free if there exists a family of sets $(X_i)_{i\in I}$ such that E is bornologicaly isomorphic to the direct sum $\oplus_{i\in I} l^1(X_i)$. A quotient bornological space $E \mid F$ is said to be standard if the b-space E is free.

It follows from [8], that to each quotient bornological space $E \mid F$, we can associate a standard quotient bornological space $E' \mid F'$ and a pseudo-isomorphism $E' \mid F' \longrightarrow E \mid F$.

If $E'_1 \mid F'_1, \ldots., E'_k \mid F'_k$ are standard quotient bornological spaces and $U \mid V$ is a quotient bornological space then the multimorphism

$$(E'_1 \mid F'_1) \times (E'_2 \mid F'_2) \times \ldots. \times (E'_k \mid F'_k) \longrightarrow U \mid V$$

is strict.

We consider a q-algebra $(E \mid F, m)$ which is isomorphic to a standard quotient bornological space $E' \mid F'$, by the structure transport we define a multiplication m' on $E' \mid F'$ such that $(E' \mid F', m')$ is a standard q-algebra. The multiplication m' is strict, induced by a mapping $m_1' : E' \times E' \longrightarrow E'$. So E' equipped with the multiplication m_1' is a b-algebra. We say that $(E' \mid F', m')$ is a strict q-algebra.

Look now at the b-subspace F' of E', we see that F' has a boundedness such that $m_1'(B_1, B_2)$ and $m_1'(B_2', B_1')$ are bounded in F'. It follows that F' is a two-sided b-ideal of E'.

Instead of writing $(E \mid F, m)$, we shall usually write $\mathcal{A} \mid \alpha$, where $\mathcal{A}$ is a b-algebra, its multiplication is . and α is a two-sided b-ideal of $\mathcal{A}$.

3. Associative q-algebras

A q-algebra $(E \mid F, m)$ is said to be associative if $m(m, Id) = m(Id, m)$. We start with a result about associative q-algebras.

Proposition 3.1. *A strict q-algebra $(sl\mathcal{A} \mid \alpha, .)$ is associative iff the "associator" $\mathcal{A}\times\mathcal{A}\times\mathcal{A} \longrightarrow \alpha$, $(x, y, z) \longmapsto (x, y).z - x.(y.z)$ is a bounded trilinear mapping.*

Proof. We put $m_1(x, y) = x.y$. Then we see that $m(m, Id)$ is a strict trilinear morphism induced by the mapping

$$m_1(m_1, Id) : (x, y, z) \longmapsto (x.y).z.$$

In a similar way, the trimorphism $m(Id, m)$ is induced by the mapping

$$m_1(Id, m_1) : (x, y, z) \longmapsto x.(y.z).$$

Saying that the q-algebra is associative means that the associator

$$\mathcal{A}\times\mathcal{A}\times\mathcal{A} \longrightarrow \alpha,\ (x, y, z) \longmapsto (x.y).z - x.(y.z)$$

is a trilinear bounded mapping. The proof is finished.

Now if $(E \mid F, m)$ and $(E' \mid F', m')$ are two q-algebras, a morphism of q-algebras $u : (E \mid F, m) \longrightarrow (E' \mid F', m')$ is a morphism of quotient bornological spaces $u : E \mid F \longrightarrow E' \mid F'$ such that

$$u(m(x, y)) = m'(u(x), u(y)).$$

The following property of morphism of q-algebras is clear.

Proposition 3.2. *Let $(sl\mathcal{A} \mid \alpha, .)$ and $(sl\mathcal{A}' \mid \alpha', .)$ be two strict q-algebras. A strict morphism of q-algebras $(sl\mathcal{A} \mid \alpha, .) \longrightarrow (sl\mathcal{A}' \mid \alpha', .)$ is a strict morphism of quotient bornological spaces $u : sl\mathcal{A} \mid \alpha \longrightarrow \mathcal{A}' \mid \alpha'$ induced by a bounded linear mapping $u_1 : \mathcal{A} \longrightarrow \mathcal{A}'$ such that the mapping $\mathcal{A}\times\mathcal{A} \longrightarrow \alpha$, $(x, y) \longmapsto u_1(x.y) - u_1(x)u_1(y)$ is a bounded bilinear mapping.*

The next result describes the image of a strict morphism of q-algebras.

Proposition 3.3. *Let $sl\mathcal{A} \mid \alpha$ and $\mathcal{A}' \mid \alpha'$ be two associative q-algebras, and let $u : sl\mathcal{A} \mid \alpha \longrightarrow \mathcal{A}' \mid \alpha'$ be a strict morphism of q-algebras induced by a bounded linear mapping $u_1 : \mathcal{A} \longrightarrow \mathcal{A}'$. Then $u_1(sl\mathcal{A}) + \alpha'$ is a b-subalgebra of $\mathcal{A}'$.*

Proof. It is clear that $u_1(sl\mathcal{A}) + \alpha'$ is a subalgebra of $\mathcal{A}'$. We must show that $u_1(sl\mathcal{A}) + \alpha'$ is a b-subalgebra of $\mathcal{A}'$. Consider two bounded subsets M and N of $u_1(sl\mathcal{A}) + \alpha'$. They are contained respectively in $u_1(B) + C$ and $u_1(B') + C'$, where B, B' are bounded subsets in $\mathcal{A}$ and C, C' are bounded subsets in α. The product of M and N is contained in

$$u_1(B).u_1(B') + u_1(B).C' + C.u_1(B') + C.C'$$

and $u_1(B).u_1(B')$ is contained in the sum of

$$u_1(B.B') \text{ and } D = \{u_1(b).u_1(b') - u_1(b.b') : \ b \in B, \ b' \in B'\}.$$

Also, the set $u_1(B.B')$ is bounded in $u_1(sl\mathcal{A})$ and the sets $u_1(B).C'$, $C.u_1(B')$, $C.C'$ and D are bounded in α'. Therefore the product $(u_1(B)+C).(u_1(B')+C')$ is bounded in $u_1(sl\mathcal{A})+\alpha'$. The result is proved.

Now we gives a result which characterizes the associative q-algebras.

Theorem 3.4. *An associative q-algebra is isomorphic to the quotient of an associative b-algebra modulo a two-sided b-ideal.*

Proof. We shall prove the result when the associative q-algebra has a unit. When it does not have a unit, we can end the proof by adding a unit to the original algebra.

We begin with a strict unital q-algebra $sl\mathcal{A} \mid \alpha$ with $\mathcal{A}$ unital. We shall use the regular left representation of $sl\mathcal{A} \mid \alpha$ into $\mathbf{q}^1(sl\mathcal{A} \mid \alpha, sl\mathcal{A} \mid \alpha) \mid \mathbf{q}^o(sl\mathcal{A} \mid \alpha, sl\mathcal{A} \mid \alpha)$. This morphism is induced by

$$t_1 : \mathcal{A} \longrightarrow \mathbf{q}^1(sl\mathcal{A} \mid \alpha, sl\mathcal{A} \mid \alpha)$$

which maps every $a \in \mathcal{A}$ onto $t_1 a$ such that $t_1 a(x) = a.x$. The mapping $t_1 a$ maps every bounded subset of $\mathcal{A}$ onto a bounded subset of $\mathcal{A}$, and every bounded subset of α onto a bounded subset of α, i.e. $t_1 a$ induces a strict morphism $sl\mathcal{A} \mid \alpha \longrightarrow sl\mathcal{A} \mid \alpha$.

Further, t_1 maps each bounded subset of $\mathcal{A}$ onto a bounded subset of $\mathbf{q}^1(sl\mathcal{A} \mid \alpha, sl\mathcal{A} \mid \alpha)$, and its restriction maps every bounded subset of α onto a bounded subset of $\mathbf{q}^o(sl\mathcal{A} \mid \alpha, \ sl\mathcal{A} \mid \alpha)$. So t_1 induces a strict morphism

$$sl\mathcal{A} \mid \alpha \longrightarrow \mathbf{q}^1(sl\mathcal{A} \mid \alpha, \ sl\mathcal{A} \mid \alpha) \mid \mathbf{q}^o(sl\mathcal{A} \mid \alpha, \ sl\mathcal{A} \mid \alpha)$$

(in the linear sense).

The bounded linear mapping t_1 induces even an algebra morphism. We want to prove that the subset

$$\{t_1(x.y) - t_1(x).t_1(y) : x \in B, \ y \in B\}$$

is bounded in $\mathbf{q}^o(sl\mathcal{A} \mid \alpha, \ sl\mathcal{A} \mid \alpha)$. Of course if $z \in \mathcal{A}$, then

$$t_1(x.y)(z) = (x, y).z$$

and

$$t_1(x)\,t_1(y)(z) = x.(y.z).$$

We assumed that the q-algebra is associative, here the mapping

$$\mathcal{A} \times \mathcal{A} \times \mathcal{A} \longrightarrow \alpha, \ (x, y, z) \longmapsto (x.y).z - x.(y.z)$$

is bounded. Therefore the subset

$$\{t_1(x.y) - t_1(x)\,t_1(y) : x \in B, \ y \in B\}$$

is bounded in $\mathbf{q}^o(sl\mathcal{A} \mid \alpha, \ sl\mathcal{A} \mid \alpha)$. Hence t_1 induces a strict morphism

$$sl\mathcal{A} \mid \alpha \longrightarrow \mathbf{q}^1(sl\mathcal{A} \mid \alpha, sl\mathcal{A} \mid \alpha) \mid \mathbf{q}^o(sl\mathcal{A} \mid \alpha, sl\mathcal{A} \mid \alpha).$$

The space $\mathbf{q}^1(sl\mathcal{A} \mid \alpha, sl\mathcal{A} \mid \alpha)$ is an associative b-algebra and the subspace $\mathbf{q}^o(sl\mathcal{A} \mid \alpha, sl\mathcal{A} \mid \alpha)$ is a two-sided b-ideal. The morphism is monic because $sl\mathcal{A} \mid \alpha$ is unital. It is epic onto $t_1(sl\mathcal{A}) + \mathbf{q}^o(sl\mathcal{A} \mid \alpha, sl\mathcal{A} \mid \alpha)$. Therefore the morphism

$$sl\mathcal{A} \mid \alpha \longrightarrow t_1(sl\mathcal{A}) + \mathbf{q}^o(sl\mathcal{A} \mid \alpha, sl\mathcal{A} \mid \alpha) \mid \mathbf{q}^o(sl\mathcal{A} \mid \alpha, sl\mathcal{A} \mid \alpha).$$

is an isomorphism. And this proves the theorem.

4. The center of a q-algebra and commutative q-algebra.

To define the center of a q-algebra, we observe that the two quotient bornological spaces $\mathbf{q}_2(E_1 \mid F_1, E_2 \mid F_2; E \mid F)$ and $\mathbf{q}_2(E_2 \mid F_2, E_1 \mid F_1; E \mid F)$ are naturally isomorphic. The isomorphism

$$\mathbf{q}_2\left(E_1 \mid F_1, E_2 \mid F_2; E \mid F\right) \longrightarrow \mathbf{q}_2\left(E_2 \mid F_2, E_1 \mid F_1; E \mid F\right)$$

is called the "permutation of the variables" isomorphism.

In this way, we have an isomorphism

$$\mathbf{q}_2\left(E \mid F, E \mid F; E \mid F\right) \longrightarrow \mathbf{q}_2\left(E \mid F, E \mid F; E \mid F\right)$$

We remember also that $\mathbf{q}_2\left(E \mid F, E \mid F; E \mid F\right)$ is defined as $\mathbf{q}\left(E \mid F, \mathbf{q}\left(E \mid F, E \mid F\right)\right)$.

The multiplication m of our q-algebra is mapped onto an element $m' \in \sigma\mathbf{q}\left(E \mid F, \mathbf{q}\left(E \mid F, E \mid F\right)\right)$. The swapping of the variables maps m' onto an element $m'' \in \sigma\mathbf{q}\left(E \mid F, \mathbf{q}\left(E \mid F, E \mid F\right)\right)$.

Now, we define the center of an associative q-algebra $(E \mid F, m)$ as the kernel of the morphism $m'' - m'$. And a q-algebra $(E \mid F, m)$ is said to be commutative if $m' = m''$.

Recall that if $E \mid F$ is a quotient bornological space and E_0 a b-subspace of E such that F is a b-subspace of E_0, the natural injection $E_0 \longrightarrow E$ induces a strict morphism $E_0 \mid F \longrightarrow E \mid F$, we say that the quotient bornological space $E_0 \mid F$ is a subquotient of $E \mid F$.

Let $sl\mathcal{A} \mid \alpha$ be an associative q-algebra, we define $Z_\alpha\left(sl\mathcal{A}\right)$ as the space of $x \in \mathcal{A}$ such that the mapping $\mathcal{A} \longrightarrow \alpha$, $y \longmapsto [x, y]$ is bounded. It is a b-space for the following boundedness : a subset B of $Z_\alpha\left(sl\mathcal{A}\right)$ is bounded if it is bounded in $\mathcal{A}$, and the set of mappings, $\mathcal{A} \longrightarrow \alpha$, $y \longmapsto [x, y]$ with $x \in B$, is equibounded.

The next property about the center of an associative q-algebra is clear.

Proposition 4.1. *The center of an associative q-algebra is a q-subalgebra. If the q-algebra is "strict", isomorphic to $sl\mathcal{A} \mid \alpha$ where $\mathcal{A}$ is an associative b-algebra, then, its center is $Z_\alpha\left(sl\mathcal{A}\right) \mid \alpha$.*

We continue with another property of the center of an associative q-algebra.

Proposition 4.2. *An associative q-algebra is isomorphic to a q-algebra $\mathcal{B} \mid \beta$ where $\mathcal{B}$ is an associative b-algebra and the center of $\mathcal{B} \mid \beta$ is such that $Z_\beta\left(\mathcal{B}\right) = Z\left(\mathcal{B}\right) + \beta$ where $Z\left(\mathcal{B}\right)$ is the center of $\mathcal{B}$.*

We shall not prove the result. A proof of a more difficult result can be found in [7], there the algebra $\mathcal{A}$ is Banach, we find a Banach algebra $\mathcal{B}$ such that $\mathcal{B} \mid \beta$ is isomorphic to $sl\mathcal{A} \mid \alpha$. It is easier to find a b-algebra $\mathcal{B}$ and a b-ideal β such that the q-algebra $\mathcal{B} \mid \beta$ is isomorphic to $sl\mathcal{A} \mid \alpha$ and $Z_\beta\left(\mathcal{B}\right) = Z\left(\mathcal{B}\right) + \beta$ than in the Banach case to find a Banach algebra and a Banach ideal solving the problem.

Proposition 4.3. *An associative and commutative q-algebra is isomorphic to the quotient of an associative and commutative b-algebra modulo a b-ideal.*

Proof. Accepting the proposition 4.1, we see that the algebra is isomorphic to $sl\mathcal{A} \mid \alpha$ where $\mathcal{A}$ is an associative b-algebra and $\mathcal{A} = Z\left(sl\mathcal{A}\right) + \alpha$ (the algebra is commutative and is equal to its center). Therefore, the q-algebras $sl\mathcal{A} \mid \alpha$ and $Z\left(sl\mathcal{A}\right) \mid Z\left(sl\mathcal{A}\right) \cap \alpha$ are isomorphic (this follows from the isomorphism theorem [7]).

5. Morphisms of a nuclear b-algebra into a q-algebra

Recall that a bounded linear mapping $u : E \longrightarrow F$, between two Banach spaces, is nuclear if there exist $(x_n^{'}) \subset E^{'}$, $(y_n) \subset F$ and $(\lambda_n) \subset l^1$ such that for all $x \in E$ we have

$$u(x) = \sum_{n=1}^{+\infty} \lambda_n x_n^{'}(x) y_n.$$

A b-space G is nuclear if each bounded completant subset B of G is included in a bounded completant subset A of G such that the inclusion $i_{AB} : G_B \longrightarrow G_A$ is a nuclear mapping. For more information about nuclear b-spaces the reader is referred to [2].

The next result gives an interesting property of nuclear b-algebras.

Proposition 5.1. *Let N be a nuclear b-algebra and $\mathcal{A} \mid \alpha$ be a strict q-algebra. Then a morphism $\varphi : N \longrightarrow \mathcal{A} \mid \alpha$ can be induced by a family of linear mappings $(\varphi_B)_B$, where B ranges over the bounded completant subsets of N, such that :*

1. each $\varphi_B : N_B \longrightarrow \mathcal{A}$ is a bounded.

2. if B' and B are two bounded completant subsets of N such that $B' \subset B$, the linear mapping $\varphi_{B'_{|N_B}} - \varphi_B : N_B \longrightarrow \alpha$ is bounded.

3. if B_1, B_2 and B' are bounded completant subsets of N such that $B_1 \subset B'$ and $B_2 \subset B'$, then the bilinear mapping $N_{B_1} \times N_{B_2} \longrightarrow \alpha$, $(x, y) \longmapsto \varphi_{B'}(x.y) - \varphi_{B_1}(x) . \varphi_{B_2}(y)$ is bounded.

Further, two such systems $(\varphi_B)_B$ and $(\varphi'_B)_B$ induce the same morphism $N \longrightarrow \mathcal{A} \mid \alpha$ iff for each bounded completant subset B of N, the linear mapping $\varphi_B - \varphi'_B : N_B \longrightarrow \alpha$ is bounded.

Proof. Consider an algebra morphism $\varphi : N \longrightarrow \mathcal{A} \mid \alpha$. In particular, φ is a linear morphism. On the other hand, the nuclear b-algebra N is an union $\cup_i N_{1i}$ where each N_{1i} is isometrically isomorphic to the Banach space ℓ^1 [2].

Since any morphism induced by a bounded linear mapping from the free Banach space ℓ^1 is strict [6], we have a bounded linear mapping $\varphi_{1i} : N_{1i} \longrightarrow \mathcal{A}$ which induces the restriction of φ to N_{1i} if $N_{1i} \subset N_{1j}$. It is clear that the linear mmapping $\varphi_{1j|N_{1i}} - \varphi_{1i} : N_{1i} \longrightarrow \alpha$ is bounded.

Consider next i, j and k such that $N_{1i}.N_{1j} \subset N_{1k}$. It follows from the closed graph theorem, that the multiplication mapping $N_{1i} \times N_{1j} \longrightarrow N_{1k}$ is bounded. Now the bilinear mapping

$$N_{1i} \times N_{1j} \longrightarrow \alpha,\ (x, y) \longmapsto \varphi_{1k}(x, y) - \varphi_{1i}(x) - \varphi_{1j}(y)$$

is bounded, because the bimorphisms induced by the mappings

$$(x, y) \longmapsto \varphi_{1k}(x, y)$$

and

$$(x, y) \longmapsto \varphi_{1i}(x) . \varphi_{1j}(y)$$

are equal. This ends the proof of the proposition.

The machinery above applies to a b-algebra which is not nuclear. The following example should be given.

Let Γ be a set of sequences of positive real numbers such that $\delta_{n+n'} \leq \delta_n \delta_{n'}$ for all $(\delta_n) \in \Gamma$, and for all $(\delta_n)_n, (\delta'_n)_n$ there exists a sequence $(\delta''_n) \in \Gamma$ such that $\delta''_n \leq \min(\delta_n, \delta'_n)$ for all n.

Let $\mathcal{C}$ be the set of complex valued sequences (a_n) such that $\sum_n \delta_n |a_n| < \infty$ for all $(\delta_n) \in \Gamma$. We endow the space $\mathcal{C}$ with the boundedness for which a subset B is bounded in if there exist $(\delta_n) \in \Gamma$ and $M \in \mathbb{R}^+$ such that $\sum_n \delta_n |a_n| \leq M$ for all $(a_n) \in B$.

With the convolution, the b-space $\mathcal{C}$ is a b-algebra. In the above proposition if we replace the nuclear b-algebra N by the b-algebra $\mathcal{C}$, then it is clear that the proposition is valid, and its proof applies.

6. Application.

In the final section of this paper, we give some consequences. For this we need to recall that for a Banach algebra $\mathcal{A}$, the sequence $\left(\|a^n\|^{1/n}\right)$ converges when $n \longrightarrow \infty$, and its limit $\rho(a)$ is called the spectral raduis of a. Also, if a and b are two commutatif elements of $\mathcal{A}$, then $\rho(a.b) \leq \rho(a)\rho(b)$ and $\rho(a+b) \leq \rho(a) + \rho(b)$.

Now, we consider the b-algebra $O(U)$ of holomorphic functions on an open subset U of $\mathbb{C}^n$. It is a b-space for its von Neumann boundedness (i.e. the family of the bounded subsets for the topology of $O(U)$). It is also a nuclear b-algebra.

Let $\mathcal{A} \mid \alpha$ be a q-algebra and $a_1, ..., a_n$ be regular elements of $\mathcal{A}$. We define the polynomially convex spectrum $sp_1(a_1, ..., a_n)$ of $(a_1, ..., a_n)$ as the set of $(s_1, ..., s_n) \in \mathbb{C}^n$ such that $|P(s)| \leq \rho(P(a))$ for all P where $\rho(P(a))$ is the spectral raduis of $P(a)$.

Si $\mathcal{A}$ est une algèbre de Banach, la suite $\|a^n\|^{1/n}$ converge si $n \longrightarrow \infty$, la limite $\rho(a)$ de la suite est appelée le rayon spectral de a. Si a et b sont deux éléments qui commutent, alors $\rho(a.b) \leq \rho(a)\rho(b)$ et $\rho(a+b) \leq \rho(a) + \rho(b)$.

Soit $\mathcal{A}$ une algèbre localement convexe, et a un élément de $\mathcal{A}$, alors a est régulier s'il existe M tel que $a^n = o_{\mathcal{A}}(M^n)$, le rayon spectral de a est l'infimum des $M \in \mathbb{R}^+$ tels que $a^n = o_{\mathcal{A}}(M^n)$. On peut changer la définition, et dire que $\rho(a) = \sup_r \left(\overline{\lim}\, r(a^n)^{1/n}\right)$.

Si a et b sont réguliers et commutent, les éléments $a+b$ et $a.b$ sont réguliers et $\rho(a.b) \leq \rho(a).\rho(b)$ et $\rho(a+b) \leq \rho(a) + \rho(b)$.

Notons qu'il existe des éléments réguliers a et b non commutants tels que ni $(a+b)$ et ni $a.b$ ne sont réguliers.

The following result is proved in ([1], theorem 4.4). Its proof is more difficult, but we obtain it here as a consequence of the proposition 5.1

Corollary 6.1. *Let $\mathcal{A} \mid \alpha$ be a unital q-algebra and $a_1, ..., a_n$ be regular elements. Then there exists a q-algebra morphism $O(sp_1(a_1, ..., a_n)) \longrightarrow \mathcal{A} \mid \alpha$ which maps the constant function 1 onto the unit of $\mathcal{A}$ and the variable function z_i onto a_i where z_i is the mapping $\mathbb{C}^n \longrightarrow \mathbb{C}$ such that $z_i(s_1, ..., s_n) = s_i$.*

Finally, we end this paper by the construction with the Arens and Calderon trick.

Let $\mathcal{A} \mid \alpha$ be a unital q-algebra and $a_1, ..., a_n$ be regular elements of the b-algebra $\mathcal{A}$. The joint spectrum of $(a_1, ..., a_n)$ is the set of $(s_1, ..., s_n) \in \mathbb{C}^n$ such that for all regular elements $u_1, ..., u_n$ of $\mathcal{A}$, we have $\sum_i (a_i - e.s_i) u_i \neq 1$ where e is the unit of $\mathcal{A}$.

The next result is proved in ([1], theorem 5.2), we obtain it here as a simple consequence of the proposition 5.1. In fact, we have

Corollary 6.2. *Let $\mathcal{A} \mid \alpha$ be a unital q-algebra and $a_1, ..., a_n$ be regular elements of $\mathcal{A}$. Then a q-algebra morphism $O(sp(a_1, ..., a_n)) \longrightarrow \mathcal{A} \mid \alpha$ exists which maps the function z_i onto the element a_i and maps the constant function 1 onto the unit of $\mathcal{A} \mid \alpha$.*

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